# Statistical symmetries of the multi-point equations and what they imply for turbulent scaling laws

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**Martin Oberlack**

**George Khujadze**

**Andreas Rosteck**

**Chair of Fluid Dynamics**

**Technische Universität Darmstadt**

**Darmstadt/Germany**

# Content

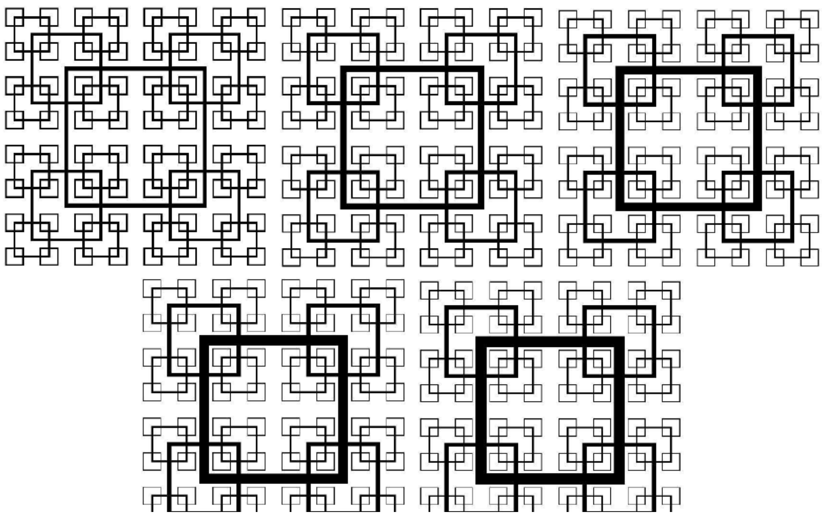


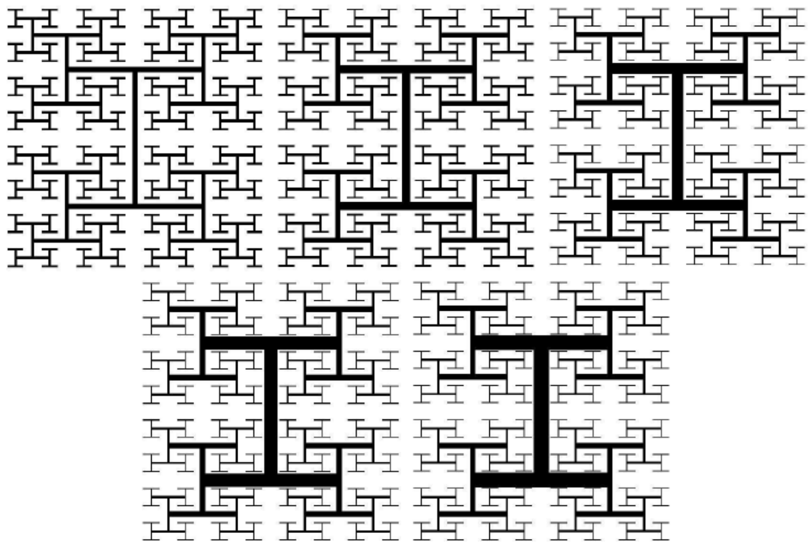
- What triggered the analysis?
- Symmetry, symmetry breaking and invariance
- Symmetries of Euler and Navier-Stokes equations
- Multi-point correlation equations
- New statistical symmetries
- New scaling laws
- Conclusion

# Fractal generated turbulence I

- Fractal grid generated turbulence (Vassilicos et al.)
- Experimental series with different
  - grids forms
  - grid bar stretching factors

