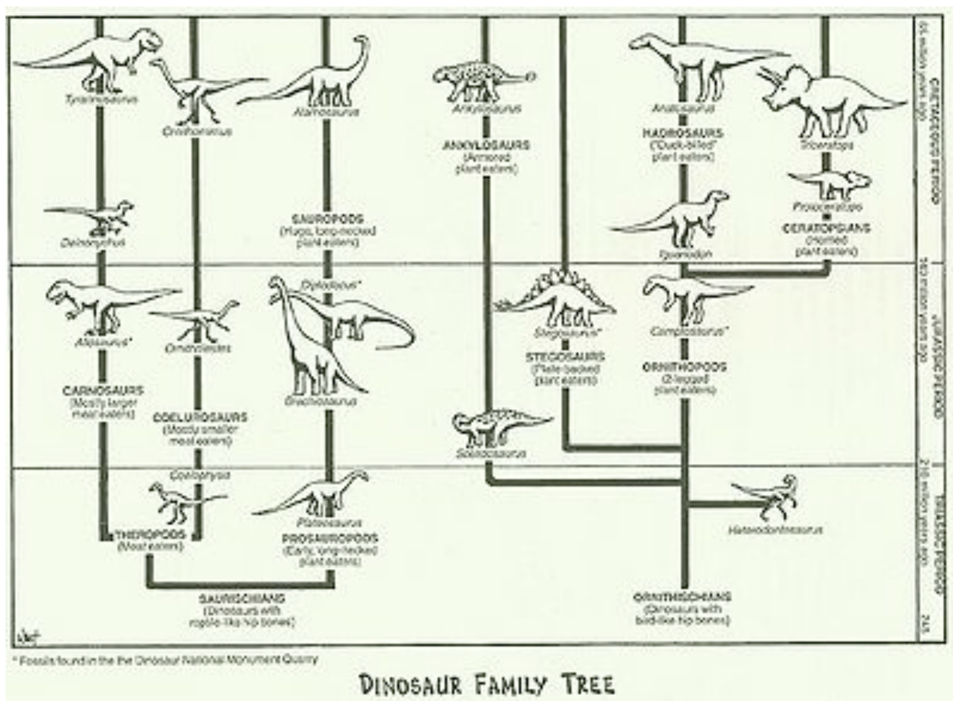
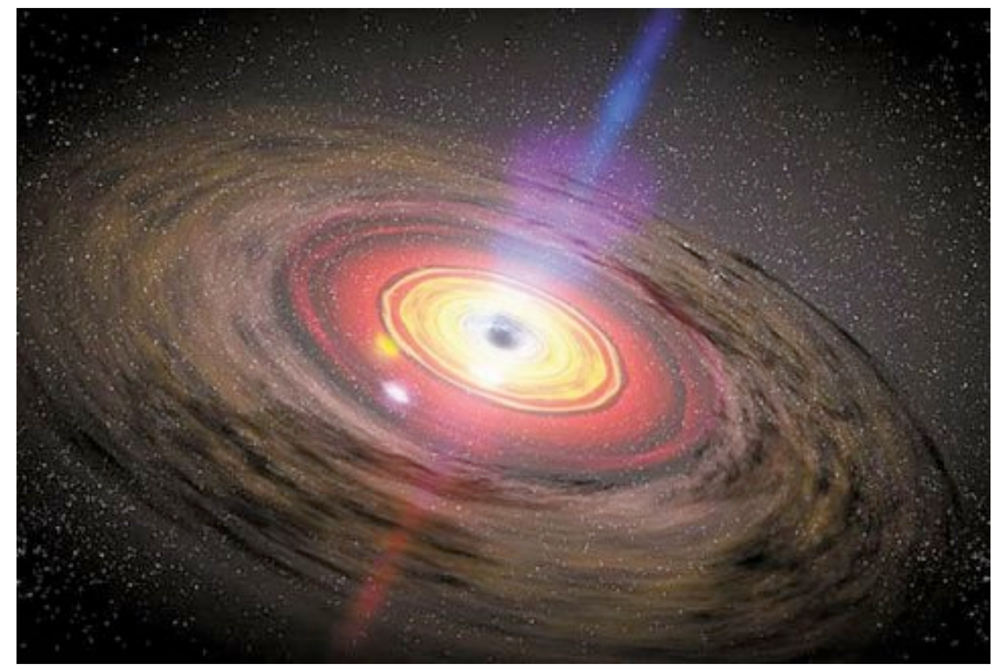
Towards a taxonomy of extremal black brane horizons



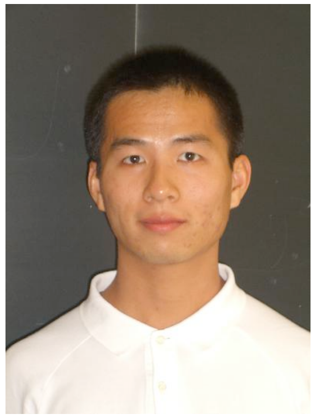


Shamit Kachru
(Stanford & SLAC)

NATIONAL ACCELERATOR LABORATORY

My recent collaborators on related subjects

Stanford/SLAC team:



X. Dong

S. Harrison

G. Torroba

H. Wang

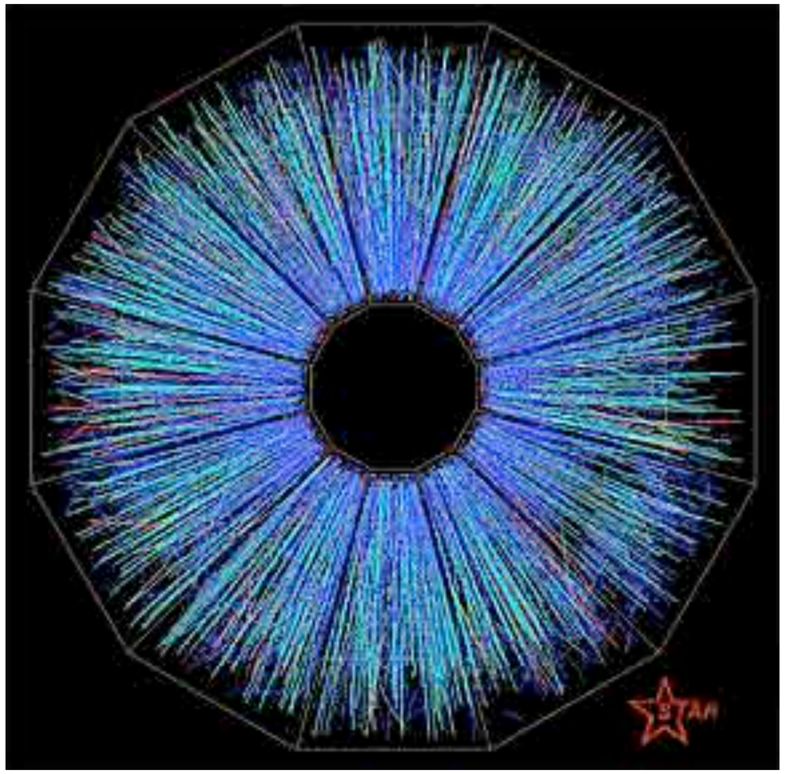
TIFR team:

