On the applicability of linearized Einstein's equations in the studies of holographic isotropization

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based on **arXiv:1202.0981 [hep-th] and arXiv:1211.???? [hep-th]**

with David Mateos, Diego Trancanelli, Wilke van der Schee and Miquel Triana

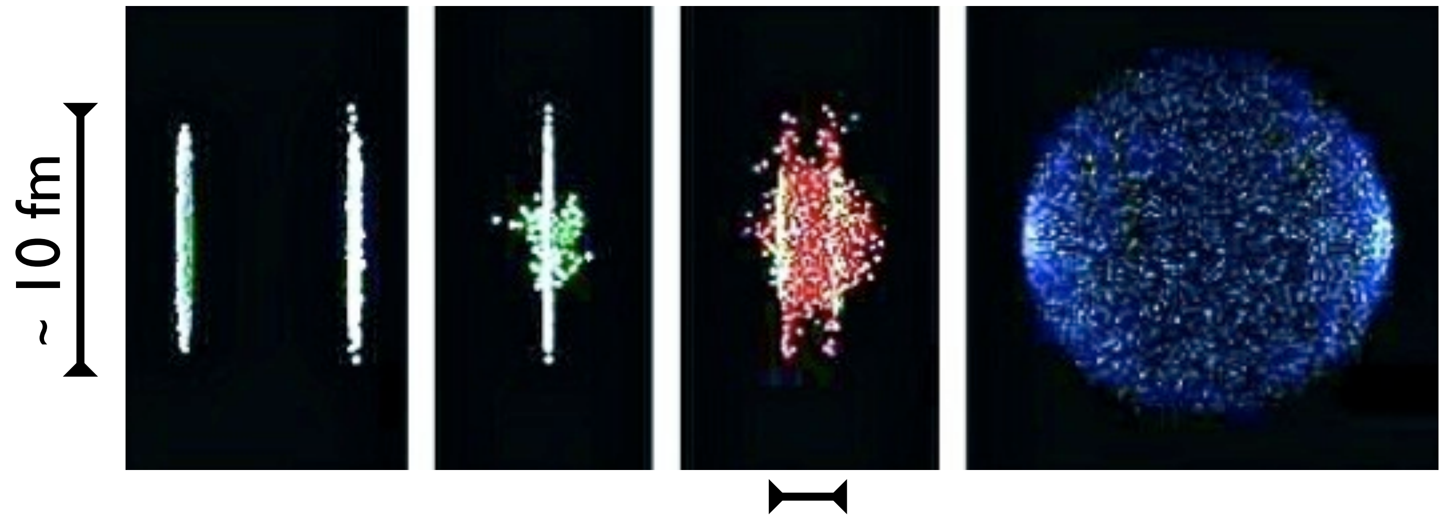
Motivation

Motivation I: fast thermalization at RHIC

There are significant evidences that relativistic heavy ion collision program at RHIC (now also at the LHC) created strongly coupled quark-gluon plasma (sQGP).

Successful description of experimental data is based on hydrodynamic simulations of an almost perfect fluid of $\eta/s = O(1/4\pi)$ starting on very early (< 1 fm/c).

Heinz (2004)



„thermalized" after < 1 fm/c

This very fast „thermalization" is a puzzle!!!

||

the stress tensor is described by hydrodynamics

Holographic media seem to **always** „thermalize" that quickly. This leads to questions

- **(AdS/CFT and gravity)** understand mechanisms ensuring fast „thermalization" on the gravity side;
- **(phenomenology)** derive predictions for HIC „thermalization" assuming strong coupling.

Motivation II: close-limit approximation

Holographic thermalization is a process in which a part of spacetime, dual to a non-equilibrium state, evolves to become (a patch of) a static black hole*.

If the non-equilibrium state is described by a slightly perturbed black hole solution, thermalization process is captured by QNMs/linearized Einstein's equations (easy).

Horowitz & Hubeny (1999)

RHIC
0.5 fm/c × 350 MeV = T t_{iso} = **0.63**

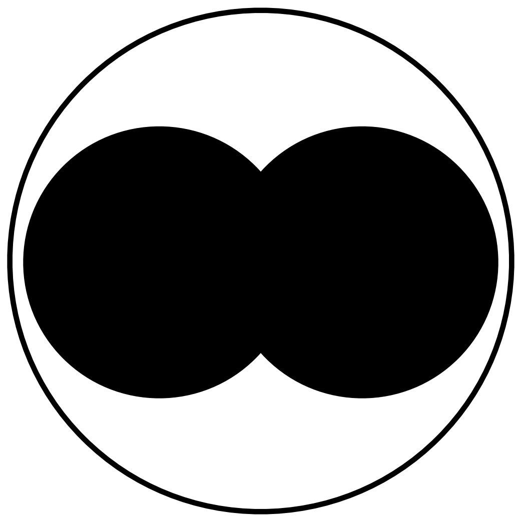
lowest QNM of AdS BH
T / |Im(frequency)| = **O(1)**

Existing studies of holographic thermalization are however based on solving numerically time-dependent Einstein's equation in the nonlinear regime (hard).

Chesler & Yaffe (2008, 2009, 2010), Heller, Janik & Witaszczyk (2011) and other studies

In black hole mergers, as soon as single horizon forms, linearized Einstein's equations give a good approximation of radiation pattern at infinity (close-limit approximation)

Price & Pullin (1994)

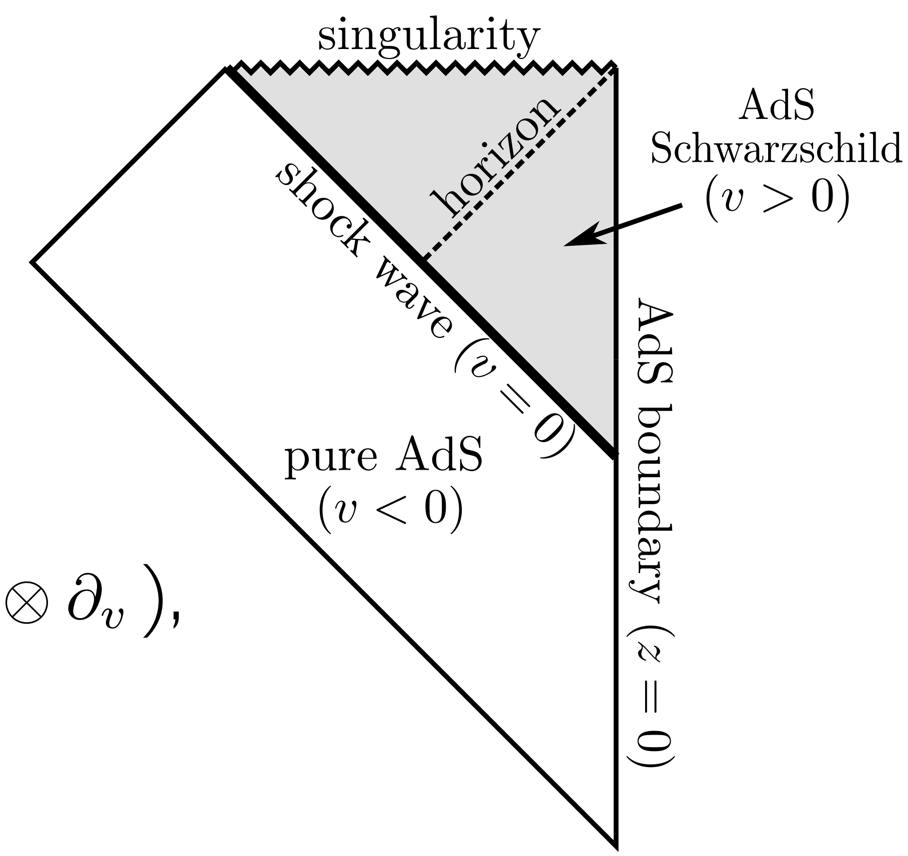


Motivation III: going beyond „the Vaidya paradigm”

The general lore is that time-dependent black branes require complicated numerics

Omnipresent toy-model in the literature is the AdS-Vaidya spacetime

$$ds^2 = 2drdv - r^2\left(1 - \frac{M(v)}{r^4}\right)dv^2 + r^2 d\vec{x}^2$$



This is a solution of $R_{ab} - \frac{1}{2}Rg_{ab} - 6g_{ab} = T_{ab}$ with

T^{ab} of **non-dynamical** null dust ($T^{ab}\,\partial_a \otimes \partial_b = \frac{3}{2r^3}M'(v)\,\partial_v \otimes \partial_v$),

e.g. $M(v) = \frac{M}{2}(1 + \tanh v/v_0)$

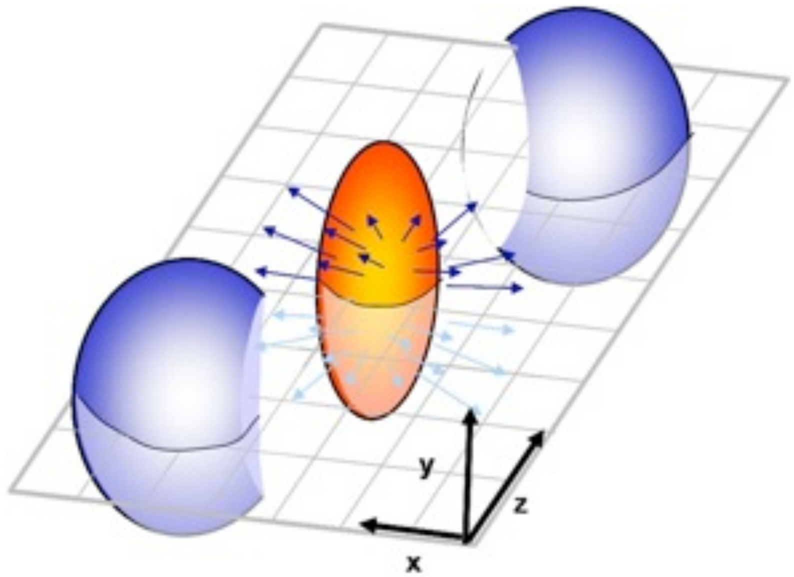
The main issues with the AdS-Vaidya are

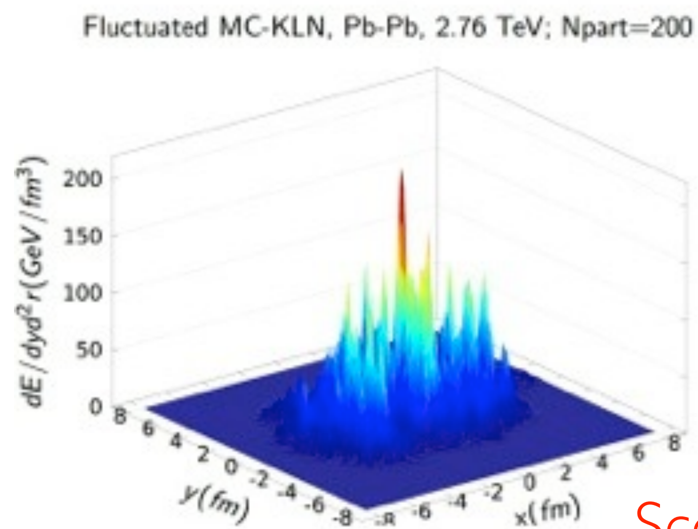
- it is not dynamical (we can take any M(v) as long as M’(v)>0)
- confusing „instantaneous thermalization” of local observables and no hydro tail

Can one one do better by simple means, at least in some cases?

Motivation IV: simplifying gravity description

Ultimate goals of holographic thermalization pheno



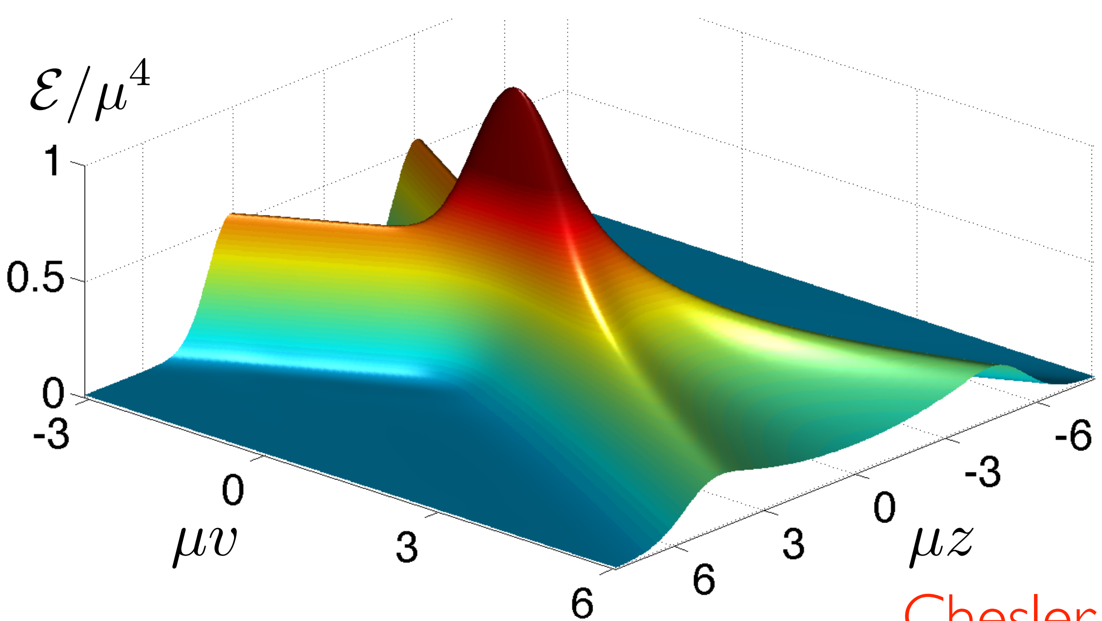


Scott Moreland (QM 2012)

Collide 3-dimensional „heavy nuclei" using 4+1D numerical relativity in AdS

Understand the imprint of color field fluctuations on the initial data for hydro

Current state of the art



Chesler & Yaffe (2010)

Collision of two infinite sheets using 2+1D simulation in AdS

Key question

How complicated is holographic thermalization?

or more concretely, to which extend do solutions of linearized Einstein's equations („easy") reproduce the full nonlinear result (hard)?