**Hurst & Vassilicos**
**Phys. Fluids 2007**
**Seoud & Vassilicos**
**Phys. Fluids 2007**

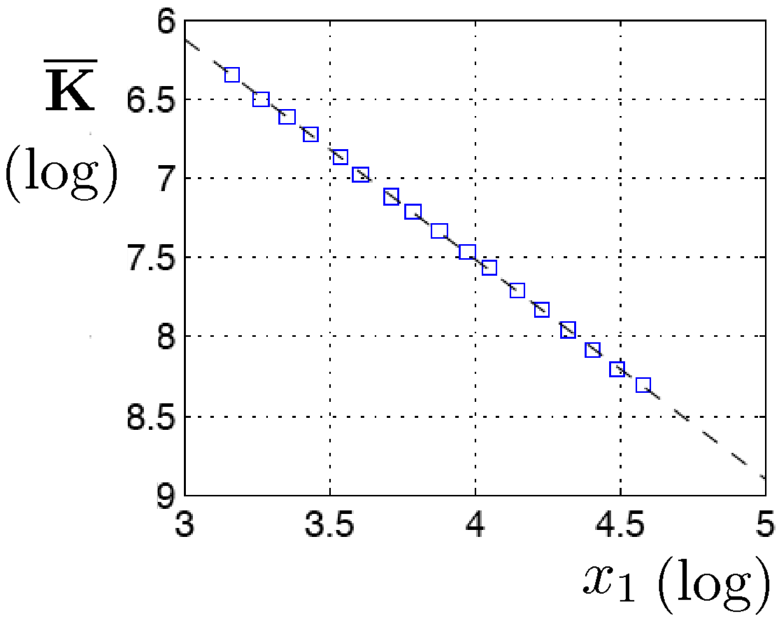




# Fractal generated turbulence II

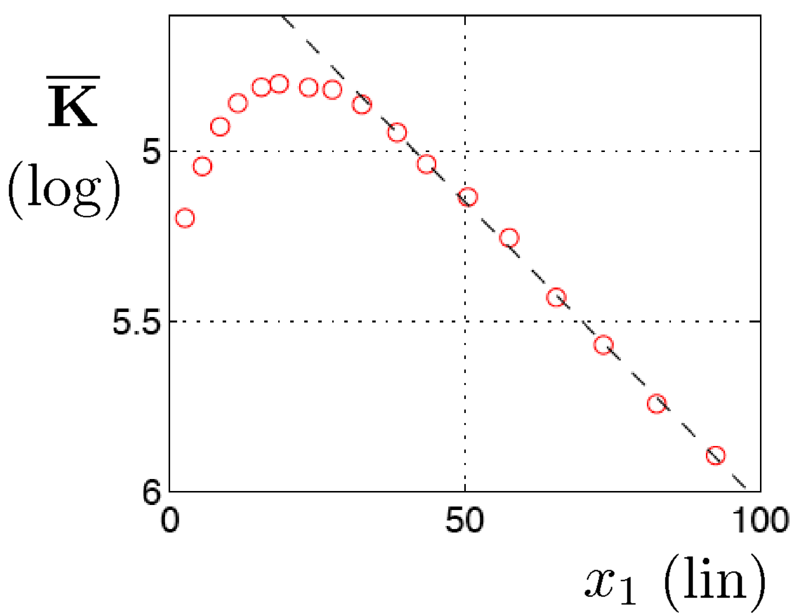
- Square grid vs. fractal grid generated turbulence
  - **Square grid**

$$\overline{\mathbf{K}} \sim {x_1}^{-n}$$




  - **Fractal grid**

$$\overline{\mathbf{K}} \sim e^{-x_1/a}$$



# Key questions & observations

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- Scaling symmetries from invicid equations (von Karman-Howarth - > large scale) **only** leads to an algebraic decay law
- **No** combination may lead to an exponential decay!
- In Khujadze, Oberlack (TCFD 2004) a **new scaling group** was observed in the two-point correlation equation with no correspondence in Euler or Navier-Stokes equation!

- Do new *statistical symmetries* physically make sense?
- Are there more *statistical symmetries* in the multi-point equations?