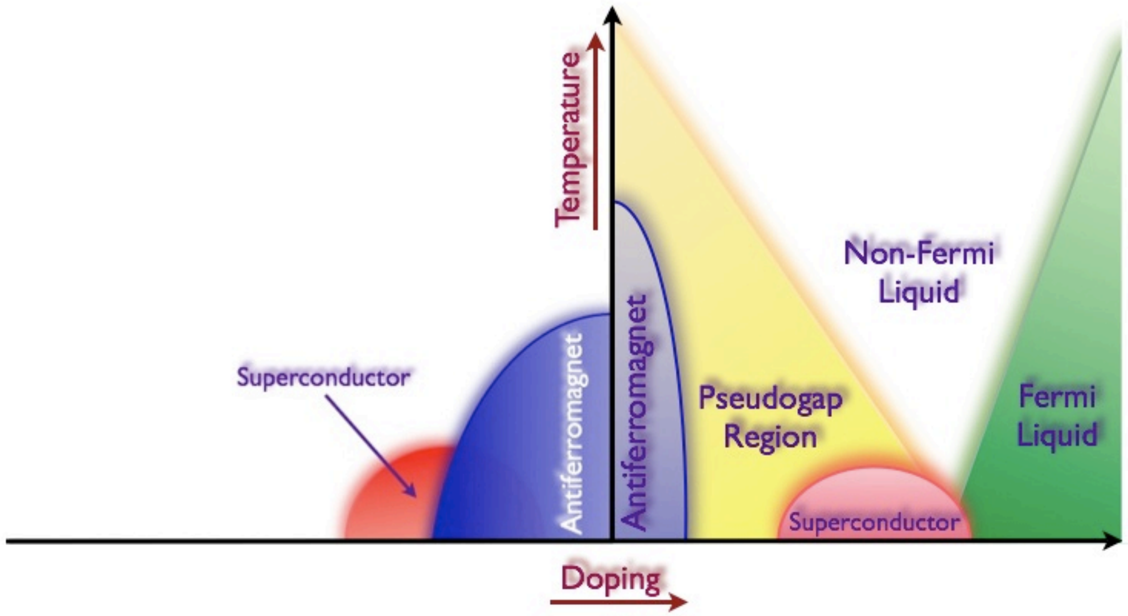
S. Trivedi
(fearless leader)

arXiv 1201.1095, 1201.4861, 1202.6635 & to apppear

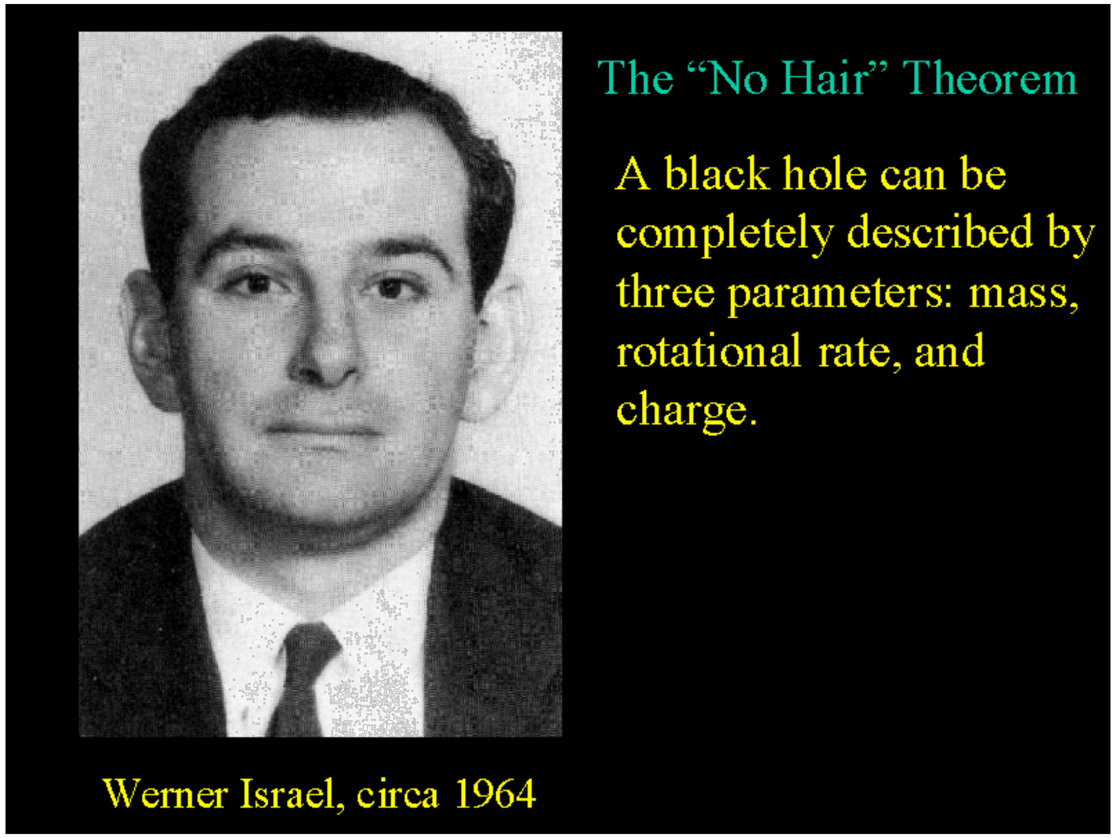
I. Introduction

An ab initio interesting question in theoretical physics is the classification of states of matter. Interesting new states are thought to arise in finite density QCD; in materials of modern interest in condensed matter physics; and probably other places as well.





An a priori different question is the classification of black hole solutions in general relativity (or its extensions with simple matter sectors). In asymptotically flat 4d Minkowski space, impressive results were obtained by the early 1970s.



Gauge/gravity duality invites us to connect the two questions.

Finite temperature field theory is represented by the AdS/ Schwarzschild black brane:

$$ds^2 = \frac{L^2}{z^2}(-f dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f})$$

$$f = 1 - \frac{z^d}{z_H^d}$$

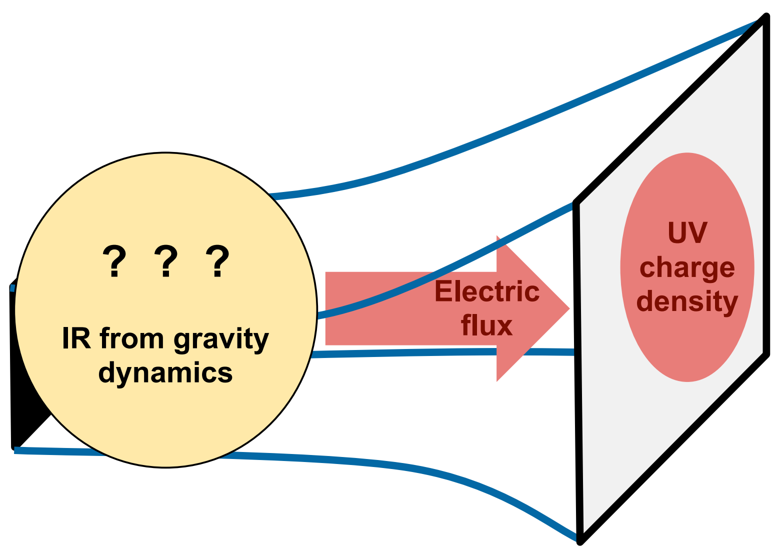
"Doping" by adding charge density, one is led to study instead a charged black brane. It is familiar from flat space relativity that there are extremal avatars of such branes:

$$ds^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2 d\Omega^2$$

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2} \ .$$

$$GM^2 = p^2 + q^2$$

The no-hair theorems fail in AdS space. Charged black brane horizons can take a variety of forms: Gubser; ...



