Remarks on 3d flat cosmological horizon holographic descriptions

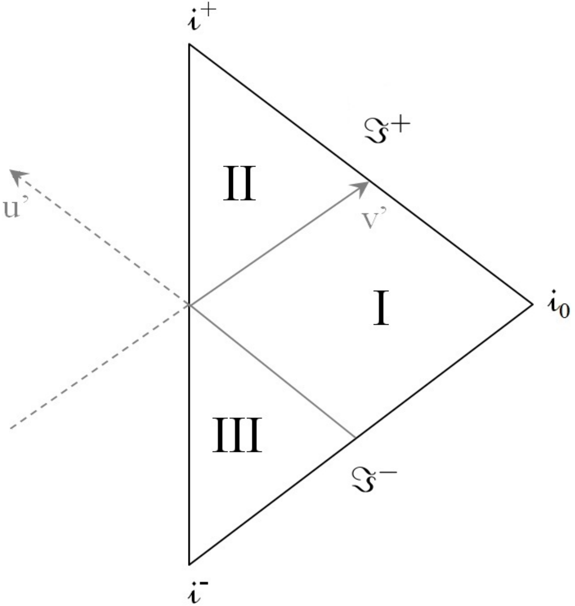
Joan Simón

University of Edinburgh and Maxwell Institute of Mathematical Sciences

Nordita, Stockholm, October 18, 2012

Based on work with A. Bagchi, S. Detournay and R. Fareghbal, arXiv:1208.4372 (see also G. Barnich, arXiv:1208.4371)

Holography : AdS vs Minkowski



Symmetries : a pure kinematic approach

Following Brown-Henneaux, find a set of boundary conditions defining a proper variational principle and giving rise to a closed algebra of conserved charges.

When this is done in $\mathbb{R}^{1,2}$, one obtains the 3d analogue of the Bondi-Metzner-Sachs (BMS) algebra. [Ashtekar,Bicak,Schmidt] whose central charge extended version is [Barnich,Compere]

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + c_{\mathrm{LL}}\, m(m^2-1)\delta_{m+n,0}, \\
[L_m, M_n] &= (m-n)M_{m+n} + c_{\mathrm{LM}}\, m(m^2-1)\delta_{m+n,0}, \\
[M_n,\, M_m] &= 0\,.
\end{aligned}
$$

1. L_m : diffeomorphisms of spatial circle at null infinity
2. M_n : angular dependent supertranslations and translations.

Some order zero kinematic questions

Questions one may ask are

- Is BMS_3 relevant for quantum gravity in $\mathbb{R}^{1,2}$?
- If so, can we use it for anything ?
- What is the connection between BMS_3 and $\mathrm{Vir} \oplus \mathrm{Vir}$ in AdS_3/CFT_2 ?
- If such a connection exists, can we infer the properties a QFT invariant under BMS_3 must satisfy ?
  (i.e. some analogue of modular invariance for 2d CFTs)

Remark

We would like to understand flat holography in any number of dimensions. We will make some progress in 3d because of the infinite number of symmetries existent in AdS_3/CFT_2.

The reason I am partially giving this talk

Assume a 2d CFT with its couple of Virasoro algebras

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}\,,$$
$$[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12} m(m^2-1)\delta_{m+n,0}$$

with central charges

$$c = \bar{c} = \frac{3\ell}{2G}\,.$$

Consider the contraction $\ell \to \infty$ with G fixed [Bagchi; Bagchi-Fareghbal]]

$$L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, M_n = \epsilon\left(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}\right), \; \epsilon = G/\ell \to 0$$

This gives rise to the centrally extended BMS_3 algebra

$$[L_m, L_n] = (m-n)L_{m+n}\,, \quad [M_n,\, M_m] = 0$$
$$[L_m, M_n] = (m-n)M_{m+n} + \frac{1}{4}\, m(m^2-1)\delta_{m+n,0}$$

with $c_{\mathrm{LL}} = 0, c_{\mathrm{LM}} = 1/4$.

Order zero "dual" considerations

AdS$_3$/CFT$_2$ set-up

The 2d CFT is defined on its timelike boundary, a cylinder.