# Concept of symmetries

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- Analogy between symmetric object and differential equation:

| | | |
|---|---|---|
| geometrical object | $\leftrightarrow$ | differential equation |
| virtual change | $\leftrightarrow$ | transformation of variables |

- Example: heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

- Symmetry transformation I: scaling of space and time

$$S_\alpha^{(1)}: \quad x^* = \mathrm{e}^\alpha x \ , \quad t^* = \mathrm{e}^{2\alpha} t \ , \quad u^* = u$$

- into heat equation:

$$\Rightarrow \ \mathrm{e}^{2\alpha}\frac{\partial u^*}{\partial t^*} = \mathrm{e}^{2\alpha}\frac{\partial^2 u^*}{\partial {x^*}^2}$$

# Invariants

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- *An invariant is a quantity which under a given symmetry transformation does not change ist functional form*

**Example**

$$S_{\varepsilon_1,\varepsilon_2}: \quad x^* = \mathrm{e}^{\varepsilon_1} x \;, \quad t^* = \mathrm{e}^{2\varepsilon_1} t \;, \quad u^* = \mathrm{e}^{\varepsilon_2} u$$

Invariants:

1) $$\gamma = \frac{x}{\sqrt{t}} = \frac{\mathrm{e}^{-\varepsilon_1} x^*}{\sqrt{\mathrm{e}^{-2\varepsilon_1} t}} = \frac{x^*}{\sqrt{t^*}} = \gamma^*$$

2) $$\Delta = \frac{u}{t^{\frac{\varepsilon_2}{2\varepsilon_1}}} = \frac{\mathrm{e}^{-\varepsilon_2} u^*}{\left(\mathrm{e}^{-2\varepsilon_1} t^*\right)^{\frac{\varepsilon_2}{2\varepsilon_1}}} = \frac{u^*}{t^{*\frac{\varepsilon_2}{2\varepsilon_1}}} = \Delta^*$$

# Invariant solutions

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**Implementing the invariants admitted by the DE into the DE leads to a (similarity) reduction**

**Example: Heat equations**

Invariants under the combined group of scaling:

$$\gamma = \frac{x}{\sqrt{t}} \qquad \Delta = \frac{u}{t^{\frac{\varepsilon_2}{2\varepsilon_1}}}$$

into the heat equation

$$\Rightarrow \quad \frac{d^2\Delta}{d\gamma^2} + \frac{1}{2}\gamma\frac{d\Delta}{d\gamma} - \frac{\varepsilon_2}{2\varepsilon_1}\Delta = 0$$

# Symmetry breaking

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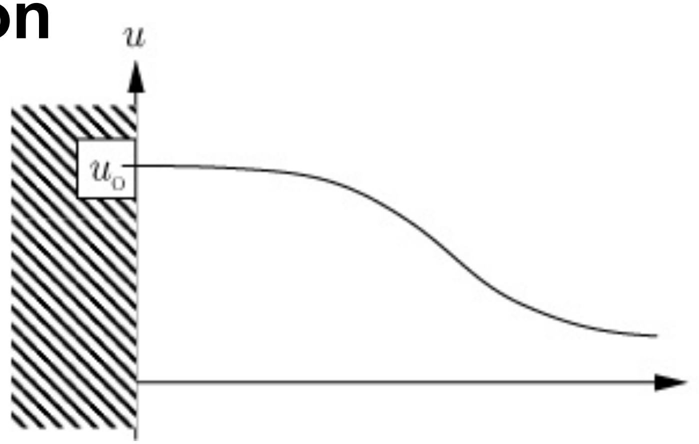
- Specific boundary conditions may adjust certain parameters

**Example: Heat equation**

BC: $x = 0 : u = u_0$

$$\Rightarrow \ \mathrm{e}^{-\varepsilon_1} x^* = 0 : \mathrm{e}^{-\varepsilon_2} u^* = u_0$$




Implementing the symmetries: $\Rightarrow \quad \varepsilon_2 = 0 \ , \quad \varepsilon_1$ is arbitrary

**$u_0$ is symmetriy breaking since $u$ cannot be scaled**

# Symmetries of the Euler and Navier-Stokes equations

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- Scaling of space:

$$T_{s_1}: \quad t^* = t \ , \quad \boldsymbol{x}^* = \mathrm{e}^{a_2}\boldsymbol{x} \ , \quad \boldsymbol{u}^* = \mathrm{e}^{a_2}\boldsymbol{u} \ , \quad p^* = \mathrm{e}^{2a_2}p$$