If some aspect of it is linear, this might lead to significant simplifications.

Holographic thermalization setup

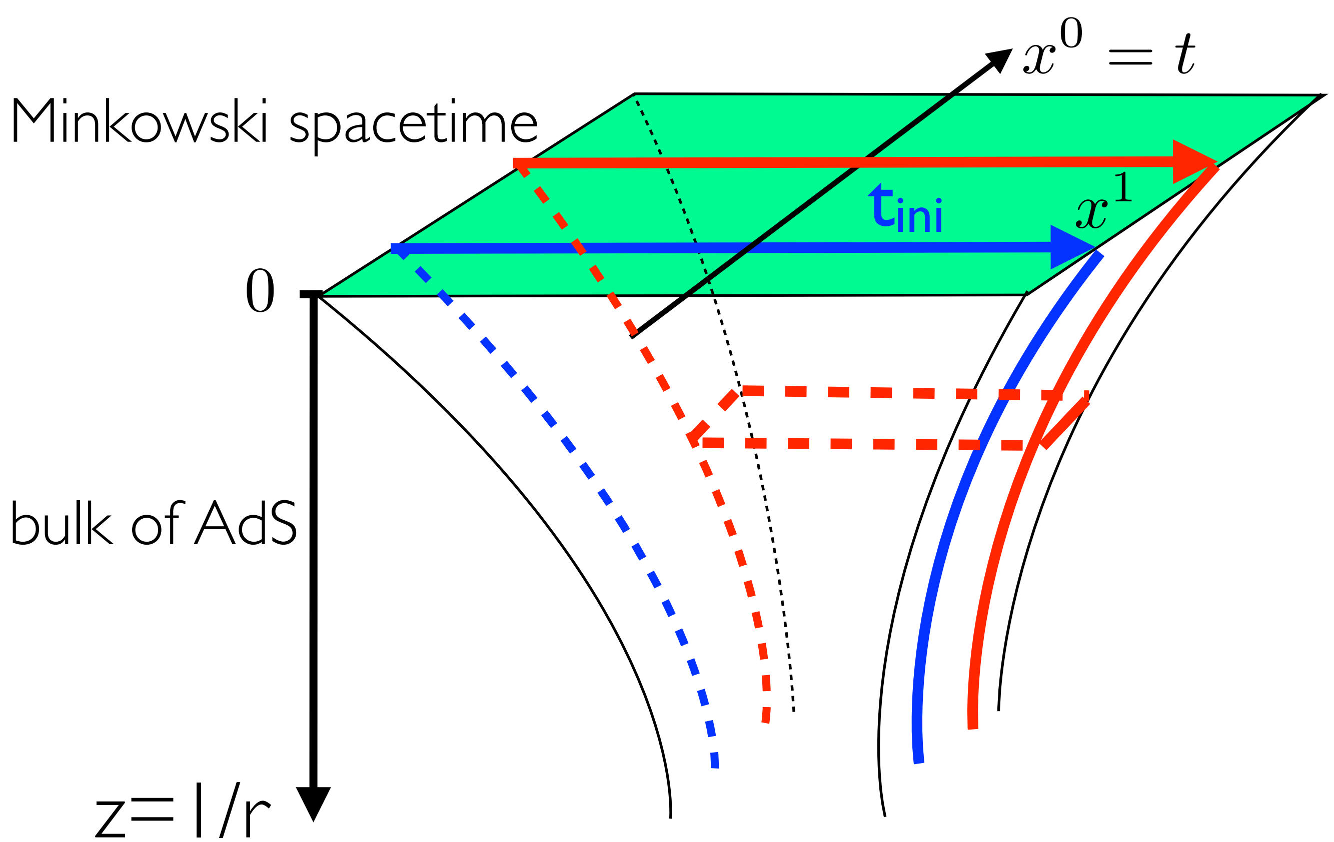
AdS/CFT correspondence and thermalization

From applicational perspective AdS/CFT is a tool for computing correlation functions in certain strongly coupled gauge theories, such as $\mathcal{N}=4$ SYM at large N_c and λ

In its simplest instance, considered also here, AdS/CFT maps the dynamics of the stress tensor of a holographic CFT$_{1+3}$ into (1+4)-dimensional asymptotically AdS geometry being a solution of

$$R_{ab} - \frac{1}{2} R\, g_{ab} - 6\, g_{ab} = 0$$

Of interest are geometries which interpolate between far-from-equilibrium states at the boundary at initial time t_{ini} and thermalized ones at (some) larger time t_{iso}



The stress tensor is read off from near-boundary expansion of dual solution

Skenderis et al. (2000)

The criterium for (local) thermalization is that the stress tensor is to a good accuracy described by hydrodynamics

Setup (field theory)

The field theory dynamics of interest is **isotropization of stress tensor (without any sources).** The matter fills the whole spacetime and is translationally invariant.

The most general stress tensor retaining (for simplicity) SO(2) symmetry reads

$$\langle T_{\mu\nu} \rangle = \mathrm{diag}\left\{ \epsilon(t),\, P_L(t),\, P_T(t),\, P_T(t) \right\}$$

Imposing conservation and tracelessness (CFT!) reduces it to

$$\langle T_{\mu\nu} \rangle = \mathrm{diag}\left\{ \epsilon,\, \frac{1}{3}\epsilon - \frac{2}{3}\Delta P(t),\, \frac{1}{3}\epsilon + \frac{1}{3}\Delta P(t),\, \frac{1}{3}\epsilon + \frac{1}{3}\Delta P(t) \right\}$$

Field theory state in this sector of dynamics is thus specified by the energy density ϵ (constant in time) and a single function of time measuring pressure anisotropy $\Delta P(t)$

There are two simplifying features intrinsic to this setup:

1) The final configuration is known precisely from the start;

2) Due to translational invariance no hydrodynamic modes are excited.

Thermalization criterium is thus based on the smallness of pressure anisotropy $\Delta P(t)$

Setup (gravity side)

The symmetries of boundary stress tensor dictate the following metric ansatz

$$ds^2 = -f_{tt}dt^2 + 2f_{tr}dtdr + f_{rr}dr^2 + \Sigma^2 e^{-2B}dx_1^2 + \Sigma^2 e^B(dx_2^2 + dx_3^2)$$

where there is a redundancy in the choice of $f_{tr}(t,r)$, $f_{tt}(t,r)$ and $f_{rr}(t,r)$

Following Chesler and Yaffe (2008) we choose $f_{tr}(t,r)=1$ and $f_{rr}(t,r)=0$ being generalized ingoing Eddington-Finkelstein coordinates.

$$ds^2 = 2dtdr - Adt^2 + \Sigma^2 e^{-2B}dx_1^2 + \Sigma^2 e^B(dx_2^2 + dx_3^2)$$

The coordinates are regular at the horizon and extend also behind it. Ingoing radial light rays propagate along curves of constant t, x^1, x^2, x^3.

The unique regular time-independent solution of Einstein's equations with negative cc is isotropic and is just the usual AdS-Schwarzschild black brane reading

Janik & Witaszczyk (2008)

$$A = r^2(1 - \frac{\pi^4 T^4}{r^4}), \qquad \Sigma = r \quad \text{and} \quad B = 0$$

A patch of this solution will be the end point of studied isotropization process.

Solving Einstein's equations in time Chesler & Yaffe (2008)

Let's look closer at Einstein's equations

dynamical equations (EOMs)

$$\begin{aligned} 0 &= \Sigma\,(\dot{\Sigma})' + 2\Sigma'\,\dot{\Sigma} - 2\Sigma^2 \\ 0 &= \Sigma\,(\dot{B})' + \tfrac{3}{2}\left(\Sigma'\dot{B} + B'\,\dot{\Sigma}\right) \\ 0 &= A'' + 3B'\dot{B} - 12\Sigma'\,\dot{\Sigma}/\Sigma^2 + 4 \end{aligned}$$

if obeyed at const. r + EOMs then obeyed everywhere

if obeyed at const. t + EOMs then obeyed everywhere

constraints

$$0 = \ddot{\Sigma} + \frac{1}{2}(\dot{B}^2\Sigma - A'\dot{\Sigma})$$

$$0 = \Sigma'' + \frac{1}{2}(B')^2\Sigma$$

$$\left(h' = \partial_r h(t,r)\;, \quad \dot{h} = \partial_t h(t,r) + \frac{1}{2}A(t,r)\partial_r h(t,r) \right)$$

On a constant t slice Σ and B are related by a constraint $\Sigma'' + \frac{1}{2}(B')^2\Sigma = 0$. As B appears quadratically (important later on), we choose it to characterize initial state.

B is not completely arbitrary - it needs to satisfy near-boundary (large r) expansion with AdS asymptotics (no sources - flat boundary metric, see the next slide).

Once B and Σ are known on a given time slice, one can use EOMs to obtain first $\dot{\Sigma}$, then $\dot{B}$ and finally A. Having those guys one can solve $\dot{B}$ and $\dot{\Sigma}$ for $\partial_t B$ and $\partial_t \Sigma$ and choose your favorite finite difference scheme for making a step in time

The only remaining issue is the choice of the bulk cut off for radial integration. By trials and errors we put it behind the event horizon on the initial time hypersurface.

What's new?

Holographic isotropization was considered before in Chesler and Yaffe (2008)

$$ds^2 = -dt^2 + e^{B_0(t)}\, d\boldsymbol{x}_\perp^2 + e^{-2B_0(t)}\, dx_{||}^2$$

$$B_0(t) = \tfrac{1}{2}\, c\,[1 - \tanh(t/\tau)]$$

$\lvert c \rvert$	1	1.5	2	2.5	3	3.5	4
$\tau\, T$	0.23	0.31	0.41	0.52	0.65	0.79	0.94
$\tau_{\rm iso}\, T$	0.67	0.68	0.71	0.92	1.2	1.5	1.8
$\tau_{\rm iso}/\tau$	3.0	2.2	1.7	1.8	1.8	1.9	1.9

TABLE I: Final equilibrium temperature T and isotropization time $\tau_{\rm iso}$ (in units of T^{-1} or τ), for various values of c. The isotropization time $\tau_{\rm iso}$ is the time at which the pressures deviate from their equilibrium values by less than 10%.

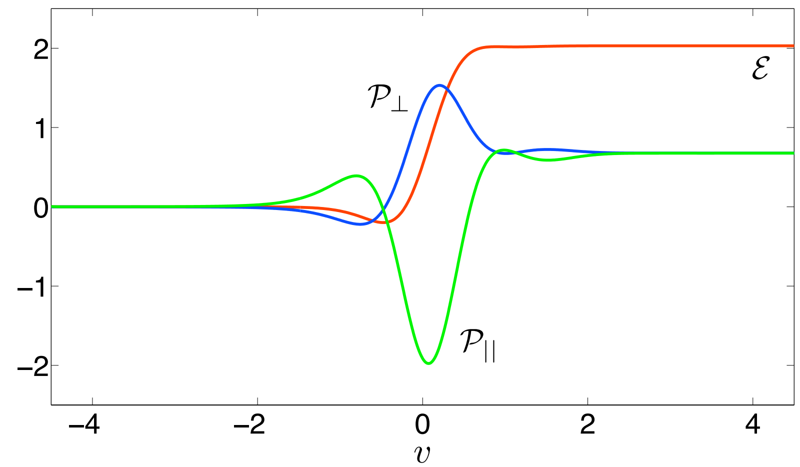


FIG. 1: Energy density, longitudinal and transverse pressure, all divided by $N_c^2/2\pi^2$, as a function of time for $c = 2$.

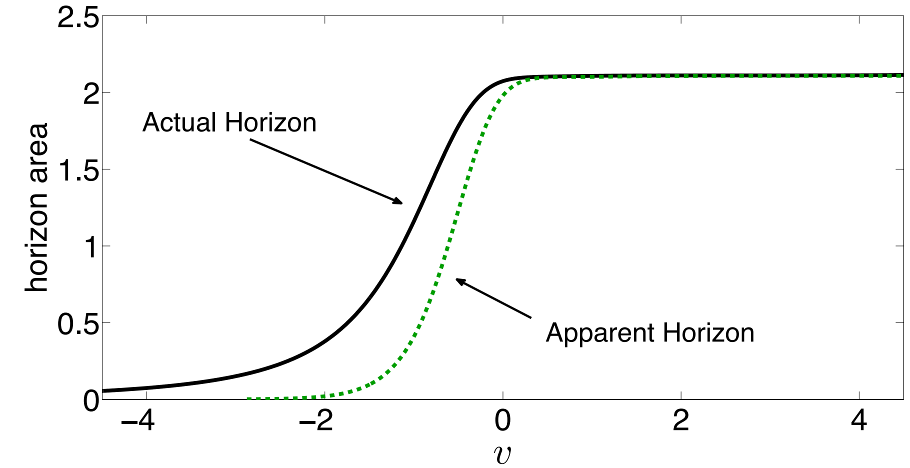


FIG. 3: Area elements of the true event horizon and the apparent horizon as a function of time.

The key innovations of our paper are

- clean separation between creation of non-equilibrium state and its subsequent equilibration

- evolving large number of initial states (O(2000) vs. O(10) in all previous studies!)