The problem of classifying low-energy phases of doped matter can be mapped to the problem of classifying extremal black brane geometries.

While attempts at classification (of 2d cfts, string vacua, etc.) have a checkered history, we will put aside suspicion and pursue this line of thought:

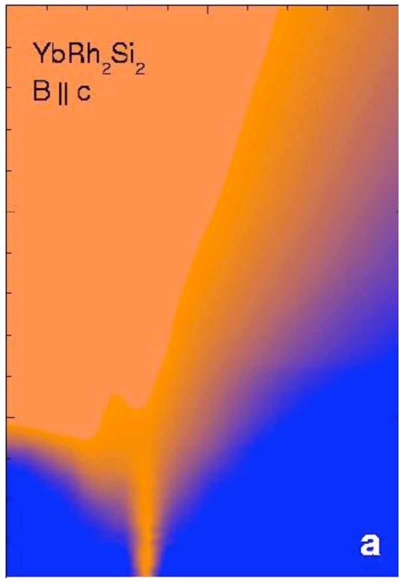
II. The basic horizon
   A. Near-horizon limit
   B. Interesting physics

III. More general homogeneous, isotropic geometries
   A. Lifshitz geometries
   B. Hyperscaling violation & entanglement
   C. Instabilities

IV. Homogeneous, anisotropic geometries
   A. Bianchi horizons
   B. More general 4-algebras

At various times, we will take detours to explore the physical properties of the horizons we've discovered.

We will often comment on their possible relationship to theories of Fermi surfaces and non-Fermi liquids.



It is important to remember that this is a problem of classical & quantum gravity in its own right, however, and should be understood on its own terms.

II. The basic horizon

In AdS/CFT, one represents a (global) U(1) symmetry of the field theory by a bulk Abelian gauge field.

So the simplest theory where one can ask about field theory at finite density is:

$$S = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left[\mathcal{R} - \frac{6}{R^2} - \frac{R^2}{g_F^2}F_{MN}F^{MN}\right]$$

This theory has simple, exact solutions which represent the planar analogue of the Reissner-Nordstrom black hole.

The AdS/RN solution has a metric:

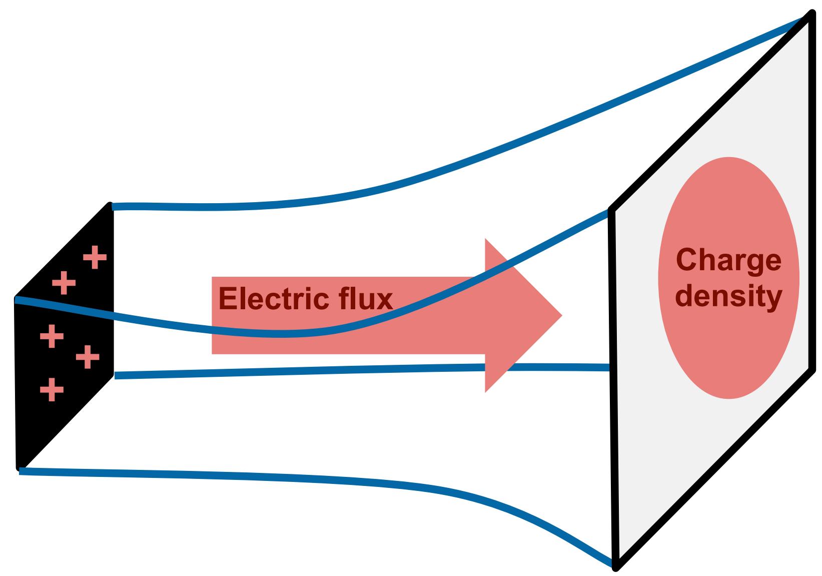
$$ds^2 \equiv g_{MN} dx^M dx^N = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}\frac{dr^2}{f}$$

Chamblin, Emparan, Johnson, Myers

with gauge field and "emblackening factor":

$$f = 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d}, \qquad A_t = \mu\left(1 - \frac{r_0^{d-2}}{r^{d-2}}\right) \ .$$

The extremal limit arises when Q reaches a value where f develops a double zero at the outer horizon. This black brane represents the zero temperature ground state.



At extremality, the near-horizon geometry simplifies to $AdS_2 \times R^2$:

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + dx^2 + dy^2$$

It has some interesting (peculiar) features:

1. The spatial slices don't contract at the horizon. This leads directly to an extensive ground state entropy.

2. A natural notion at RG fixed points is dynamical scaling. We're used to fixed points of the renormalization group with a scale invariance:

$$x \to \lambda x, \; t \to \lambda t$$

In a doped system, even if you have IR rotation invariance, you may well expect instead:

$$x \to \lambda x, t \to \lambda^z t, \quad z \neq 1$$

By a simple change of variables, you can see that the IR extremal geometry here realizes this with

$$z = \infty$$

Naively, this too could indicate that such a fixed point is fine tuned. Large z is hard to realize in weakly coupled QFT. E.g. to get z=2, one can consider:

$$\mathcal{L} = \int d^2x \; dt \; \left((\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2 \right) \; .$$

One has tuned away the standard gradient term. Higher z would require tuning away more such operators.

But, this is just weakly coupled intuition, and could be misleading at strong coupling.

Much interesting physics can arise by coupling fermions to such an IR theory. For instance, the large z critical theory easily dresses them into non-Fermi liquids. Liu, McGreevy, Vegh; S.S. Lee; Leiden group

But this isn't our focus today, and we move on to consider...

III. More general homogeneous, isotropic horizons

Gravity + Maxwell is the minimal content to study doped holographic matter.

It is quite reasonable to expect additional light fields in the bulk!

Let us consider adding a neutral scalar "dilaton" field to the system. To begin with, we consider:

A. Theories with no dilaton potential

M. Taylor; Goldstein, S.K., Prakash, Trivedi

$$S = \int d^4x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - e^{2\alpha\phi}F^2 - 2\Lambda\right) \ .$$

We restrict to metrics of the form:

$$ds^2 = -a(r)^2\ dt^2 + a(r)^{-2}\ dr^2 + b(r)^2\ (dx^2 + dy^2)$$

One can solve for the gauge field using Gauss' law:

$$e^{2\alpha\phi}F = \frac{Q}{b(r)^2}\ dt \wedge dr.$$

The result is solutions with near-horizon geometry:

$$a \sim r^{\gamma},\ b \sim r^{\beta},\ \phi = -K\ \log(r)$$

where one easily finds:

$$\gamma = 1,\ K = \frac{\frac{\alpha}{2}}{1+(\frac{\alpha}{2})^2},\ \beta = \frac{(\frac{\alpha}{2})^2}{1+(\frac{\alpha}{2})^2}\ .$$

After a simple change of variables, the metric takes the form of a "Lifshitz" metric representing emergent dynamical scaling with dynamical exponent z:

$$ds^2 = -r^2 dt^2 + r^{2z}(dx^2 + dy^2) + \frac{dr^2}{r^2}$$

SK, Liu, Mulligan

$$t \to \lambda^z t, \ (x, y) \to \lambda(x, y), r \to \frac{r}{\lambda}, z = \frac{1}{\beta}$$

The near-horizon geometry glues into asymptotically AdS space with constant dilaton.