- Scaling of time:

$$T_{s_2}: \quad t^* = \mathrm{e}^{a_3}t \ , \quad \boldsymbol{x}^* = \boldsymbol{x} \ , \quad \boldsymbol{u}^* = \mathrm{e}^{-a_3}\boldsymbol{u} \ , \quad p^* = \mathrm{e}^{-2a_3}p$$

# Symmetries of the Euler and Navier-Stokes equations II

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- Translation in time:

$$T_t: \quad t^* = t + a_1 \;,\quad \boldsymbol{x}^* = \boldsymbol{x} \;,\quad \boldsymbol{u}^* = \boldsymbol{u} \;,\quad p^* = p,$$

- Finite rotation:

$$T_{r_1} - T_{r_3}: \qquad t^* = t \;,\quad \boldsymbol{x}^* = \mathbf{a} \cdot \boldsymbol{x} \;,\quad \boldsymbol{u}^* = \mathbf{a} \cdot \boldsymbol{u} \;,\quad p^* = p,$$

Note: $\mathbf{a} \neq \boldsymbol{f}(t)$

# Symmetries of the Euler and Navier-Stokes equations III

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## Generalized Galilean invariance:

$$T_{u_1} - T_{u_3}: \quad t^* = t \;,\; \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t) \;,\; \boldsymbol{u}^* = \boldsymbol{u} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t} \;,\; p^* = p - \boldsymbol{x} \cdot \frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2}$$

- Comprises two classical cases:
  - Translation in space: $\boldsymbol{f}(t) = \boldsymbol{a}$
  - Classical Galilean invariance: $\boldsymbol{f}(t) = \boldsymbol{b}\, t$

## Pressure invariance:

where $f_4(t)$ is an arbitrary function of time.

$$T_p: \quad t^* = t \;,\; \boldsymbol{x}^* = \boldsymbol{x} \;,\; \boldsymbol{u}^* = \boldsymbol{u} \;,\; p^* = p + a_4 f_4(t),$$

**All symmetries transfer to the correlation equations!**

# Notations in turbulence

- Instantaneous value

$$\boldsymbol{U}, \quad P$$

- Mean value

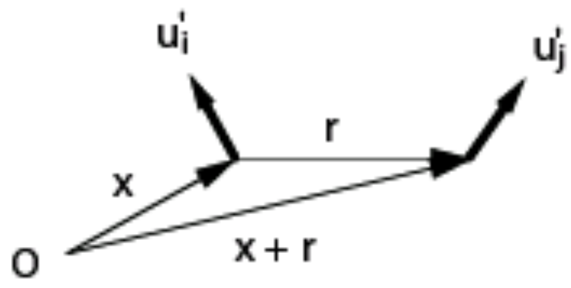
$$\bar{\boldsymbol{U}}, \quad \bar{P}$$

- Fluctuating quantities

$$\boldsymbol{u}, \quad p$$

- Two-point correlation tensor

$$R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = \overline{u_i(\boldsymbol{x}, t)\, u_j(\boldsymbol{x}+\boldsymbol{r}, t)}$$



# Multi-point correlation: fluctuation approach (classical)

- Definition multi-point tensor

$$R_{i_{\{n+1\}}} = R_{i_{(0)}i_{(1)}\dots i_{(n)}} = \overline{u_{i_{(0)}}(\mathbf{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)})}$$

$$P_{i_{\{n\}}[l]} = \overline{u_{i_{(0)}}(\mathbf{x}_{(0)}) \cdot \ldots \cdot u_{i_{(l-1)}}(\mathbf{x}_{(l-1)}) p(\mathbf{x}_{(l)}) u_{i_{(l+1)}}(\mathbf{x}_{(l+1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)})}$$

$$\Rightarrow$$

$$\begin{aligned}\mathcal{T}_{i_{\{n+1\}}} = \frac{\partial R_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \Bigg[ & \bar{U}_{k_{(l)}}(\boldsymbol{x}_{(l)}) \frac{\partial R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}} + R_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]} \frac{\partial \bar{U}_{i_{(l)}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \\ & + \frac{\partial P_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 R_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} - R_{i_{\{n\}}[i_{(l)} \mapsto \emptyset]} \frac{\partial \overline{u_{i_{(l)}} u_{k_{(l)}}}(\boldsymbol{x}_{(l)})}{\partial x_{k_{(l)}}} \\ & + \frac{\partial R_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} \Bigg] = 0\end{aligned}$$