- detailed comparison with linearized gravity and quasinormal modes decomposition

Generating a large number of states

Specifying initial states: near boundary behavior

Near-boundary expansion of warp-factors to O(1/r^8) takes the form

$$B = \frac{1}{r^4}\left\{b_4(t) + \frac{1}{r}b_4'(t) + \frac{2}{12r^2}b_4''(t) + \frac{1}{4r^3}b_4^{(3)}(t) + \ldots\right\}, \quad \Sigma = r\left\{1 - \frac{1}{7r^8}b_4(t)^2 + \ldots\right\} \text{ and}$$

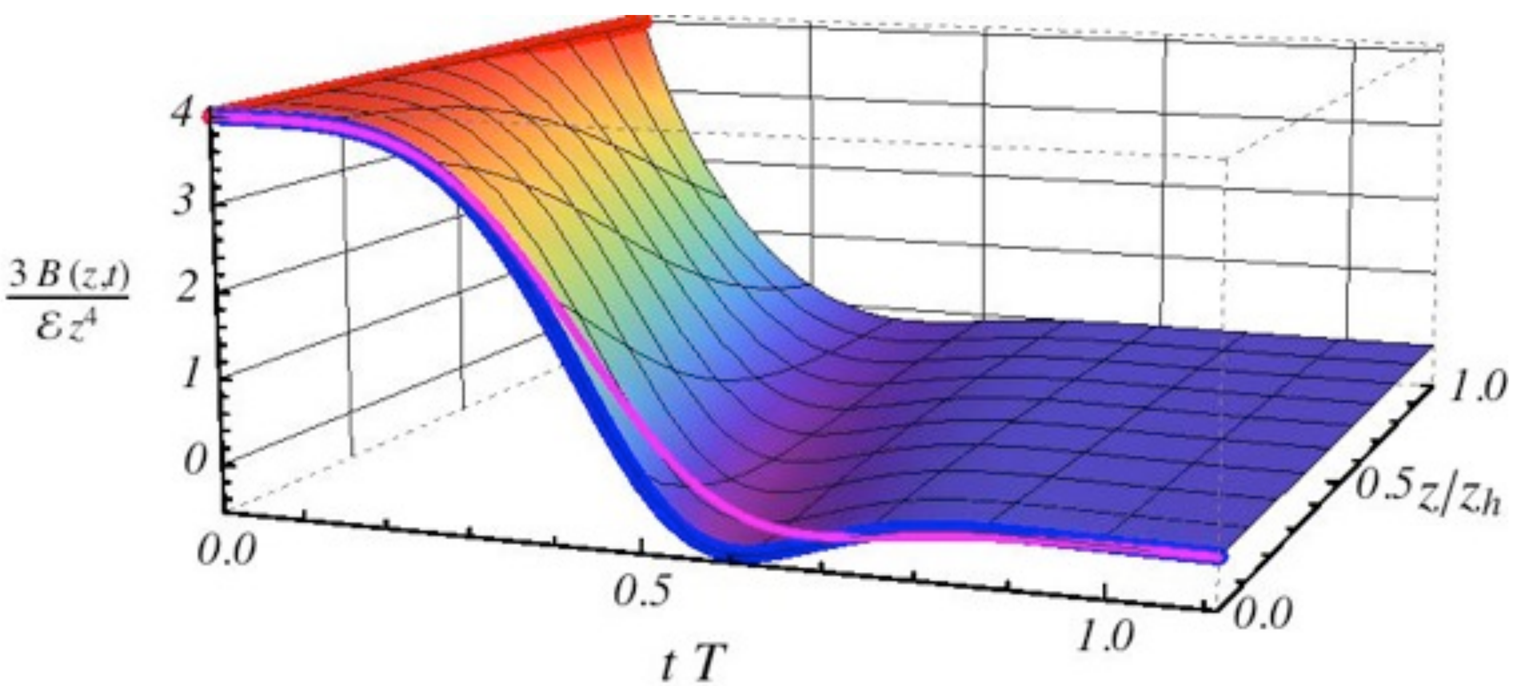
$$A = r^2\left\{1 - \frac{1}{r^4}a_4 - \frac{2}{7r^8}b_4(t)^2 - \frac{3}{7r^9}b_4(t)b_4'(t) + \ldots\right\}$$

where $\epsilon = \frac{3}{8\pi^2}N_c^2 a_4$ and $\Delta P(t) = \frac{3}{8\pi^2}N_c^2 b_4(t)$.

The initial state in the bulk contains information about all time derivatives of pressure anisotropy at a given instance of time!

see Beuf, Heller, Janik & Peschanski (2009) for a similar observation for the Bjorken flow

But that is not enough to solve the dynamics, e.g. B_{ini} = 1/r^4:



(blue curve: pressure anisotropy)

Specifying initial states: bulk analysis

As the near-bdry analysis turns out to be not enough, we have to solve for the bulk

Let's look again at Einstein's equations

dynamical equations (EOMs)

$$
\begin{aligned}
0 &= \Sigma\,(\dot{\Sigma})' + 2\Sigma'\,\dot{\Sigma} - 2\Sigma^2 \\
0 &= \Sigma\,(\dot{B})' + \tfrac{3}{2}\left(\Sigma'\dot{B} + B'\,\dot{\Sigma}\right) \\
0 &= A'' + 3B'\dot{B} - 12\Sigma'\,\dot{\Sigma}/\Sigma^2 + 4
\end{aligned}
$$

if obeyed at const. r + EOMs then obeyed everywhere

constraints

$$0 = \ddot{\Sigma} + \frac{1}{2}(\dot{B}^2\Sigma - A'\dot{\Sigma})$$

$$0 = \Sigma'' + \frac{1}{2}(B')^2\Sigma$$

if obeyed at const. t + EOMs then obeyed everywhere

$$\left(\, h' = \partial_r h(t,r)\;, \quad \dot{h} = \partial_t h(t,r) + \frac{1}{2}A(t,r)\partial_r h(t,r) \,\right)$$

On a given slice it is sufficient to know Σ or B and the energy density (a_4).

Due to $\Sigma'' + \frac{1}{2}(B')^2\Sigma = 0$ and AdS asymptotics ($\Sigma \sim r$), Σ goes to 0 for some r > 0.

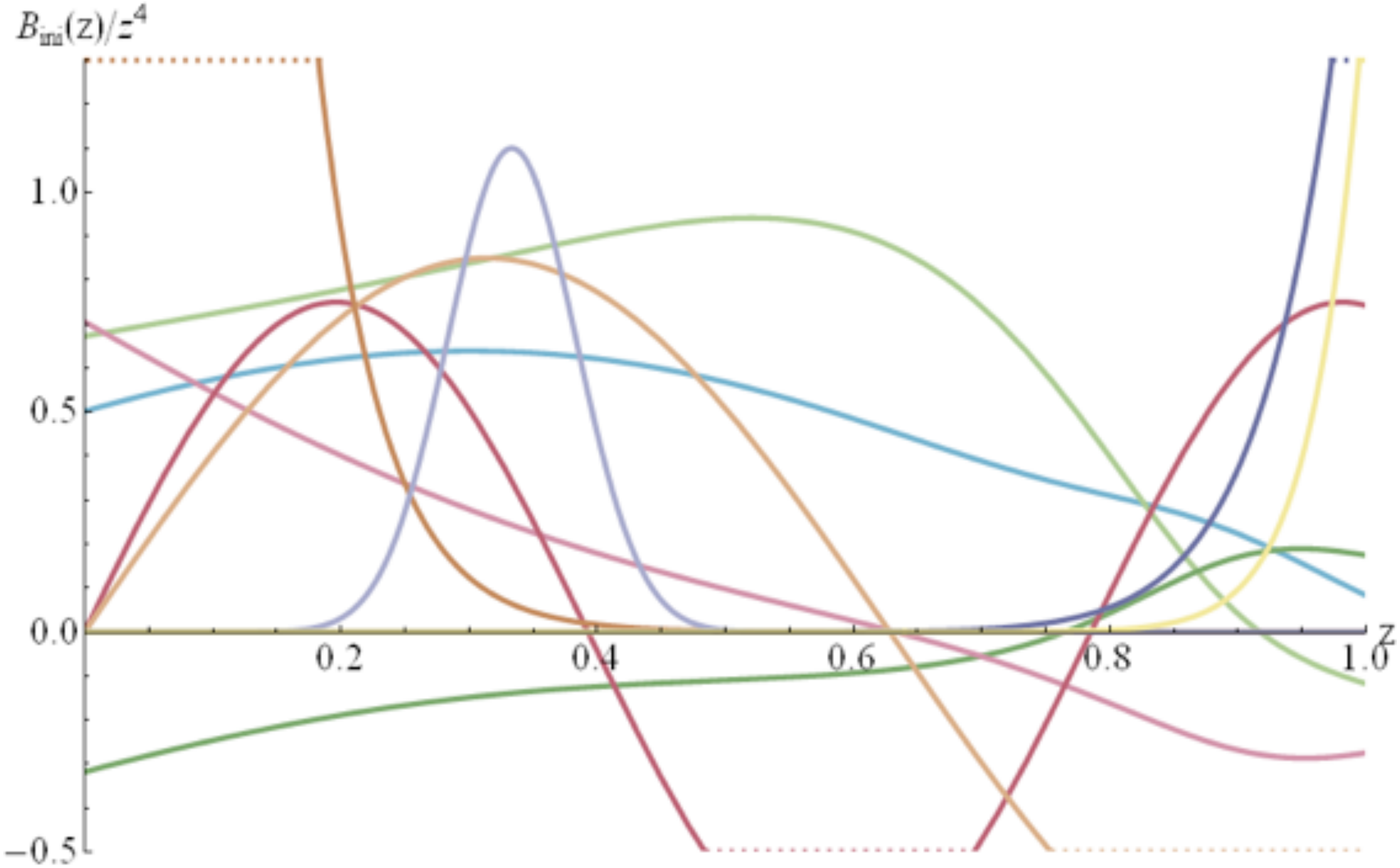
Numerical studies indicate that it is a curvature singularity (need to be behind EH).

We also found that for a given initial profile B there seems to be a minimal value of the energy density for which this singularity is covered by the event horizon.

Obtaining representative set of initial states

Setting up initial states at t_{ini} and letting them evolve unforced is more generic than quenching and can be used to obtain a variety of behaviors

see Heller, Janik & Witaszczyk (2011) for a similar approach to the holographic Bjorken flow



10 examples of initial states encoded geometrically including once supported mostly in the UV, mostly in the IR, in the middle and spread evenly between UV and IR

$(z = 1/r)$

In order to produce a large set of initial data (and so hopefully a good statistics) we

1) without any loss of generality fix units by setting $a_4=1$;

2) generate B as the ratio of two 10th order polynomials with random coefficients modulo minimal subtraction necessary for having AdS asymptotics; normalize B in a convenient way;

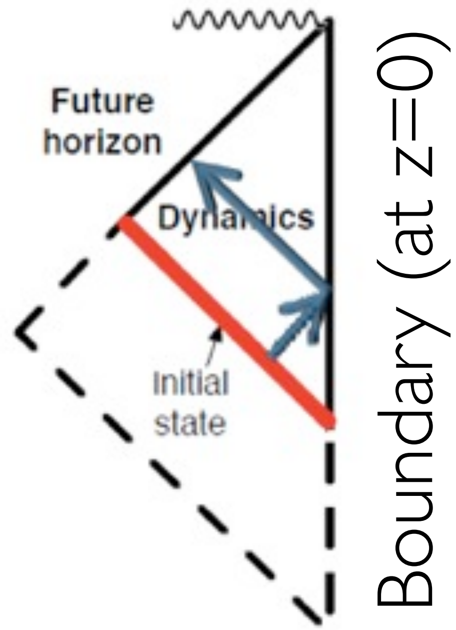
3) run simulation for a given B and store data increasing at each run B 1.15-folds until we obtain profiles close to „maximally far-from-equilibrium ones'' (typically multiplication is repeated ~ 8x);

4) return to step 2);