Comparing to AdS/RN:

1. The (tunable) value of z controls scaling laws for thermodynamic observables:

$$s \sim T^{2\beta} \mu^{2-2\beta} \ .$$

2. The finite value of z relaxes the extensive ground state entropy of the Einstein/Maxwell theory.

3. Probe fermions coupled to finite z gravity duals also realize non-Fermi liquid transport. z=2 can yield linear resistivity.

Hartnoll, Polchinski, Silverstein, Tong

B. Hyperscaling violation

So far, we considered vanishing dilaton potential. It is more reasonable to postulate:

$$\mathcal{L}_{EMD} = \frac{1}{2\kappa^2}\left(R - 2\left(\nabla\Phi\right)^2 - \frac{V(\Phi)}{L^2}\right) - \frac{Z(\Phi)}{4e^2} F_{\mu\nu}F^{\mu\nu}$$

Rather natural choices for the functions (motivated by e.g. gauged supergravity), which yield scaling solutions, are:

$$\begin{aligned} Z(\Phi) &= Z_0 \exp\left(\alpha\, \Phi\right) \\ V(\Phi) &= -V_0 \exp\left(-\beta\, \Phi\right) \end{aligned}, \qquad \text{as } \Phi \to \infty.$$

Charmousis, Gouteraux, Kim, Kiritsis, Meyer

The first parameter, in the gauge coupling function, transmuted into a dynamical critical exponent z.

With two parameters in L, we find solutions that reflect two critical exponents: z and a "hyperscaling violation" parameter θ :

$$ds^2_{d+2} = r^{-2(d-\theta)/d}\left(-r^{-2(z-1)}dt^2 + dr^2 + dx_i^2\right) \ .$$

These are conformal to scale invariant metrics:

$$x_i \to \lambda x_i \ , \ t \to \lambda^z t \ , \ r \to \lambda r \ , \ ds \to \lambda^{\theta/d} ds \, .$$

The finite temperature deformations are also known:

$$ds^2_{d+2} = \frac{R^2}{r^2}\left(\frac{r}{r_F}\right)^{2\theta/d}\left(-r^{-2(z-1)}f(r)dt^2 + \frac{dr^2}{f(r)} + dx_i^2\right) ,$$

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{d+z-\theta} .$$

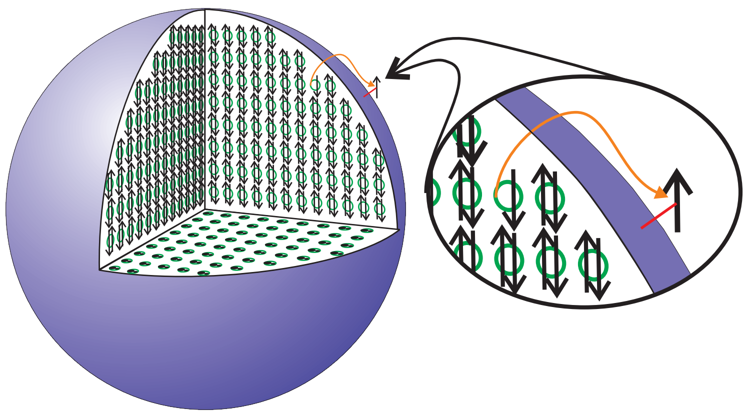
$$T = \frac{1}{4\pi}\frac{|d+z-\theta|}{r_h^z} .$$

This results in an entropy that scales like:

$$\mathcal{S}_T \sim (M_{Pl}R)^d V \frac{T^{(d-\theta)/z}}{r_F^\theta} .$$

Hyperscaling violation shifts d!

An example of a system which enjoys such “shifted d” in the real world, is a Fermi liquid. Basic picture:



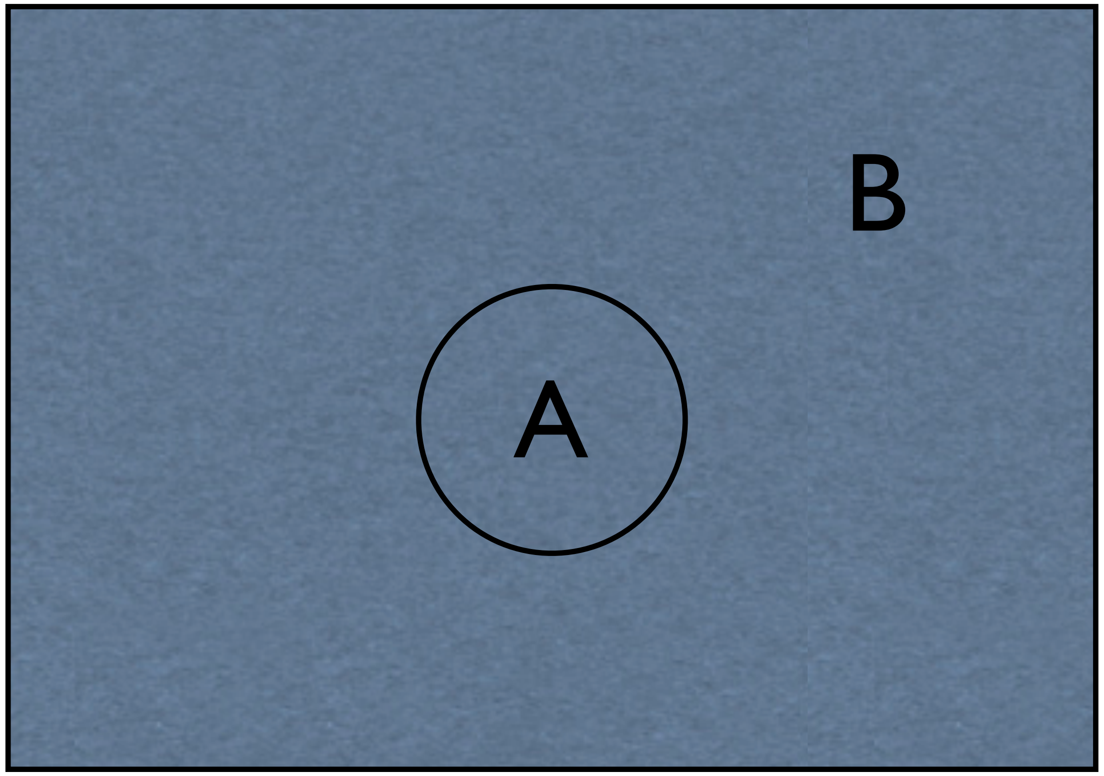
A codimension one surface in k-space divides occupied from unoccupied states. Only the orthogonal direction “scales” at a given point on the Fermi surface. It is like a surfaces worth of 1+1 dimensional CFTs.