- **Constitues a infinite set of nonlinear PDEs**

# Multi-point correlation: Instantaneous approch

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- Definition multi-point tensor

$$H_{i_{\{n+1\}}} = H_{i_{(0)}i_{(1)}\ldots i_{(n)}} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}$$

$$I_{i_{\{n\}}[l]} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)}) P(\boldsymbol{x}_{(l)}) U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}$$

$$\Rightarrow \qquad \mathcal{S}_{i_{\{n+1\}}} = \frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \left[ \frac{\partial H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} \right] = 0 \ \text{ for } \quad n = 1, \ldots, \infty \ .$$

- **Constitues a infinite set of linear PDEs (compare Hopf equation)**

# Relations

- Relations among multi-point tensor

$$
\begin{aligned}
H_{i_{(0)}} &= \bar{U}_{i_{(0)}} \\
H_{i_{(0)}i_{(1)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} + R_{i_{(0)}i_{(1)}} \\
H_{i_{(0)}i_{(1)}i_{(2)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\bar{U}_{i_{(2)}} \\
&\quad + R_{i_{(0)}i_{(1)}}\bar{U}_{i_{(2)}} + R_{i_{(0)}i_{(2)}}\bar{U}_{i_{(1)}} + R_{i_{(1)}i_{(2)}}\bar{U}_{i_{(0)}} + R_{i_{(0)}i_{(1)}i_{(2)}} \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

# New Statistical Symmetry I

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$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Translation in correlation space:

$$\bar{T}_1': \quad t^*=t,\ \ \boldsymbol{x}^*=\boldsymbol{x},\ \ \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)}+\boldsymbol{a}_{(l)},\quad \mathbf{H}^*_{\{n\}}=\mathbf{H}_{\{n\}},\ \ \mathbf{I}^*_{\{n\}}=\mathbf{I}_{\{n\}},$$

- Form of the symmetry unchanged in classical notation

$\Rightarrow$ not considered further!

# New Statistical Symmetry II

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$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Translation in function space:

$$\bar{T}'_{2_{\{n\}}}:\quad t^*=t,\ \ \boldsymbol{x}^*=\boldsymbol{x},\ \ \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)},\ \ \mathbf{H}^*_{\{n\}}=\mathbf{H}_{\{n\}}+\mathbf{C}_{\{n\}},\ \ \mathbf{I}^*_{\{n\}}=\mathbf{I}_{\{n\}}+\mathbf{D}_{\{n\}},$$

with $\mathbf{C}_{\{n\}}$ and $\mathbf{D}_{\{n\}}$ arbitrary constant tensors

- Formulation in classical notation

$$\bar{T}'_{2_{\{1\}}}:\ t^*=t,\ \ \boldsymbol{x}^*=\boldsymbol{x},\ \ \boldsymbol{r}^*_{(l)}=\boldsymbol{r}_{(l)},\ \ \bar{U}^*_{i_{(0)}}=\bar{U}_{i_{(0)}}+C_{i_{(0)}},$$
$$R^*_{i_{(0)}i_{(1)}}=R_{i_{(0)}i_{(1)}}+\bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}-\left(\bar{U}_{i_{(0)}}+C_{i_{(0)}}\right)\left(\bar{U}_{i_{(1)}}+C_{i_{(1)}}\right),\ \ \cdots$$



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# New Statistical Symmetry III

$$\frac{\partial H_{i_{\{n+1\}}}}{\partial t}+\sum_{l=0}^{n}\left[\frac{\partial H_{i_{\{n+2\}}[i_{(n+1)}\mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)}\mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}}+\frac{\partial I_{i_{\{n\}}[l]}}{\partial x_{i_{(l)}}}-\nu\frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right]=0$$