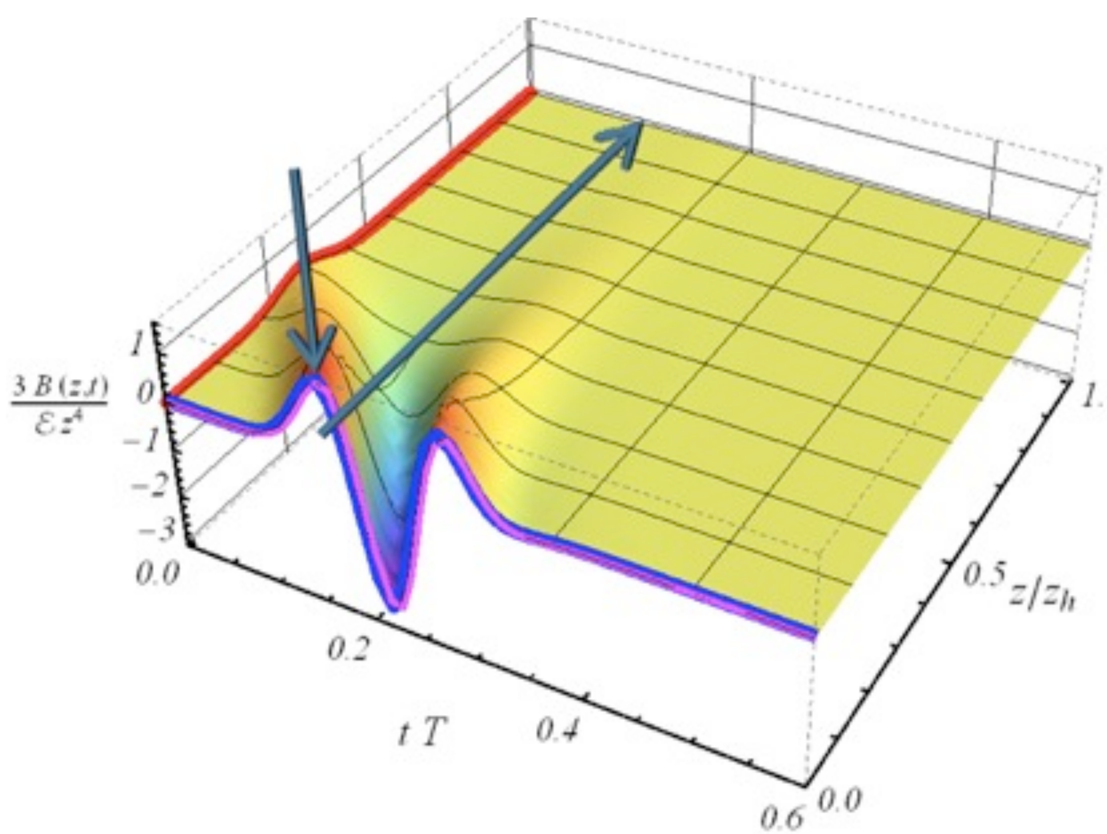
Thermalization

Typical holographic thermalization process

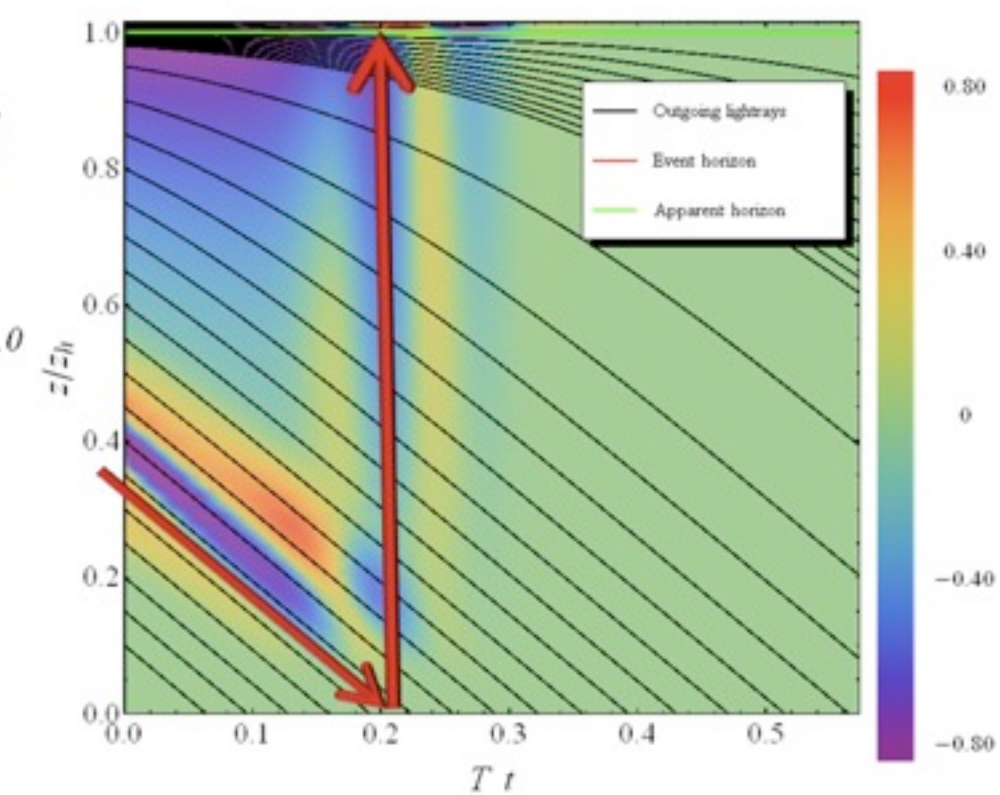
Theory:



Numerical experiment:



Curvature (BH subtracted)



Linearized Einstein's equations

Linearized approximation

Let's look yet again at Einstein's equations

dynamical equations (EOMs)

$$
\begin{aligned}
0 &= \Sigma\,(\dot\Sigma)' + 2\Sigma'\,\dot\Sigma - 2\Sigma^2 \\
0 &= \Sigma\,(\dot B)' + \tfrac{3}{2}\big(\Sigma'\dot B + B'\,\dot\Sigma\big) \\
0 &= A'' + 3B'\dot B - 12\Sigma'\,\dot\Sigma/\Sigma^2 + 4
\end{aligned}
$$

if obeyed at const. r + EOMs then obeyed everywhere

constraints

$$0 = \ddot\Sigma + \frac{1}{2}(\dot B^2\Sigma - A'\dot\Sigma)$$

$$0 = \Sigma'' + \frac{1}{2}(B')^2\Sigma$$

if obeyed at const. t + EOMs then obeyed everywhere

All the equations, but one, are quadratic in B.

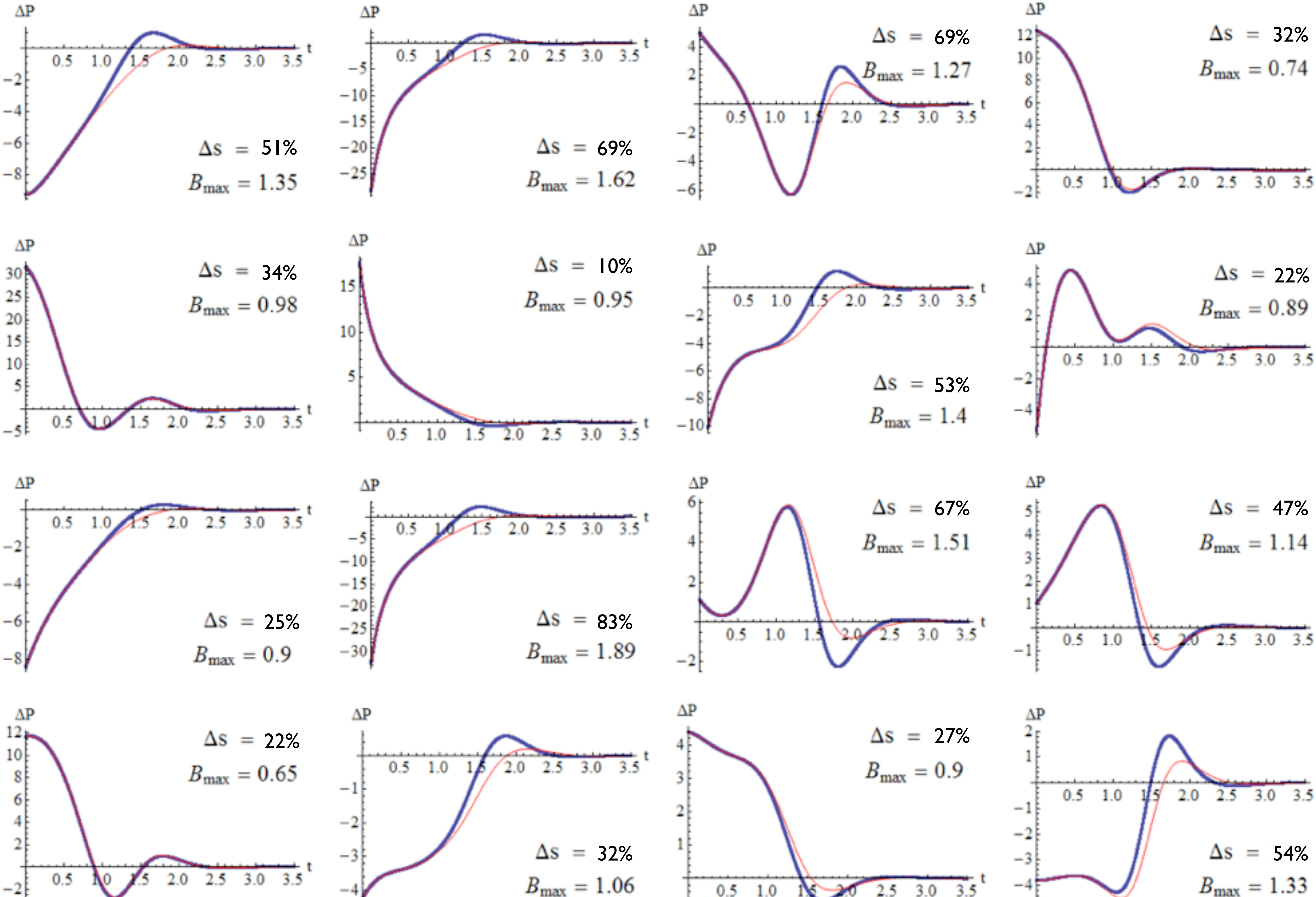
This implies that at the linear order A and Σ are that of AdS-Schwarzschild (and so do not evolve) and B undergoes decoupled dynamics captured by the equation

$$0 = \Sigma\,(\dot B)' + \tfrac{3}{2}\big(\Sigma'\dot B + B'\,\dot\Sigma\big)\,, \quad \text{where } \Sigma = r \text{ and } \dot h = \partial_t h + \frac{1}{2}r^2\Big(1 - \frac{(\pi T)^4}{r^4}\Big)\partial_r h$$

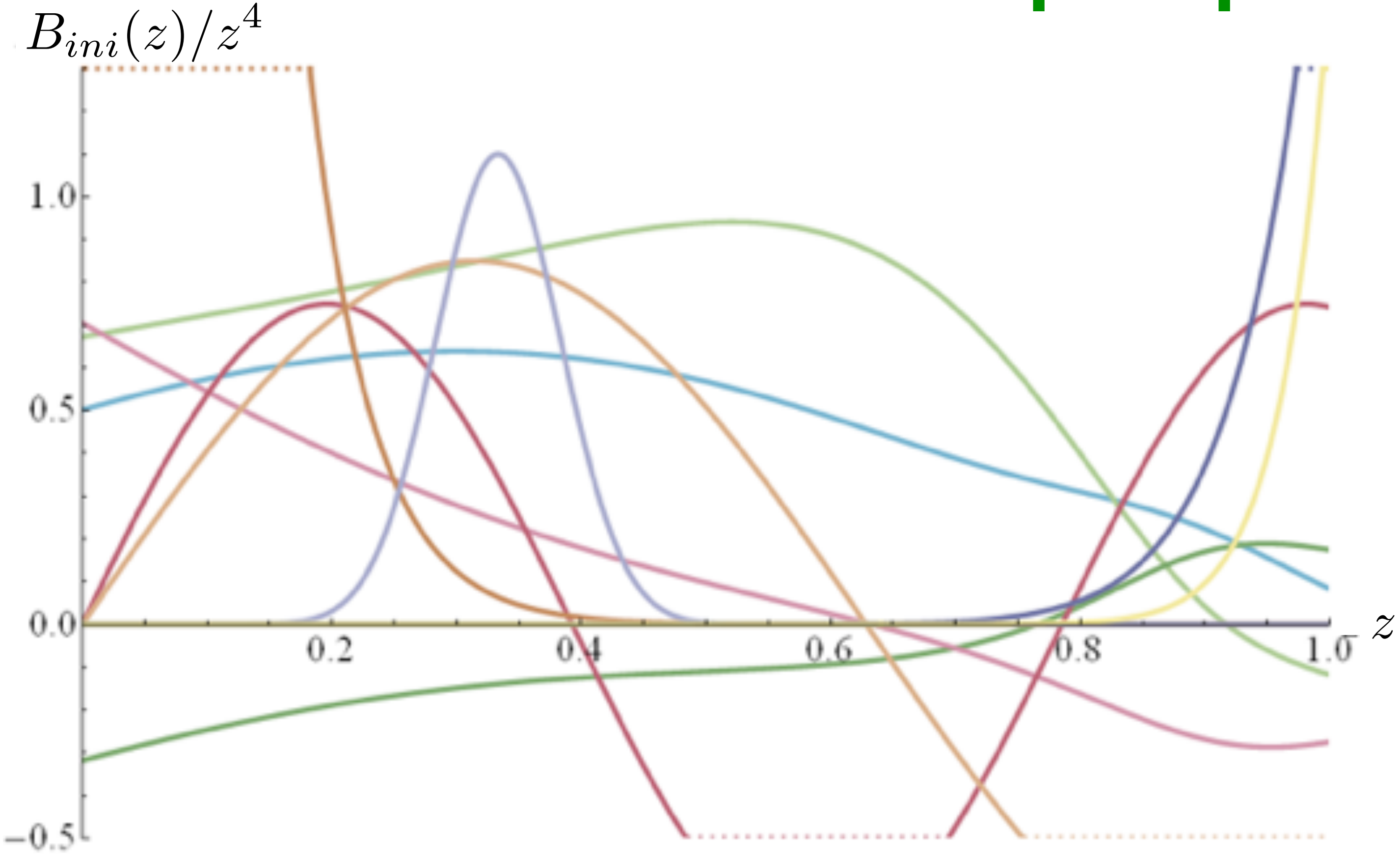
The solutions of interest are such that satisfy AdS boundary condition (no sourcing = flat boundary metric).

In the following we will scan through a large set of initial data (B's at t=0) and compare solutions of linearized Einstein's equations with solutions of the non-linear problem focusing mostly on predictions for dual stress tensor operator

Time evolution of pressure anisotropy (L/NL)