One place where this shows up quantitatively is in the entanglement entropy:

$$\rho_{tot} = |\Psi\rangle\langle\Psi|$$

$$\rho_A = \mathrm{tr}_B \ \rho_{tot},$$

$$S_A = -\mathrm{tr}_A \, \rho_A \log \rho_A.$$

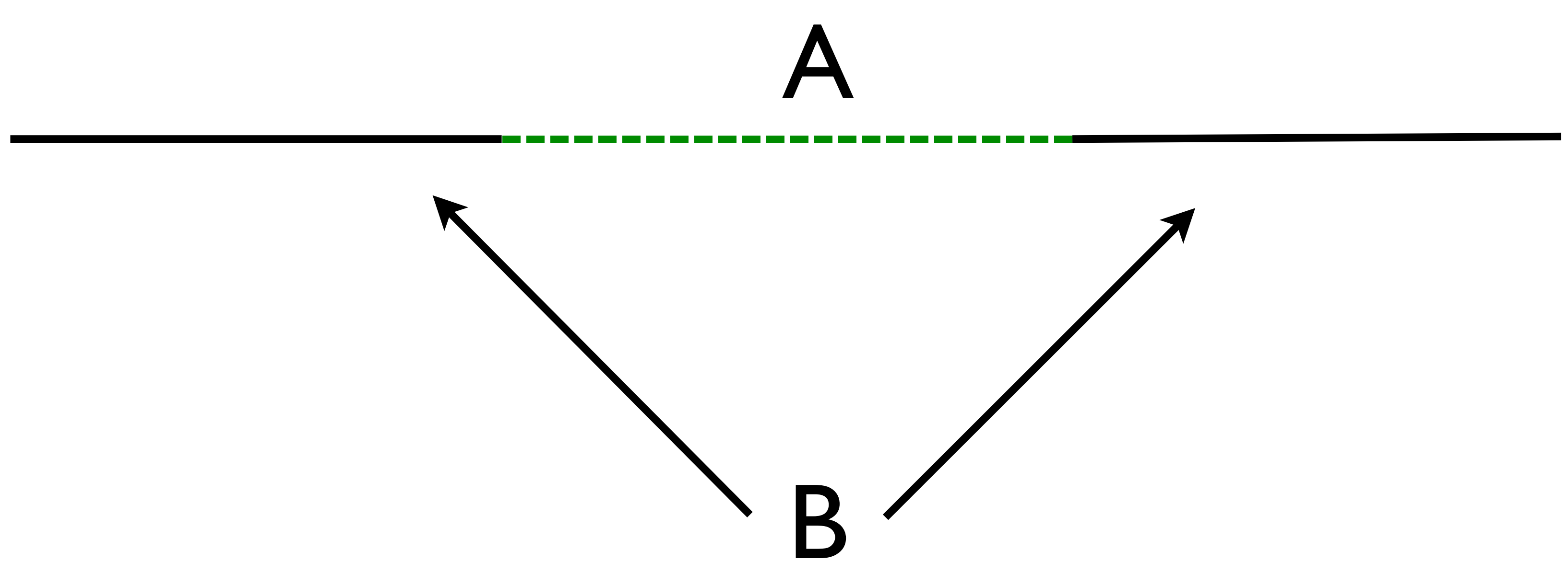


In general, one expects an "area law" for the entanglement (with a UV-cutoff dependent coefficient). This represents the entanglement of microscopic d.o.f. across the boundary.

One well-known violation of the area law occurs in 2D CFTs. There, the result is:

$$S_A \sim \frac{c}{3} \log l/a$$

Holzhey, Larsen, Wilczek



The Fermi liquid actually famously gives a logarithmic violation of the "area law" which is universal. Eg in d=2:

$$S_L \sim k_F L \, \log(L) + \frac{L}{\epsilon} + \cdots$$

By setting

$$\theta = d - 1$$

Huijse, Sachdev, Swingle

we can match the thermodynamic behavior of a Fermi surface, in terms of T-scaling of entropy, heat capacity, etc.

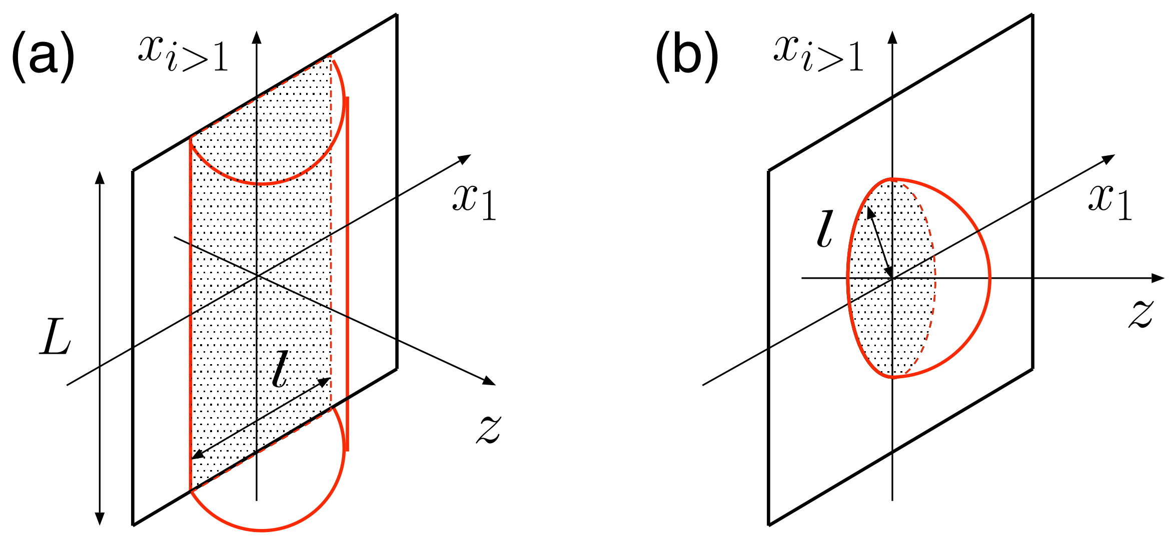
Do we also match the anomalous entanglement entropy?

Ogawa, Takayanagi, Ugajin

There is a formula for computing the holographic entanglement entropy, proposed by Ryu and Takayanagi:

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}},$$

with γ_A the minimal surface with appropriate boundary, extending into the bulk of AdS space.



Minimal surfaces for strip and circular regions A on the boundary

Simple minimal surface calculations in the hyperscaling violating metrics reveal that:

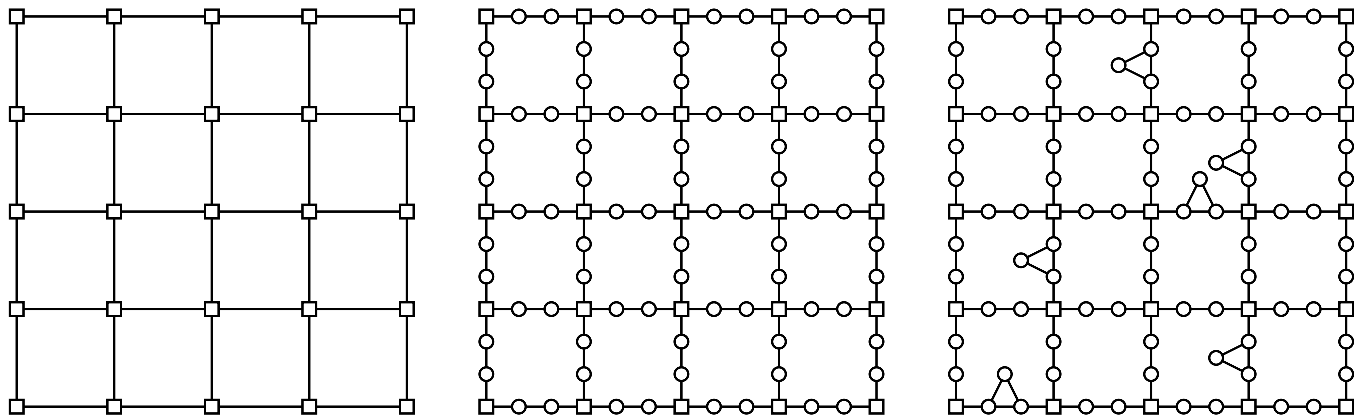
* The metrics with

$$\theta = d - 1$$

have both the entanglement and free energy scalings of a theory with a Fermi surface.