- Scaling of correlations:

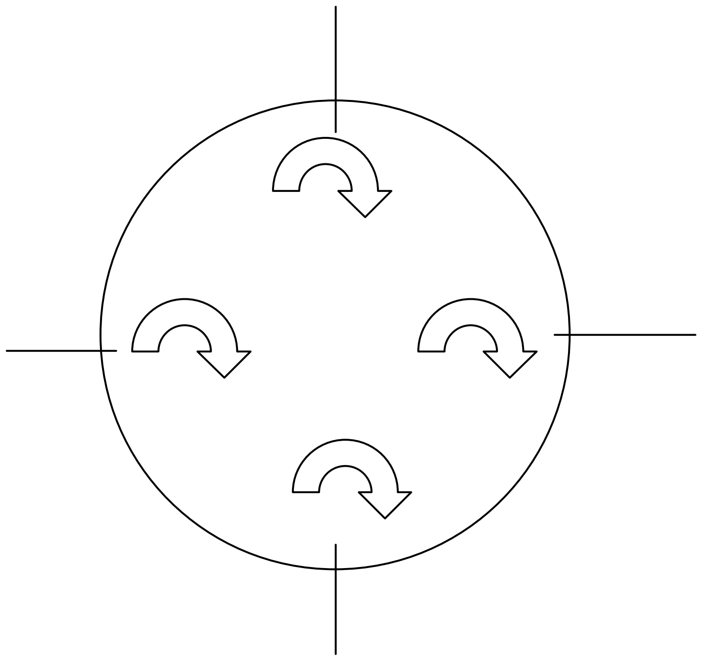
$$\bar{T}_s':\quad t^*=t,\;\; \boldsymbol{x}^*=\boldsymbol{x},\;\; \boldsymbol{r}_{(l)}^*=\boldsymbol{r}_{(l)},\quad \mathbf{H}_{\{n\}}^*=\mathrm{e}^{a_s}\mathbf{H}_{\{n\}},\;\; \mathbf{I}_{\{n\}}^*=\mathrm{e}^{a_s}\mathbf{I}_{\{n\}}$$

- Formulation in classical notation

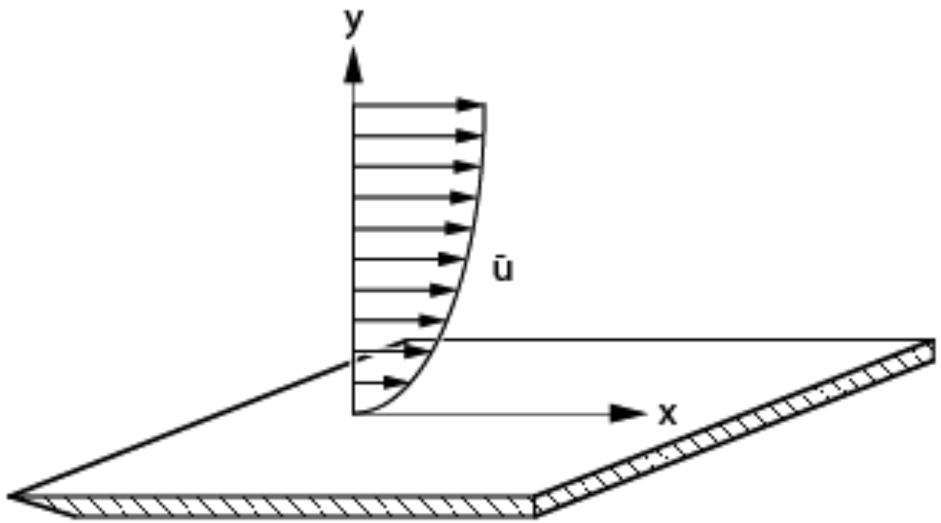
$$\bar{T}_s':\; t^*=t,\;\; \boldsymbol{x}^*=\boldsymbol{x},\;\; \boldsymbol{r}_{(l)}^*=\boldsymbol{r}_{(l)},$$
$$\bar{U}_{i_{(0)}}^*=\mathrm{e}^{a_s}\bar{U}_{i_{(0)}},\;\; R_{i_{(0)}i_{(1)}}^*=\mathrm{e}^{a_s}\left[R_{i_{(0)}i_{(1)}}+\left(1-\mathrm{e}^{a_s}\right)\bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\right],\;\;\cdots$$