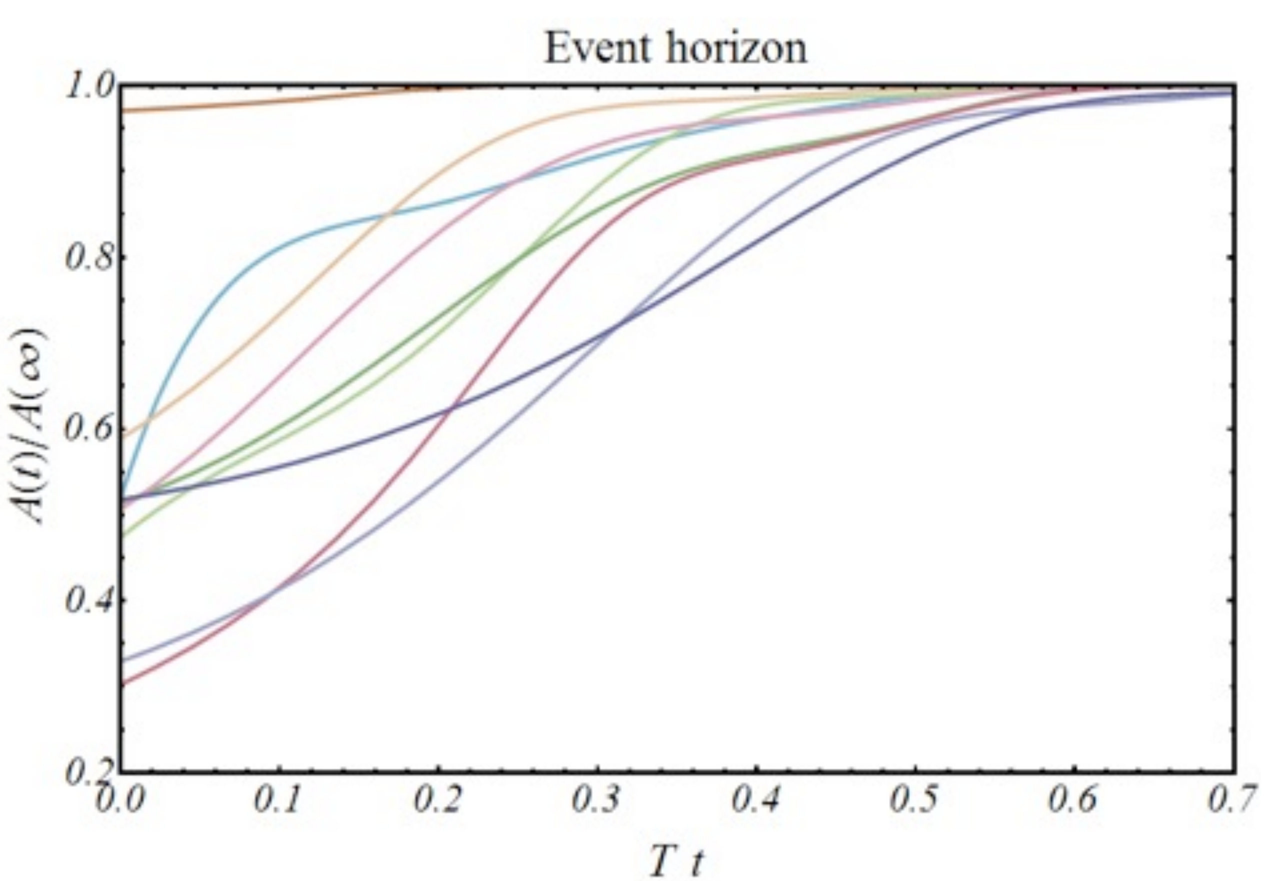


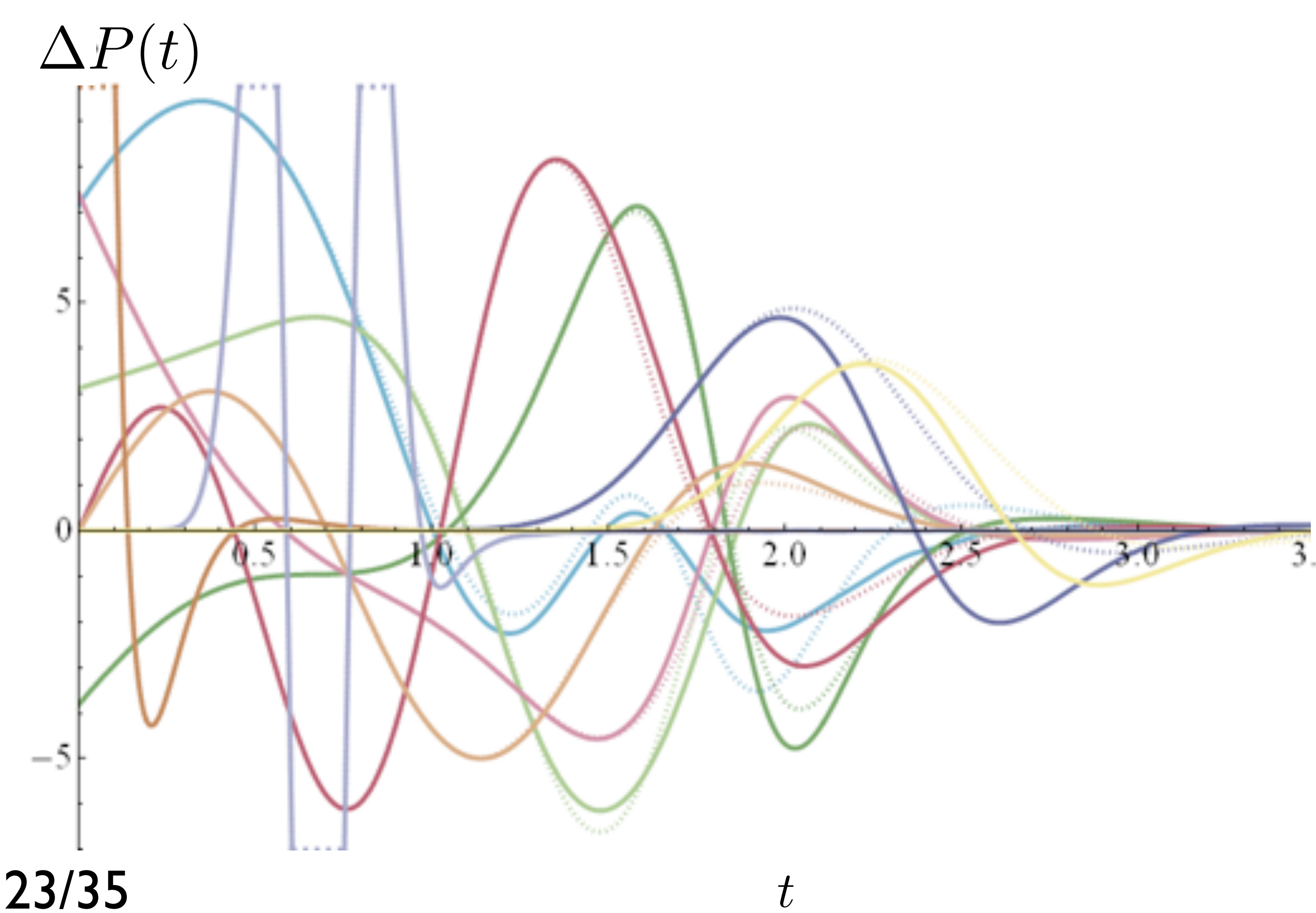
Evolution of 10 sample profiles



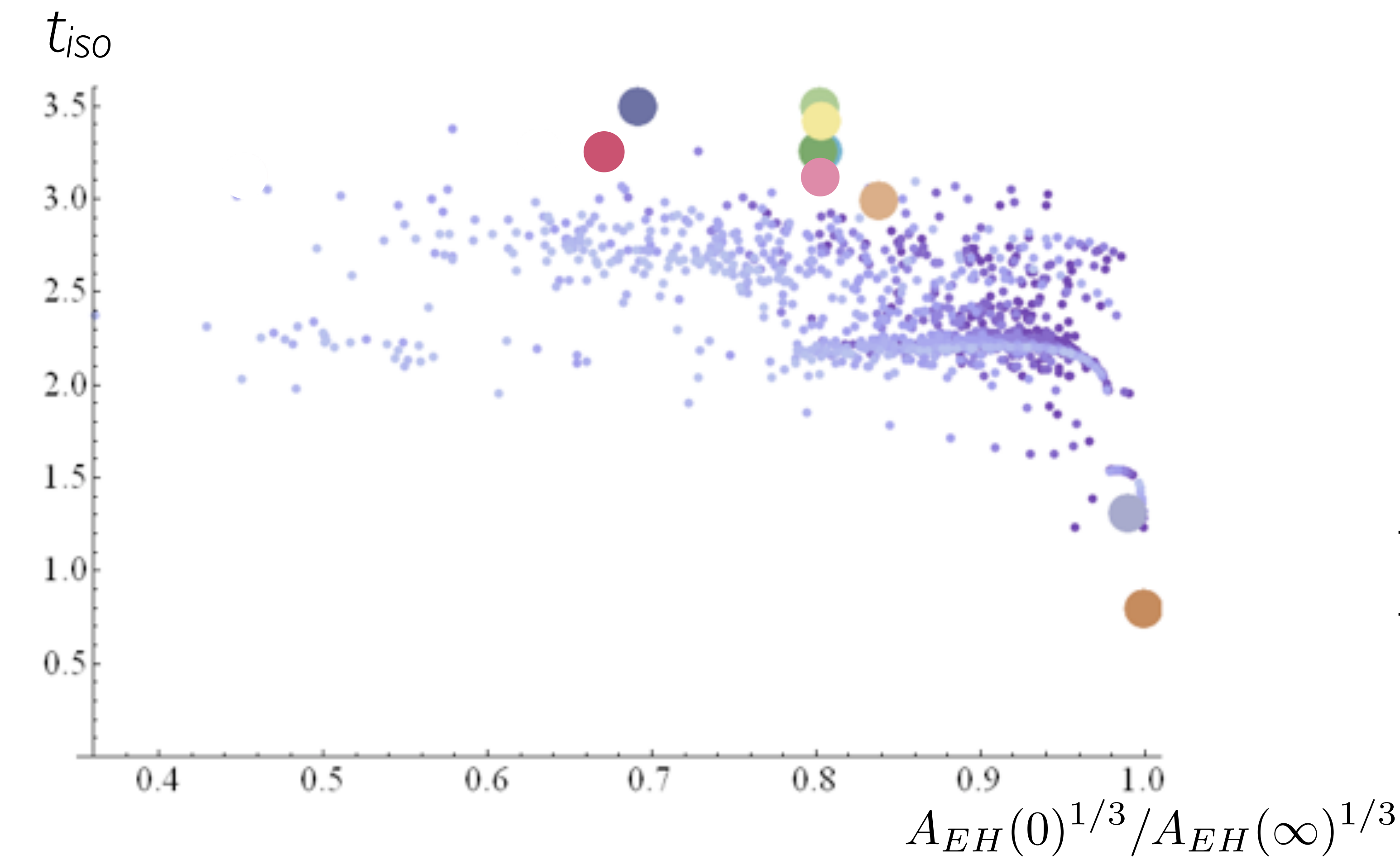
Linearized Einstein's equations again do a surprisingly good job in reproducing boundary stress tensor (dotted curves in the plot below)

(z = 1/r)





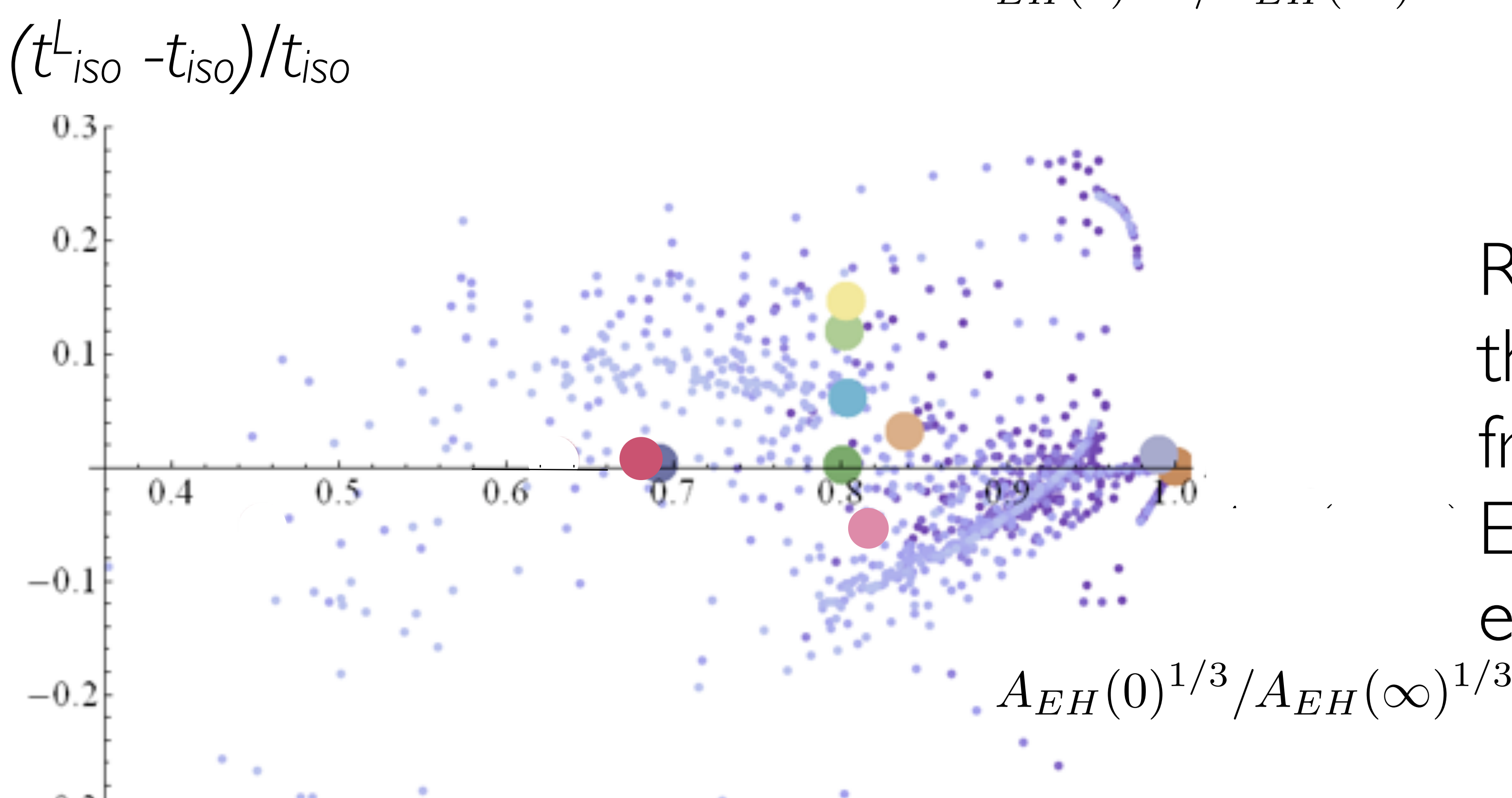
Isotropization time as a function of initial entropy



results of the analysis of 1210 different initial states

$$|\Delta P(t \geq t_{iso})| \leq 0.1\epsilon(t)$$

The closer initial entropy to the final one, the faster the thermalization (in units of a_4 ~ initial=final energy density)



Relative difference in thermalization time obtained from linearized and full Einstein's equation does not exceed 30% !!!