The flat limit as a contraction

- Geometrically, $\mathbb{R}^{1,2}$ appears as the flat limit $\ell \to \infty$ of AdS$_3$.

$$\text{bulk}: \quad \tau \to \frac{t}{\ell}, \; \rho \to \frac{r}{\ell} \qquad \text{boundary}: \quad \tau \to \frac{t}{\ell}$$

- Algebraically, the latter matches $\mathrm{SO}(2,2)$ to $\mathrm{ISO}(1,2)$ contraction.
- Since the Virasoro generators L_n and $\bar{L}_n$ are realised both in the CFT and in the bulk AdS$_3$ asymptotics, we can study how the flat limit/contraction is realised on them
  1. The boundary becomes a null line times a circle.
  2. In 2d, BMS$_3$ is isomorphic to the Galilean conformal algebra (GCA) $\Rightarrow$ relevant 2d theories must be GCA invariant. [Bagchi; Bagchi,Fareghbal]

Outline

- 3d pure Einstein gravity theory : classical phase space
- Most general zero mode solution : shift-boost orbifold $\mathbb{R}^{1,2}/\Gamma$
  1. BMS charges
  2. Global structure : $\exists$ Killing cosmological horizon
  3. Thermodynamics
  4. Flat limit of BTZ
- Partition function of 2d GCA invariant theories
- Semiclassical evaluation
  1. Derivation of a Cardy-like formula
  2. Matching of bulk entropy

Classical phase space

In the absence of sources, the most general solution to 3d pure gravity with vanishing cosmological constant is locally flat

$$ds^2 = \Theta(\psi)du^2 - 2drdu + 2[\Xi(\psi) + \frac{u}{2}\Theta']d\psi du + r^2 d\psi^2 .$$

Asymptotic null infinity preserved by a set of diffeomorphisms whose modes are

$$\ell_n = ie^{in\psi}\left(inu\partial_u - inr\partial_r + \left(1 + n^2\frac{u}{r}\right)\partial_\psi\right), \quad m_n = ie^{in\psi}\partial_u\,, \qquad n \in \mathbb{Z}$$

1. Satisfy the centerless BMS_3 algebra.
2. $\ell_{\pm 1,0}$ and $m_{\pm 1,0}$ form the $\mathfrak{iso}(2,1)$ subalgebra of BMS_3.
3. The coefficients $\{L_n,\ M_n\}$ of the Fourier mode decomposition of the corresponding asymptotically conserved charges determine the arbitrary functions $\Theta(\psi)\,,\Xi(\psi)$:

$$\Theta = -1 + 8G\sum_n M_n\, e^{-in\psi}\,, \quad \Xi = 4G\sum_n L_n\, e^{-in\psi}\,.$$

Most general zero mode solution

Consider $\Theta = 8GM \equiv \hat{r}_+^2$ and $\Xi = 4GJ$ (with $M,\, J \geq 0$ and $\sqrt{\frac{2G}{M}}|J| = r_0$)

$$ds^2 = \hat{r}_+^2 dt^2 - \frac{r^2\, dr^2}{\hat{r}_+^2(r^2 - r_0^2)} + r^2 d\phi^2 - 2\hat{r}_+ r_0 dt d\phi\,,$$

in terms of $d\psi = d\phi + \frac{r_0\, dr}{\hat{r}_+\,(r^2 - r_0^2)}$ and $du = dt + \frac{r^2\, dr}{\hat{r}_+^2\,(r^2 - r_0^2)}$.
$\exists$ Killing horizon with surface gravity κ, angular velocity Ω and entropy

$$T = \frac{\kappa}{2\pi} = \frac{\hat{r}_+^2}{2\pi\, r_0}\,, \quad \Omega = \frac{\hat{r}_+}{r_0}\,, \quad S = \frac{\pi\,|r_0|}{2G}\,.$$

satisfying a first law of thermodynamics

$$-T\, dS = dM - \Omega\, dJ$$

Zero mode solution as a shift-boost orbifold

To identify its quotient nature, consider (for $r > r_0$)

$$\begin{aligned} X^2 &= \frac{r^2 - r_0^2}{\hat{r}_+^2} \sinh^2(\hat{r}_+ \phi) \quad T^2 = \frac{r^2 - r_0^2}{\hat{r}_+^2} \cosh^2(\hat{r}_+ \phi), \\ Y &= r_0 \phi - \hat{r}_+ t \end{aligned}$$

The generator $\xi = \partial_\phi = r_0 \partial_Y + \hat{r}_+ (X \partial_T + T \partial_X)$ acts like

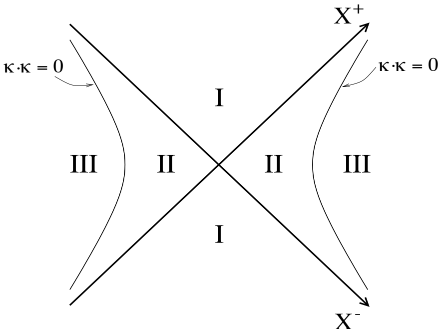
$$X^\pm \sim e^{\pm 2\pi \hat{r}_+} X^\pm, \;\; Y \sim Y + 2\pi r_0$$

Thus,

- $r_0 \neq 0$: shift-boost orbifold [Costa-Cornalba]
- $r_0 = 0$: boost orbifold and vanishing horizon area.