* For $d - 1 \leq \theta \leq d$ one can find enhancements which range from log to linear in the entanglement entropy.

Dong, Harrison, SK, Torroba, Wang



(a) The original graph Λ (b) The decorated graph Λ_3 (c) The decorated graph with quenched disorder $\Lambda_{3,t}$ with $t = 6$

Such phases were found in field theory models on "decorated lattices" after the gravity papers.

Huijse, Swingle

C. Instability of dilatonic metrics

In all of the solutions we've discussed, the dilaton is running close to the horizon:

$$\phi = -K\ \log(r)$$

The coupling takes the form:

$$g = e^{-\alpha\phi}$$

In 4d bulk, there are also magnetic solutions related to the electric solutions we've been talking about, roughly by

$$g \leftrightarrow \frac{1}{g}$$

With standard conventions, g flows to weak coupling at the "electric" horizons and strong coupling at the "magnetic" horizons. Both cases are dangerous.

a) For the electrically charged black brane (which I discussed), the coupling vanishes at the horizon.

Why is this dangerous? In a theory like string theory, there is a UV scale lower than the Planck scale:

$$M_{\mathrm{string}} = g M_{\mathrm{Planck}}$$

Higher derivative operators in Einstein frame are suppressed by powers of this scale:

$$\mathcal{L} = \sqrt{-g}\left(R + \alpha' R^2 + \cdots\right)$$

If the horizon has zero coupling, α' corrections matter.

Now, an arbitrarily small temperature serves to smooth out this issue. But for the strict zero-T solution, one expects a deformation.

Sen; Dabholkar, Kallosh, Maloney

b) We have a 4d bulk, and at leading order the theory enjoys electric-magnetic duality. We could just as well consider the magnetic solutions. There, the opposite problem occurs:

g flows to strong coupling at the horizon!

Then, we can neglect higher derivatives, but not g corrections to e.g. gauge couplings.

For magnetic branes, in some cases the "very near-horizon fate" of the emergent Lifshitz solution (with running dilaton) is as follows:

Harrison, SK, Wang

$$S = \int d^4x \, \sqrt{-g} \left(R - 2(\nabla\phi)^2 - e^{2\alpha\phi} F^2 - 2\Lambda \right) .$$

$$e^{2\alpha\phi} F^2 \rightarrow f(\phi) F^2$$

$$f(\phi) = e^{2\alpha\phi} + \xi_1 + \xi_2 \; e^{-2\alpha\phi} + \xi_3 \; e^{-4\alpha\phi} + \cdots = \frac{1}{g^2} + \xi_1 + \xi_2 \; g^2 + \xi_3 \; g^4 + \cdots$$

Result: the "attractor potential" for the dilaton now has a near-horizon minimum, and one can find flows that:

* Asymptote to AdS_4

* Exhibit Lifshitz-scaling with the expected "z" over many decades in energy

* End up in $AdS_2 \times R^2$ (with much reduced entropy)

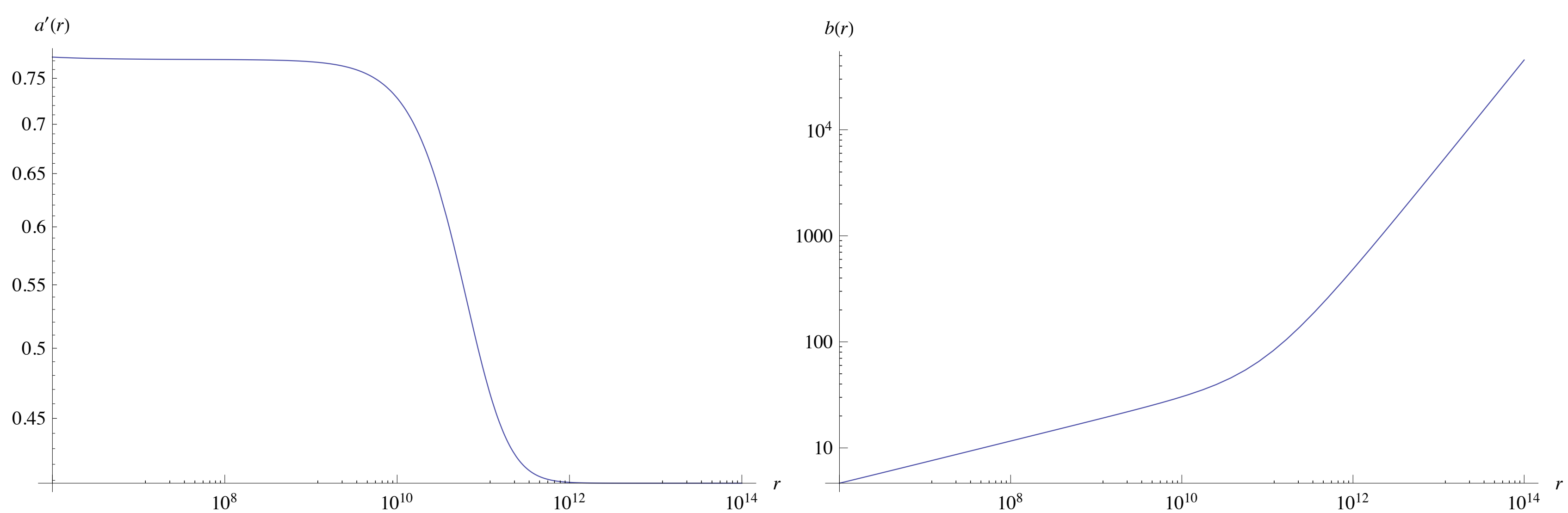


Figure 2: Here we see a log-log plot of the crossover from Lifshitz scaling to AdS_4 in the metric functions $a(r)$ and $b(r)$. The crossover occurs around $r = 10^{11}$. For $r < 10^{11}$, the Lifshitz region persists over several decades in r, while for $r > 10^{11}$, the solution becomes AdS_4. Left: $a'(r)$; right: $b(r)$. The flow in $a(r)$ just reflects the fact that the coefficient of the linear term in $a(r) \sim r$ is different in the Lifshitz and AdS_4 regions. The change in slope in the log-log plot for $b(r)$ indicates the difference between a solution with dynamical scaling ($z = 5$, for our choice of parameters) and the $z = 1$ characteristic of AdS_4.

The hyperscaling metrics can similarly be "IR completed" by AdS2 regions, in some cases.