Linearized gravity and quasinormal modes

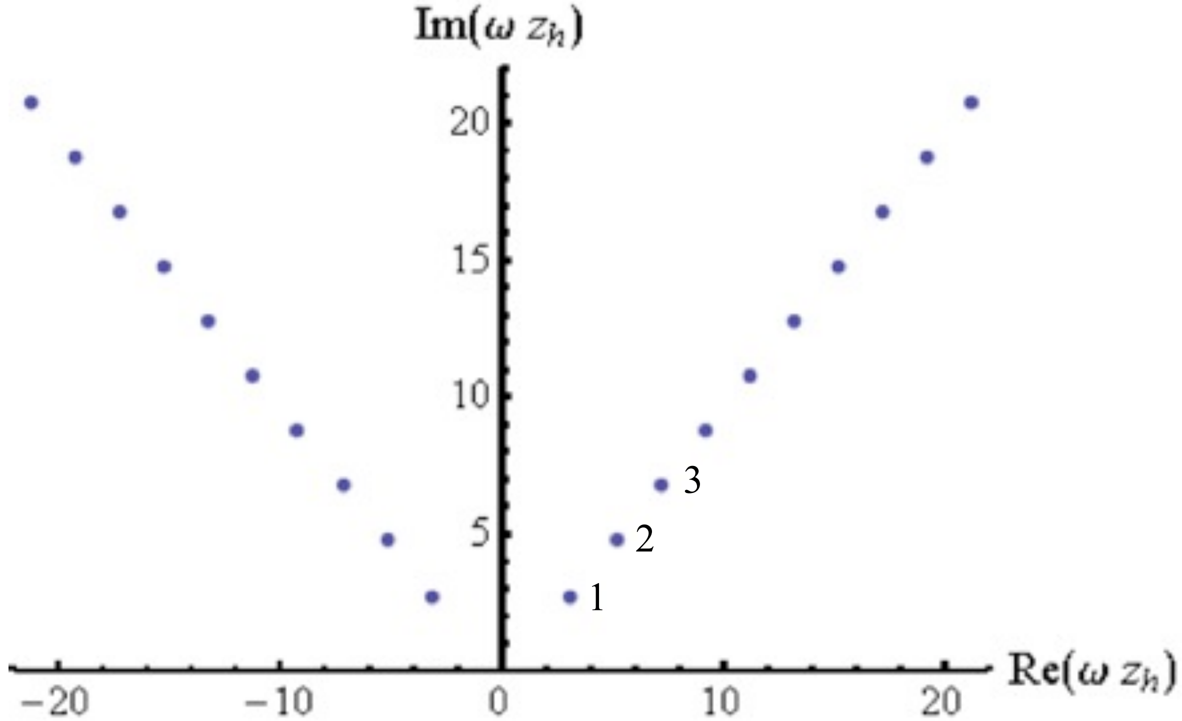
Quasinormal modes arise here as solutions of the same equation for B

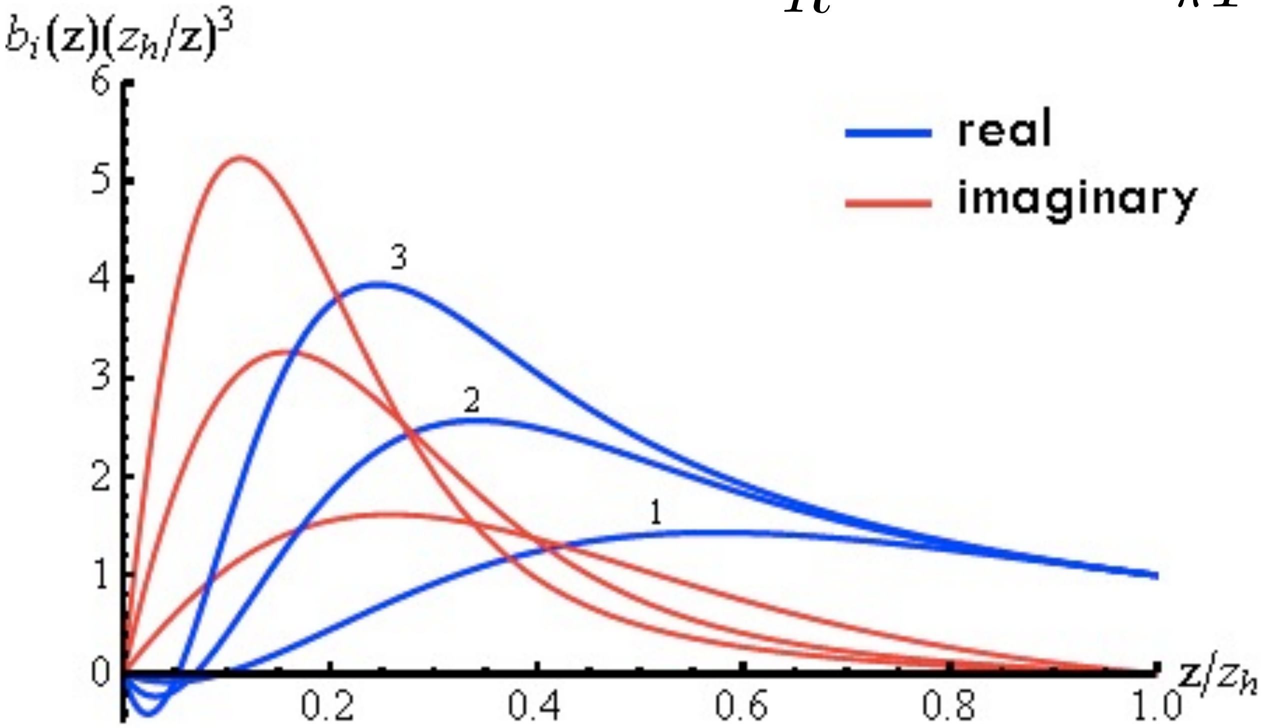
$$0 \;=\; \Sigma\,(\dot{B})' + \tfrac{3}{2}\big(\Sigma'\dot{B} + B'\,\dot{\Sigma}\big)\,, \quad \text{where} \;\; \Sigma = r \;\; \text{and} \;\; \dot{h} = \partial_t h + \frac{1}{2}r^2(1 - \frac{(\pi T)^4}{r^4})\partial_r h$$

with B written as $e^{i\omega t} f(r)$ and satisfying ingoing bdry conditions at EH

This leads to complex frequencies with Im(ω)>0 and so to the exponential decay.

$$z = \frac{1}{R} \text{ and } z_H = \frac{1}{\pi T}$$





Note that none of the modes carries momentum (follows from homogeneity)

Connection with quasinormal modes

IR profile:

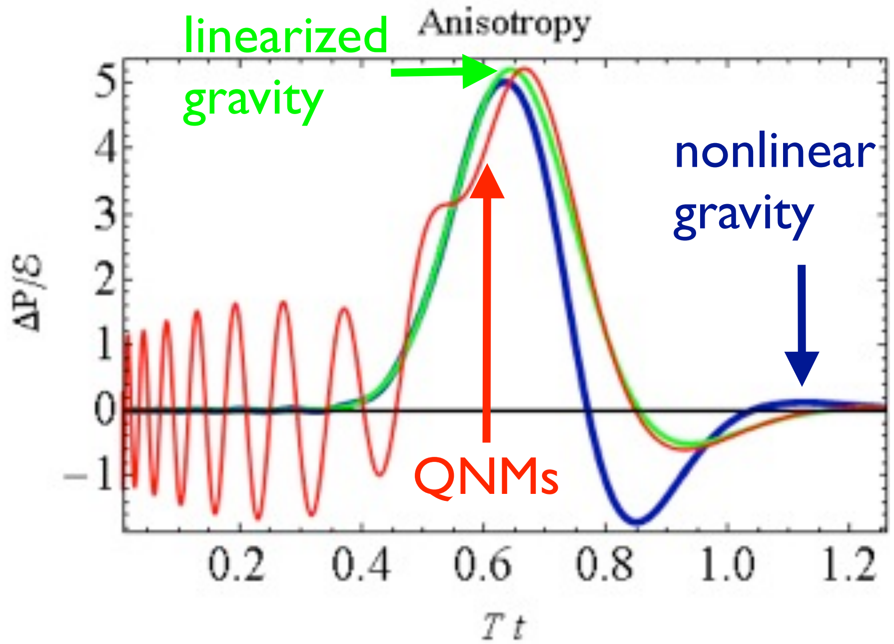
$$B_{ini}(z) = 1.6\,(z/z_h)^{20}$$

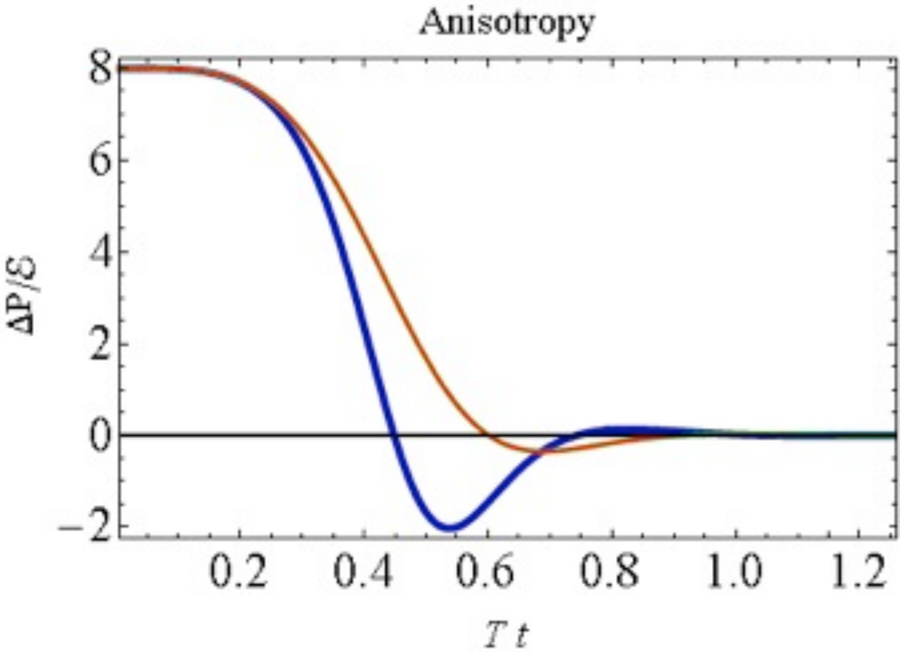
middle (spread) profile:

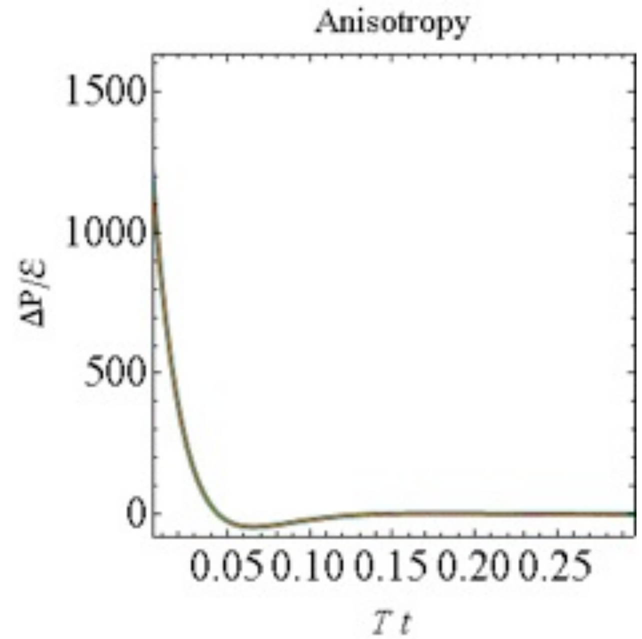
$$B_{ini}(z) = 2\,(z/z_h)^{4}$$

UV profile:

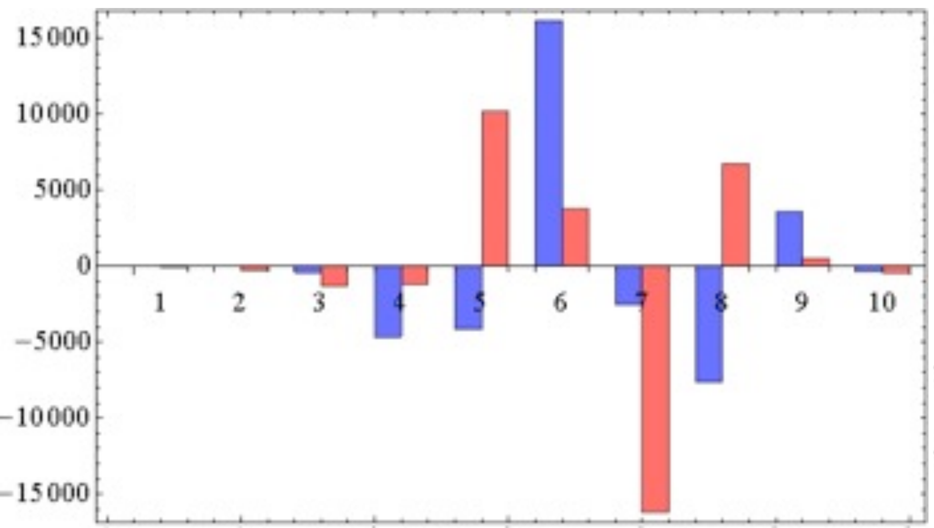
$$B_{ini}(z) = 400\,(z/z_h)^{4}\exp\left(-20\,z/z_h\right)$$