Shift-boost orbifold : global perspective

1. Time dependent background
2. Killing horizon : cosmological horizon
3. Temperature = temperature of radiation generated by particle creation [Costa-Cornalba]



A different perspective : BTZ and flat limits

Non-extremal BTZ

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2\ell^2}\,dt^2 + \frac{r^2\ell^2\,dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\left(d\phi - \frac{r_+ r_-}{r^2\,\ell}d\tau\right)^2$$

1. Conserved charges

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+ r_-}{4G\ell}.$$

2. AdS_3/Γ generated by

$$\xi_{\mathrm{BTZ}} = \frac{r_+}{\ell}J_{12} - \frac{r_-}{\ell}J_{03}$$

$$J_{12} = x\partial_u + u\partial_x\,, \qquad J_{03} = y\partial_v + v\partial_y\,.$$

when viewing AdS_3 as $-u^2 - v^2 + x^2 + y^2 = -\ell^2$ in $\mathbb{R}^{2,2}$.

3. Known CFT description.

The flat limit of BTZ black holes

The flat $\ell \to \infty$ limit corresponds to

$$r_+ \to \ell\sqrt{8GM} = \ell\,\hat{r}_+ \quad \text{and} \quad r_- \to \sqrt{\frac{2G}{M}}\,|J| = r_0$$

1. $\xi_{\text{BTZ}} \to \xi$
2. Outer horizon pushed to infinity.
3. Cosmological horizon is the "remnant" of the BTZ inner horizon
4. Thermodynamics of shift-boost is equivalent to the $\ell \to \infty$ limit of the inner horizon thermodynamics

Order zero dual theory considerations

Following the logic in this talk, quantum gravity states in 3d flat space should transform under representations of the infinite 2d GCA. These are labelled by eigenvalues of L_0, M_0
[Bagchi,Mandal;Bagchi,Gopakumar,Mandal,Miwa]

$$L_0|h_L, h_M >= h_L|h_L, h_M >, \quad M_0|h_L, h_M >= h_M|h_L, h_M >$$
$$\text{where } \; h_L = \lim_{\epsilon\to 0}(h - \bar{h}), \quad h_M = \lim_{\epsilon\to 0}\epsilon(h + \bar{h})$$

The GCA dictionary with bulk BMS charges :

$$h = \frac{1}{2}(\ell M + J) + \frac{c}{24}, \quad \bar{h} = \frac{1}{2}(\ell M - J) + \frac{\bar{c}}{24}$$
$$\Rightarrow h_L = J, \quad h_M = GM + \frac{c_{\mathrm{LM}}}{2} = GM + \frac{1}{8}\,.$$

This suggests the bound $h_M \geq 0$, saturated by $\mathbb{R}^{1,2}$ given that $GM = -1/8$ for global AdS_3 [Barnich,Gomberoff,Gonzalez]

Partition function considerations

2d CFT partition functions

$$Z_{\mathrm{CFT}} = \sum d_{\mathrm{CFT}}(h, \bar{h}) e^{2\pi i(\tau h + \bar{\tau}\bar{h})}$$

are modular invariant. In particular, they are invariant under the S transformation

$$(\tau, \bar{\tau}) \to \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right)$$

At finite ϵ, this partition function can be written as

$$Z_{\mathrm{CFT}} = \sum d(h_{\mathrm{L}}, h_{\mathrm{M}}) e^{2\pi i(\eta h_{\mathrm{L}} + \frac{\rho}{\epsilon} h_{\mathrm{M}})}$$

with S transformation

$$\eta, \rho = \frac{1}{2}(\tau \pm \bar{\tau}) \Rightarrow (\eta, \rho) \to \left(\frac{\eta}{\rho^2 - \eta^2}, \frac{-\rho}{\rho^2 - \eta^2}\right).$$