Bhattacharya, Cremonini, Sinkovics

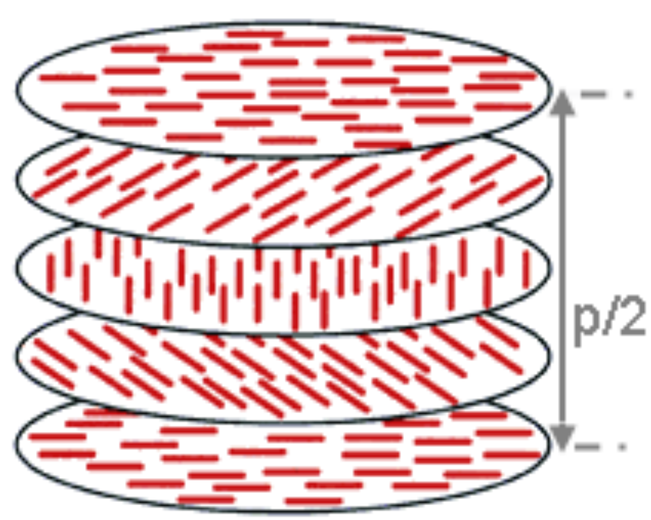
But as AdS2 itself is under suspicion of various instabilities, this leads to an Ouroborosian scenario...

Some of the instabilities of AdS2 that have been discussed in the literature, break spatial translations. This provides a natural transition to our next topic...

Gauntlett et al;...

IV. Homogeneous, anisotropic geometries

All the phases we've discussed (and most discussed in the literature) have enjoyed normal translation symmetry. But in real systems, often the low T states break naive translations:



As a starting point for a classification, we would like to classify the most general homogeneous, anisotropic extremal black brane geometries.

Iizuka, SK, Kundu, Narayan, Sircar, Trivedi

Homogeneous: For a theory in d spatial dimensions, there should be a d-dimensional group action which relates each point to its neighbors.

That, is there should be d Killing vectors whose commutators give rise to a Lie algebra. Only the trivial algebra gives "normal" translations.

Example:

Imagine in our three-dimensional space, we enjoy usual translation symmetries along two of the directions. But along the third, one must translate as well as rotating in the transverse plane, to get a symmetry.

$$\xi_1 = \partial_2, \quad \xi_2 = \partial_3,$$

$$\xi_3 = \partial_1 + x^2 \partial_3 - x^3 \partial_2.$$

← vector fields which generate our generalised translations

These generate a homogeneous space, in the sense that each point in an infinitesimal neighborhood can be transported to each other point.

However, the commutation relations define a Lie algebra which is invariantly distinct from the algebra of translations:

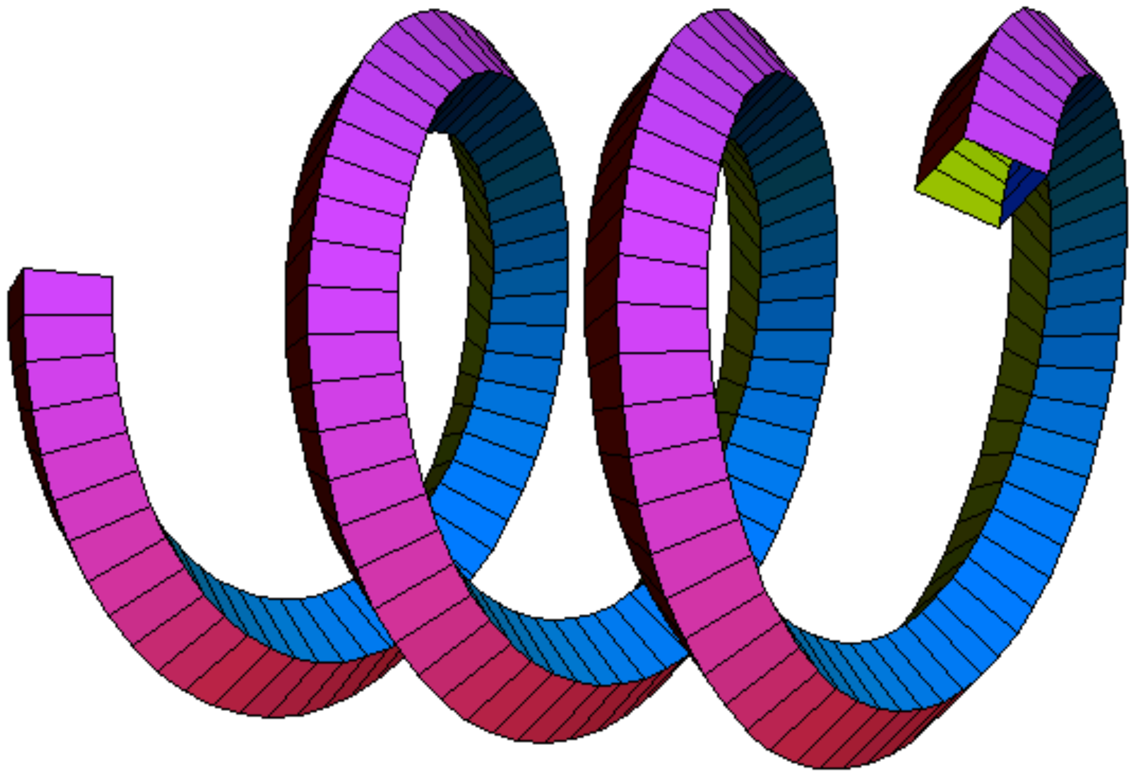
$$[\xi_1, \xi_2] = 0; \quad [\xi_1, \xi_3] = \xi_2; \quad [\xi_2, \xi_3] = -\xi_1,$$

In fact, these generalised translations could leave invariant a vector order parameter (whose expectation value "breaks" normal translations):

$$\delta V = \epsilon[\xi_i, V].$$

$$V^1 = \text{constant}, \quad V^2 = V_0 \cos(x^1 + \delta), V^3 = V_0 \sin(x^1 + \delta)$$

Or in a picture:



Happily, for the application to phases in 3 spatial dimensions, all possible algebras of this sort have been classified. This is the Bianchi classification, also of use in theoretical cosmology.

The basic mathematical structure is as follows:

* For each of the 9 inequivalent algebras, there are three "invariant one-forms" ω^i left invariant under all three isometries.

* A metric expressed in terms of these forms with constant coefficients will automatically be invariant.

* The one-forms satisfy the relations:

$$d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k$$

with C the structure constants of the algebra.

Now, in holography we really have an extra spatial dimension. We also have time. Natural assumptions:

$$r \to r + \epsilon, \;\; t \to e^{-\beta_t \epsilon} t \; ;$$
$$t \to t + \text{const} \; .$$

← Maintain as symmetries

Then the general "allowed" near-horizon metrics take the form:

$$ds^2 = R^2 [dr^2 - e^{2\beta_t r} dt^2 + \eta_{ij} e^{(\beta_i + \beta_j) r} \omega^i \otimes \omega^j]$$

I.e. given the Bianchi type, there are a finite set of constants one must solve for to get the scaling metric.

* Can find solutions in most Bianchi types in Einstein gravity + massive vector, by solving algebraic equations.

* 7 of the types, as far as we can tell, are entirely new as symmetries of black-brane solutions.

* Close analogues of type VII have been previously found as instabilities of other holographic phases. Domokos, Harvey; Nakamura, Ooguri, Park; Donos, Gauntlett

* Gluing these solutions into AdS seems possible. We have done it for type VII (numerically).

Generalization to 4-algebras:

In fact, the Bianchi classification isn't quite what we ordered for 5-geometries dual to 4d field theories. Even if we wish to study static geometries, we really have 4d spatial slices in the bulk.

We should study real 4d Lie algebras with 3d subalgebras that give the symmetry preserved in the "field theory" spatial directions.

These were classified ~60 years after Bianchi.