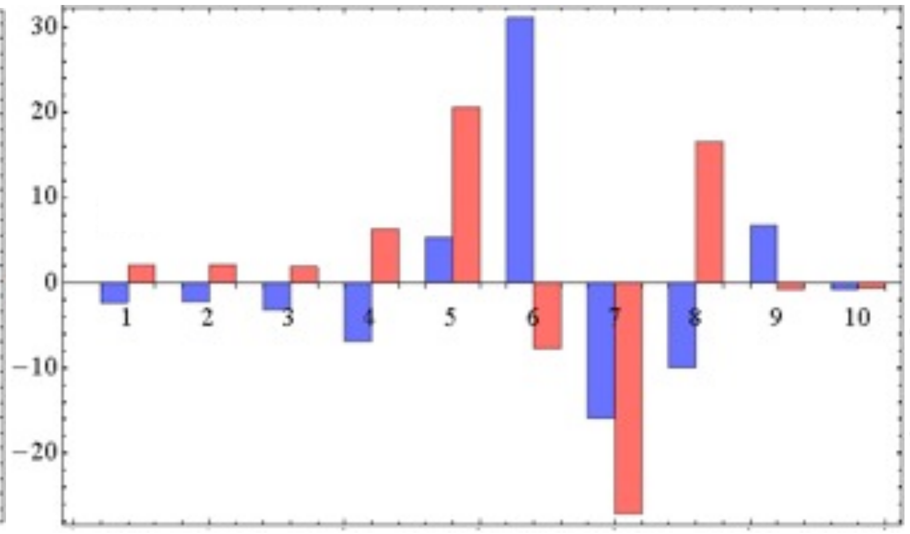


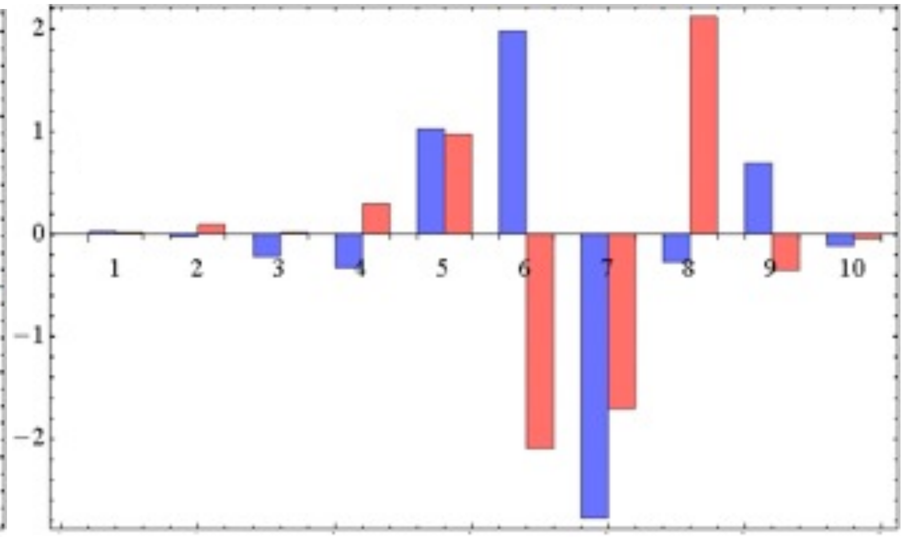




Quasinormal modes fit using the 10 lowest modes (least squares fit to B_{ini}):







Corrections to linearized Einstein's equations

Linearized approximation in the bulk

We saw that linearized gravity reproduces well $\langle T_{\mu\nu} \rangle$ also when $\Delta s = O(50\%)$

But Δs is not included at first order: $A(t,r) = A_{BH}(t,r)$ and $\Sigma(t,r) = \Sigma_{BH}(t,r)$ then.

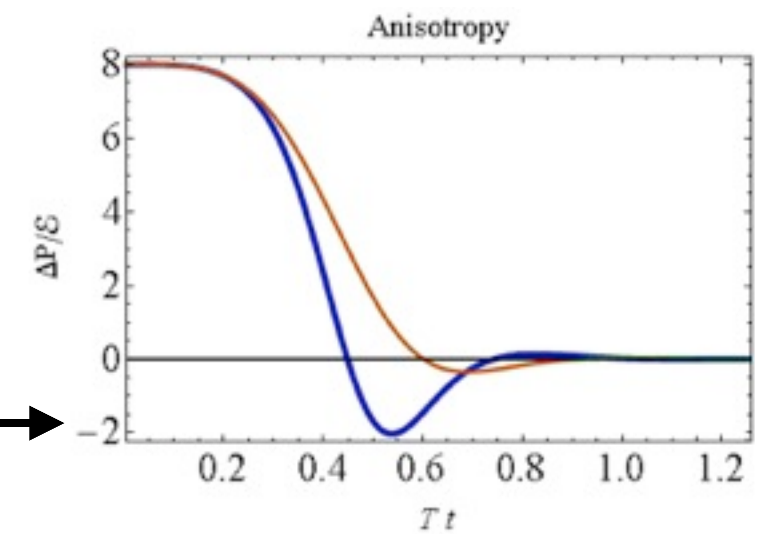
That is the chief motivation for going to the third order:

linearized gravity

$$B = \epsilon\, \delta B_1 + \epsilon^3 \delta B_3 + O(\epsilon^4)$$

NLO correction to $\Delta P(t)$

idea: do better than linearized gravity



leading order modification of EH position

$$A = A_{BH}(t,r) + \epsilon^2 \delta A_2 + O(\epsilon^4)$$

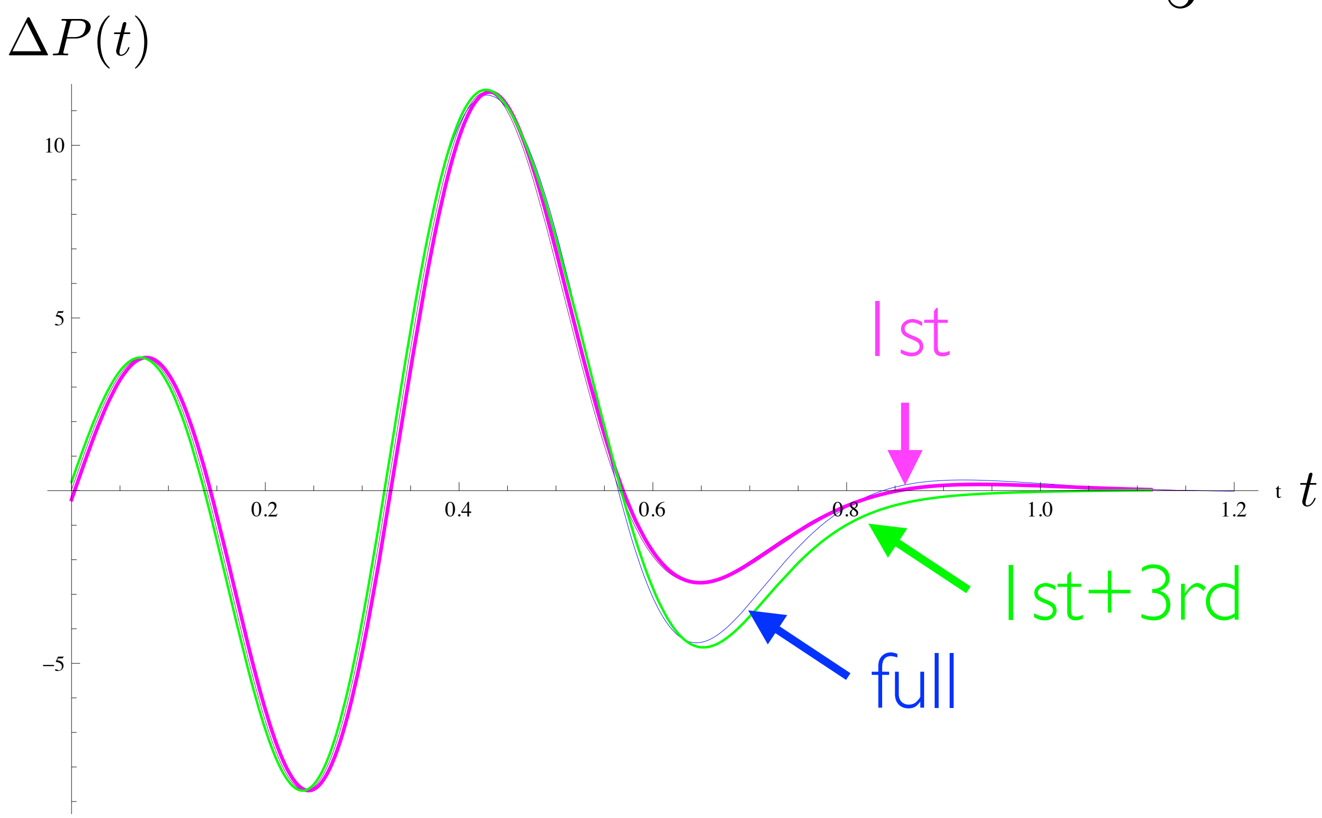
leading order modification of EH area

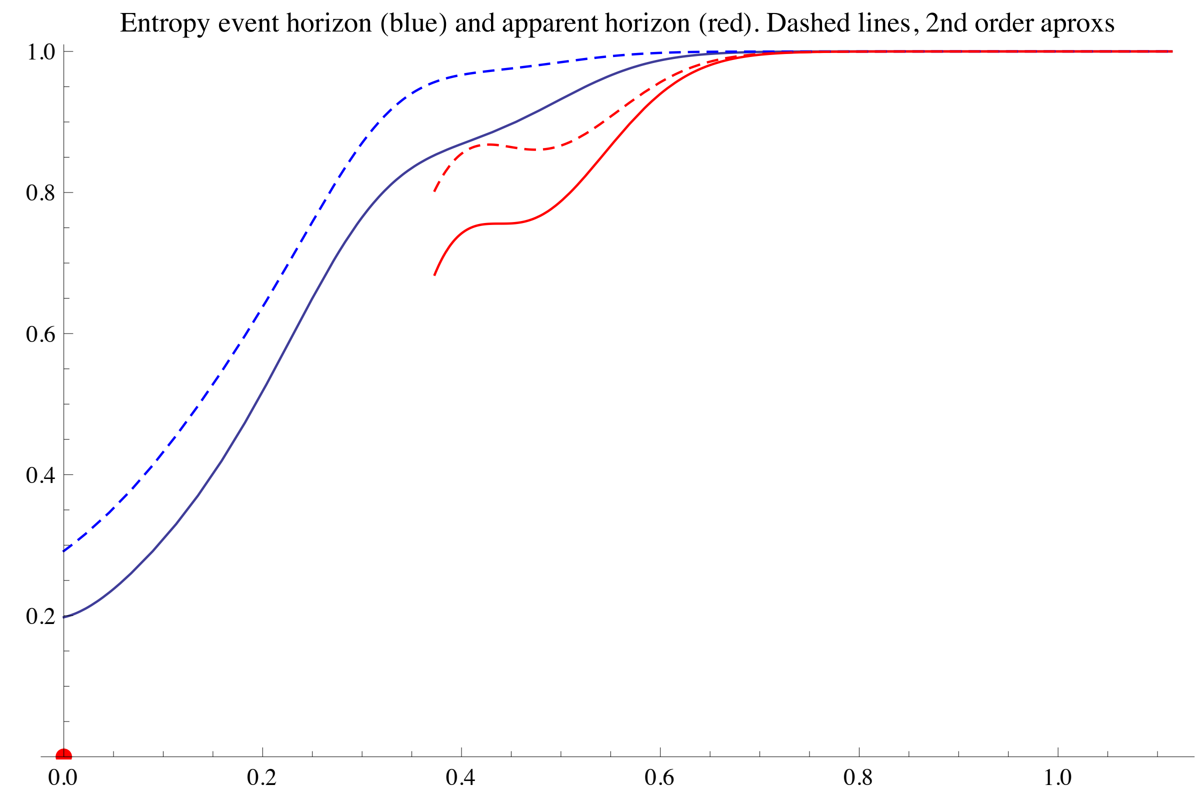
$$\Sigma = \Sigma_{BH}(t,r) + \epsilon^2 \delta \Sigma_2 + O(\epsilon^4)$$

idea: do better than linearized gravity

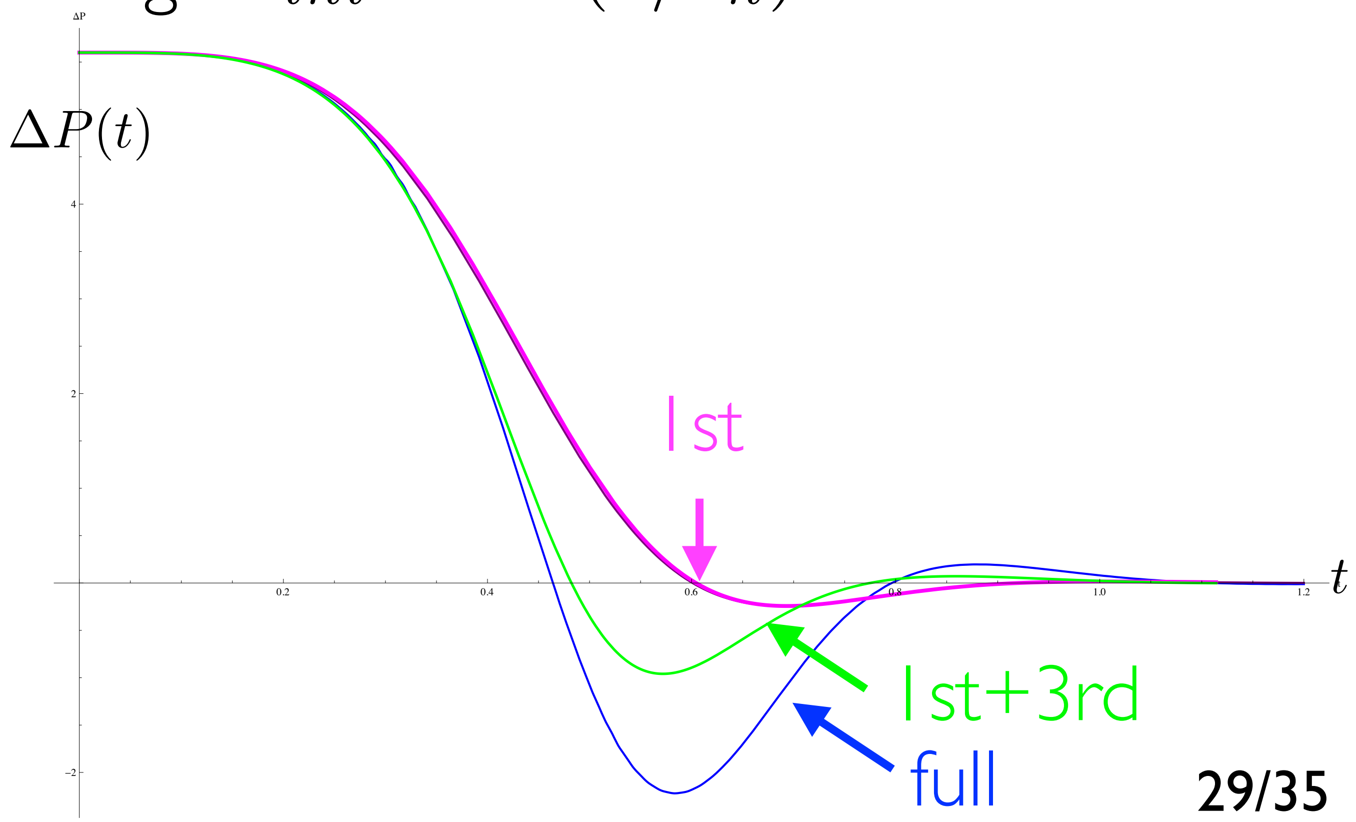
Linearized approximation in the bulk / NLO $\Delta P(t)$