BMS/GCA invariant partition function

The flat limit requires the temperature to rescale, i.e. $\rho \to \epsilon\rho$

$$Z_{\mathrm{GCFT}}(\eta, \rho) = \sum d(h_{\mathrm{L}}, h_{\mathrm{M}}) e^{2\pi i(\eta h_{\mathrm{L}} + \rho h_{\mathrm{M}})}$$

The finite S transformation remaining after this limit is :

$$(\eta, \rho) \to (-1/\eta, \rho/\eta^2)$$

We will require GCA partition functions to be *invariant* under the latter

$$\boxed{Z^0_{\mathrm{GCFT}}(\eta, \rho) = Z^0_{\mathrm{GCFT}}(-\frac{1}{\eta}, \frac{\rho}{\eta^2})}$$

where Z^0_{GCFT} is the analogue of the 2d CFT partition function on the plane.

Consistency with BMS symmetries

The above S-transformation can be derived from BMS_3 symmetries and euclidean torus considerations.

Partition function : semiclassical evaluation

More precisely,

$$Z^0_{\mathrm{GCFT}}(\eta,\rho) = e^{-\pi i\rho c_{\mathrm{LM}}} Z_{\mathrm{GCFT}}(\eta,\rho),$$

Using the S-transformation leads to

$$Z_{\mathrm{GCFT}}(\eta,\rho) = e^{i\pi c_{LM}\rho(1-\frac{1}{\eta^2})}\; Z_{\mathrm{GCFT}}\Big(-\frac{1}{\eta},\frac{\rho}{\eta^2}\Big).$$

By doing an inverse Laplace transformation, the density of states equals

$$d(h_L,h_M) = \int d\eta d\rho\;\; e^{2\pi i f(\eta,\rho)} Z_{\mathrm{GCFT}}(-1/\eta,\rho/\eta^2)$$

$$\text{where}\;\; f(\eta,\rho) = \left(\frac{c_{\mathrm{LM}}}{2} - \frac{c_{\mathrm{LM}}}{2\eta^2} - h_M\right)\rho - h_L\eta\,.$$

In the limit of large charges, there is a saddle at $\eta \approx i\sqrt{c_{\mathrm{LM}}}/\sqrt{2h_M}$ whenever $Z_{\mathrm{GCFT}}(-1/\eta,\rho/\eta^2)$ is slowly varying, which occurs at positive $i\rho$, i.e. negative GCFT temperature. The 2d GCFT entropy is then

$$S_{\mathrm{GCFT}} = \log d(h_L,h_M) = 2\pi\, h_L\sqrt{\frac{c_{\mathrm{LM}}}{2h_M}}$$

Matching of entropies

Using our previous dictionary

$$\boxed{S_{\mathrm{GCFT}} = \frac{\pi|J|}{\sqrt{2GM}} = S\,.}$$

GCFT reproduces the bulk thermodynamics but with a negative temperature

$$\frac{\partial S}{\partial h_L} = \frac{\Omega}{T}\,, \quad \frac{1}{T_{\mathrm{GCFT}}} \equiv \frac{\partial S}{\partial h_M} = -\frac{1}{G\,T}\,.$$

Remark on cosmological description

As in BTZ, we can ask what the relation is between the quantum numbers measured by a cosmological observer, who sees a contraction and an expansion of the universe, and the quantum numbers at null infinity. The frequencies measured by the former agree, up to a $\hat{r}_+$ rescaling with the ones measured by the latter. This is consistent with the work of Costa-Cornalba.

Lessons & open questions

- BMS3 as a symmetry of quantum gravity in $\mathbb{R}^{1,2}$ $\Rightarrow$ dual theory must be GCA invariant
  1. Representation theory ?
- Derivation of a Cardy-like formula for GCA theories reproducing the semiclassical (gravity) result.
  1. Corrections to our Cardy formula given details of the representation theory.
- Purely kinematic approach : dynamical approach ?
  1. Connection with S-matrix formulation ?
- Higher dimensions ? String theory embedding ? Susy extensions ?
- Thermodynamics : $\exists$ of a Hawking-Page like phase transition between hot $\mathbb{R}^{1,2}$ and the shift-boost orbifold ?
  1. Hawking-Gibbons term reproduces the thermodynamics described today
  2. This suggests no further boundary counterterms are required
  3. Comparing free energies, the phase transition may exist $\Rightarrow$ consequences for 2d GCA theories