There are 12 4-algebras and a myriad of possible embeddings of 3-algebras within.

MacCallum;
Patera, Winternitz

- $A_{4,1}$: $C^1_{24} = 1,\ C^2_{34} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3 \quad e_4 = x_2\partial_1 + x_3\partial_2 + \partial_4$
  $\omega_1 = dx^1 - x_4 dx^2 + \frac{1}{2}x_4^2 dx^3 \quad \omega_2 = dx^2 - x_4 dx^3 \quad \omega_3 = dx^3 \quad \omega_4 = dx^4$

- $A^a_{4,2}$: $C^1_{14} = a,\ C^2_{24} = 1,\ C^2_{34} = 1,\ C^3_{34} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3 \quad e_4 = ax_1\partial_1 + (x_2 + x_3)\partial_2 + x_3\partial_3$
  $\omega_1 = e^{-ax_4}dx^1 \quad \omega_2 = e^{-x_4}(dx_2 - x_4 dx_3) \quad \omega_3 = e^{-ax_4}dx^3 \quad \omega_4 = dx_4$

- $A_{4,3}$: $C^1_{14} = 1,\ C^2_{34} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3 \quad e_4 = x_1\partial_1 + x_3\partial_2 + \partial_4$
  $\omega_1 = e^{-x_4}dx^1 \quad \omega_2 = dx^2 - x_4 dx^3 \quad \omega_3 = dx^3 \quad \omega_4 = dx^4$

- $A_{4,4}$: $C^1_{14} = 1,\ C^1_{24} = 1,\ C^2_{24} = 1,\ C^2_{34} = 1,\ C^3_{34} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3$
  $e_4 = (x_1 + x_2)\partial_1 + (x_2 + x_3)\partial_2 + x_3\partial_3 + \partial_4$
  $\omega_1 = e^{-x_4}(dx^1 - x^4 dx^2 + \frac{1}{2}x_4^2 dx^3) \quad \omega_2 = e^{-x_4}(dx^2 - x_4 dx^3) \quad \omega_3 = e^{-x_4}dx^3 \quad \omega_4 = dx_4$

- $A^{a,b}_{4,5}$: $C^1_{14} = 1,\ -C^2_{24} = a,\ C^3_{34} = b$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3 \quad e_4 = x_1\partial_1 + ax_2\partial_2 + bx_3\partial_3 + \partial_4$
  $\omega_1 = e^{-x_4}dx^1 \quad \omega_2 = e^{-ax_4}dx^2 \quad \omega_3 = e^{-bx_4}dx^3 \quad \omega_4 = dx^4$

- $A^{a,b}_{4,6}$: $C^1_{14} = a,\ C^2_{24} = b,\ C^3_{24} = -1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3$
  $e_4 = ax_1\partial_1 + (bx_2 + x_3)\partial_2 + (bx_3 - x_2)\partial_3 + \partial_4$
  $\omega_1 = e^{-ax_4}dx^1 \quad \omega_2 = e^{-bx_4}[cos(x_4)dx^2 - sin(x_4)dx^3]$
  $\omega_3 = e^{-bx_4}(cos(x_4)dx^3 + sin(x_4)dx^2) \quad \omega_4 = dx^4$

- $A_{4,7}$: $C^1_{14} = 2,\ C^2_{24} = 1,\ C^2_{34} = 1,\ C^3_{34} = 1,\ C^1_{23} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \quad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$
  $e_4 = 2x_1\partial_1 + (x_2 + x_3)\partial_2 + x_3\partial_3 + \partial_4$
  $\omega_1 = e^{-2x_4}(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2) \quad \omega_2 = e^{-x_4}(dx^2 - x_4 dx^3)$
  $\omega_3 = e^{-x_4}dx^3 \quad \omega_4 = dx^4$

- $A_{4,8}$: $C^1_{23} = 1,\ C^2_{24} = 1,\ C^3_{34} = -1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \quad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$
  $e_4 = x_2\partial_2 - x_3\partial_3 + \partial_4$
  $\omega_1 = dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}dx^3 dx^2 \quad \omega_2 = e^{-x_4}dx^2 \quad \omega_3 = e^{x_4}dx^3 \quad \omega_4 = dx^4$

- $A^b_{4,9}$: $C^1_{23} = 1,\ C^1_{14} = 1 + b,\ C^2_{24} = 1,\ C^3_{34} = b$
  $e_1 = \partial_1 \quad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \quad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$
  $e_4 = (1 + b)x_1\partial_1 + x_2\partial_2 + bx_3\partial_3 + \partial_4$
  $\omega_1 = e^{-(b+1)x_4}(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2) \quad \omega_2 = e^{-x_4}dx^2 \quad \omega_3 = e^{-bx_4}dx^3 \quad \omega_4 = dx^4$

- $A_{4,10}$: $C^1_{23} = 1,\ C^3_{24} = -1,\ C^2_{34} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1 \quad e_3 = \partial_3 + \frac{1}{2}x_2\partial_1$
  $e_4 = -x_2\partial_3 + x_3\partial_2 + \partial_4$
  $\omega_1 = dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2 \quad \omega_2 = cos(x_4)dx^2 - sin(x_4)dx^3 \quad \omega_3 = cos(x_4)dx^3 + sin(x_4)dx^2$
  $\omega_4 = dx^4$

- $A^a_{4,11}$: $C^1_{23} = 1,\ C^1_{14} = 2a,\ C^2_{24} = a,\ C^3_{24} = -1,\ C^2_{34} = 1,\ C^3_{34} = a$
  $e_1 = \partial_1 \quad e_2 = \partial_2 - \frac{1}{2}x_3\partial_1$
  $e_3 = \partial_3 + \frac{1}{2}x_2\partial_1 \quad e_4 = 2ax_1\partial_1 + (ax_2 + x_3)\partial_2 + (ax_3 - x_2)\partial_3 + \partial_4$
  $\omega_1 = e^{-2ax_4}(dx^1 + \frac{1}{2}x_2 dx^3 - \frac{1}{2}x_3 dx^2) \quad \omega_2 = e^{-ax_4}(cos(x_4)dx^2 - sin(x_4)dx^3)$
  $\omega_3 = e^{-ax_4}(cos(x_4)dx^3 + sin(x_4)dx^2) \quad \omega_4 = dx^4$

- $A_{4,12}$: $C^1_{13} = 1,\ C^2_{23} = 1,\ C^2_{14} = -1,\ C^1_{24} = 1$
  $e_1 = \partial_1 \quad e_2 = \partial_2 \quad e_3 = \partial_3 + x_1\partial_1 + x_2\partial_2$
  $e_4 = \partial_4 + x_2\partial_1 - x_1\partial_2$
  $\omega_1 = e^{-x_3}(cos(x_4)dx^1 - sin(x_4)dx^2) \quad \omega_2 = e^{-x_3}(cos(x_4)dx^2 + sin(x_4)dx^1)$
  $\omega_3 = dx^3 \quad \omega_4 = dx^4$

What we can hope to accomplish in the near future:

* Refine classification of homogeneous but possibly anisotropic solutions using 4-algebras. Show that new horizons can be supported by reasonable bulk matter and embedded into asymptotically AdS space.

* Try to relate to symmetries of observed phases.

* Construct more examples of inhomogeneous phases in Einstein gravity; hopefully some that are analytically tractable.

c.f. Horowitz, Santos, Tong