Often works very well, e.g. $B_{ini}(z) = \frac{4}{5}(z/z_h)^4 \sin(8\,z/z_h)$





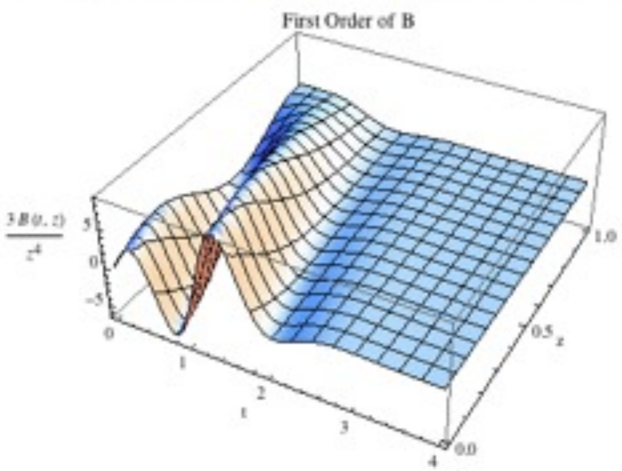
but e.g. $B_{ini} = 1.4(z/z_h)^4$



Holographic isotropization simplified?

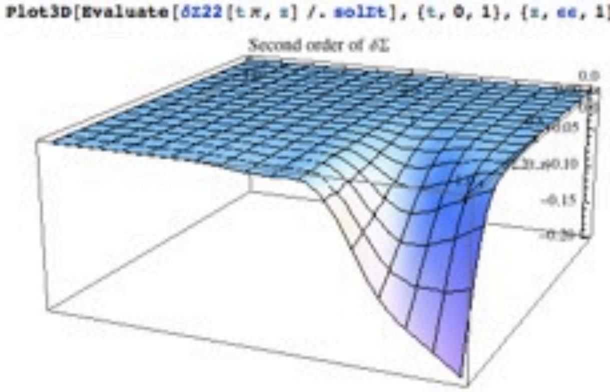
Mathematica code for simplified isotropization

```
eqnn = Simplify /@ Collect[((3 δB11^(0,1)[t, z])/(2 z) + 1/2 z^3 δB11^(0,1)[t, z] - 1/2 δB11^(0,2)[t, z] + 1/2 z^4 δB11^(0,2)[t, z] - (3 δB11^(1,0)[t, z])/(2 z) + δB11^(1,1)[t, z]) (-2/z),
  {Bn^(0,2)[t, z], Bn^(1,0)[t, z], Bn^(1,1)[t, z], Bn^(0,1)[t, z], Bn[t, z]}]

-3 (1 + 3 z^4) Bn[t, z] - z (-3 + 7 z^4) Bn^(0,1)[t, z] - z^2 (-1 + z^4) Bn^(0,2)[t, z] - 3 z Bn^(1,0)[t, z] - 2 z^2 Bn^(1,1)[t, z]

soln = NDSolve[{eqnn == 0, Bn[0, z] == 0.8 * z * Sin[8 z], Bn[t, 0] == 0}, Bn, {t, 0, 4}, {z, 0, 1.005}, MaxStepSize → 0.001, AccuracyGoal → 8, PrecisionGoal → 8]

Plot3D[Evaluate[3 z^-1 Bn[t, z] /. soln], {t, 0, 4}, {z, 0, 1}, PlotRange → All, AxesLabel → {"t", "z", "(3 B (t, z))/z^4"}, PlotLabel → "First Order of B"]
```



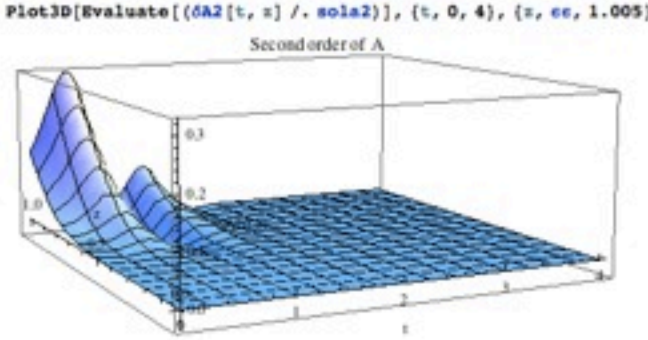
δB_1

```
solΣt = NDSolve[{-3/2 (D[z^3 Bn[t, z] /. soln, z])^2 - 6 D[δΣ22[t, z], z] - 3 z D[δΣ22[t, z], z, z] == 0, δΣ22[t, ee] == 0, δΣ22^(0,1)[t, ee] == 0}, δΣ22, {t, 0, 4}, {z, ee, 1.005},
  WorkingPrecision → 8, MaxStepSize → 0.01, AccuracyGoal → 8, PrecisionGoal → 8]

Plot3D[Evaluate[δΣ22[t π, z] /. solΣt], {t, 0, 1}, {z, ee, 1}, PlotRange → All, AxesLabel → {"t", "z", "δΣ2(t,z)"}, AxesOrigin -> {0, 0, 0}, PlotLabel → "Second order of δΣ"]
```



$\delta\Sigma_2$

```
sola2 =
 NDSolve[
  {-(3 (δΣ22[t, z] /. solΣt))/z - 3 z^3 (δΣ22[t, z] /. solΣt) - 1/2 z δA2^(0,1)[t, z] - 3 D[(δΣ22[t, z] /. solΣt), z] - 3 z^4 D[(δΣ22[t, z] /. solΣt), z] + 1/2 z^2 δA2^(0,2)[t, z] -
    3/2 D[z^3 Bn[t, z] /. soln, z] D[z^3 Bn[t, z] /. soln, t] - 3 z D[(δΣ22[t, z] /. solΣt), z, t] == 0, δA2[t, ee] == 0, δA2^(0,1)[t, ee] == 0}, δA2, {t, 0, 4}, {z, ee, 1.005},
  WorkingPrecision → 8, PrecisionGoal → 8, AccuracyGoal → 8, MaxStepSize → 0.01]

Plot3D[Evaluate[(δA2[t, z] /. sola2)], {t, 0, 4}, {z, ee, 1.005}, PlotRange → All, AxesLabel → {"t", "z", "δA2(t,z)"}, AxesOrigin -> {0, 0, 0}, PlotLabel → "Second order of A"]
```



δA_2

Comments and thoughts

Wider applicability of linearized gravity

Holographic isotropization we considered is an example in which

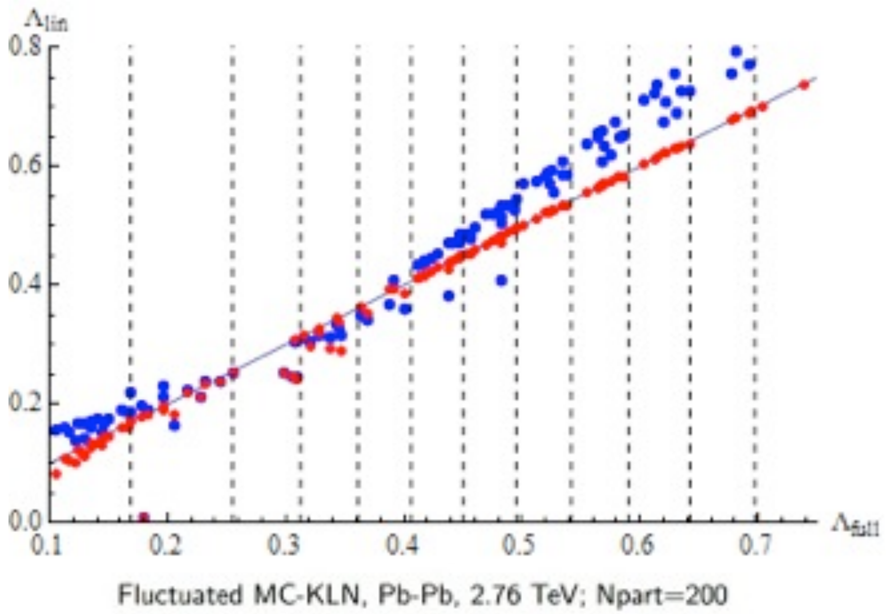
- we know the final state from the start and it is obvious to expand around this particular background
- dissipation is built-in by the final state horizon

These features are special

- in general we are interested in expanding plasmas, diluting with time in a way determined by dynamics
- the natural starting point **IS NEITHER** Poincare patch of **NOR** AdS-Schwarzchild

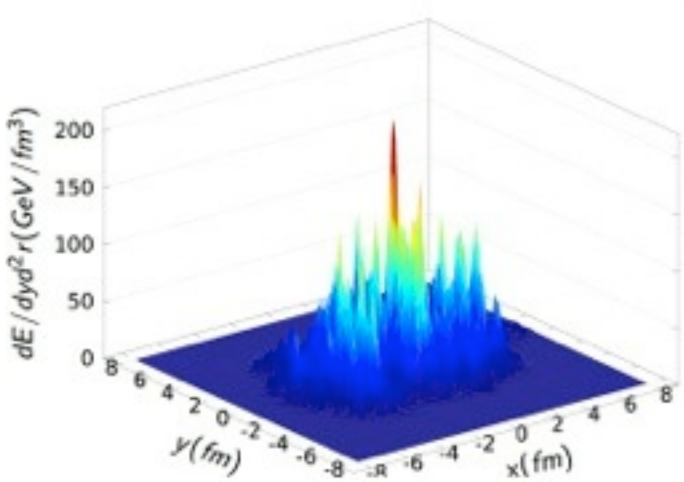
NDSolve is unlikely to help then, but it is possible to make progress

Now:



ongoing work on the boost-invariant flow with David Mateos, Wilke van der Schee and Michał Spaliński

Future:



???

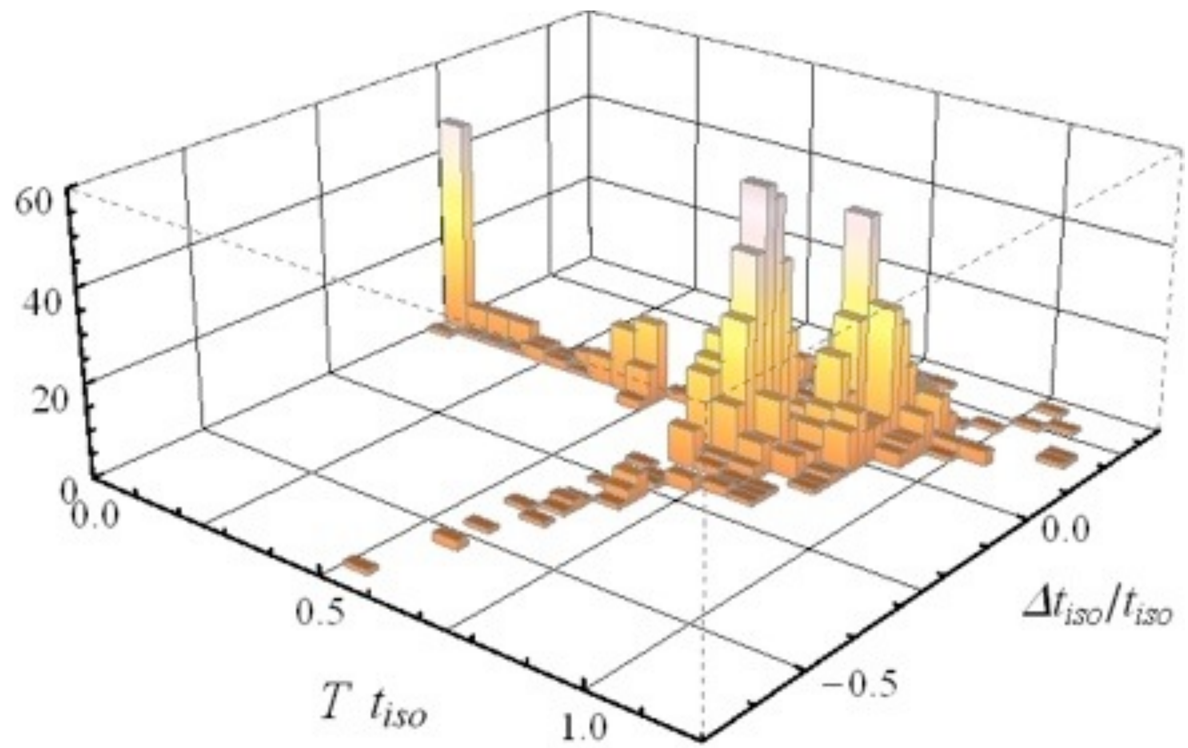
Summary

Summary

General theme: AdS/CFT leads to short thermalization times

$$0.5 \text{ fm/c} \times 350 \text{ MeV} = T\, t_{iso} = 0.63$$

Novelty I: we verified this on a large set on initial conditions (more than 2000!!!).



Novelty II: quite surprisingly, linearized gravity reproduces well both qualitative and quantitative (within 30%) the dynamics of the stress tensor of isotropizing plasma!!!

Message: sometimes no need for hardcore numerics, `NDSolve` will do the job!!!

Future: use linearized gravity to get a handle on much more complicated dynamics.