# Two test cases

- Temporally decaying turbulence



Near-wall scaling laws



# Invariant solution and symmetry breaking

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- Invariance condition (*here infinitesimal form*)

$$\frac{dt}{a_3 t + a_4} = \frac{dr_i}{a_2 r_i} = \frac{dR_{ij}}{[2(a_2 - a_3) + a_s]R_{ij}} = \cdots$$

- Symmetry breaking
  - Breaking scaling of space:

$$a_2 = 0: \quad R^\star_{ij} = e^{2(\overbrace{a_2}^{=0} - a_3) + a_s} R_{ij}; \quad t^\star = e^{a_3} t; \quad \mathbf{r}^\star = e^{\overbrace{a_2}^{=0}} \mathbf{r}; \ldots$$

  - Breaking scaling of time:

$$a_3 = 0: \quad R^\star_{ij} = e^{2(a_2 - \overbrace{a_3}^{=0}) + a_s} R_{ij}; \quad t^\star = e^{\overbrace{a_3}^{=0}} t; \quad \mathbf{r}^\star = e^{a_2} \mathbf{r}; \ldots$$

# Case1: Decaying turbulence I

„Regular“ decaying turbulence

- Characteristic variable $a_2 \neq a_3 \neq a_s$
  - Birkhoff integral, Loyatsinski integral, … ,:
- Scaling symmetries

$$R_{ij}^\star = e^{2(a_2-a_3)+a_s} R_{ij}; \quad t^\star = e^{a_3} t; \quad \mathbf{r}^\star = e^{a_2}\mathbf{r}; \ldots$$

- Invariant solution

$$R_{ij} = t^{-n} \tilde{R}_{ij}(r/t^m) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim t^{-n} \\ \ell_t \sim t^m \end{array}}$$

# Case1: Decaying turbulence II

Constant length scale $\ell_t$ decaying turbulence

- Characteristic variable

$$\ell_t = const. \Longrightarrow a_2 = 0$$

- Scaling symmetries:

$$R^\star_{ij} = e^{2(\overbrace{a_2}^{=0} - a_3) + a_s} R_{ij}; \quad t^\star = e^{a_3} t; \quad \mathbf{r}^\star = e^{\overbrace{a_2}^{=0}} \mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = t^{-2+\gamma} \tilde{R}_{ij}(\mathbf{r}) \quad \Longrightarrow \quad \boxed{\begin{array}{l} \overline{\mathbf{K}} \sim t^{-2+\gamma} \\ \ell_t \sim const. \end{array}}$$

# Case1: Decaying turbulence III

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Constant time scale $\tau$ decaying turbulence

- Characteristic variable

$$\tau = const. \quad \Rightarrow \quad a_3 = 0$$

- Scaling symmetries

$$R^\star_{ij} = e^{2(a_2 - \overbrace{a_3}^{=0}) + a_s} R_{ij}; \quad t^\star = e^{\overbrace{a_3}^{=0}} t; \quad \mathbf{r}^\star = e^{a_2}\mathbf{r}; \ldots$$

- Invariant solution:

$$R_{ij} = t^{-2t/t_0} \tilde{R}_{ij}(r/e^{t/t_0}) \quad \Longrightarrow$$

$$\overline{\mathbf{K}} \sim e^{-2t/t_0}$$

$$\ell_t \sim e^{t/t_0}$$



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# Case1: Decaying turbulence IV

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Constant space and „time scale"

- Characteristic variable
  - All space and time scales are broken:
    *Fractal Grid generated turbulence* $\Rightarrow \quad a_2 = a_3 = 0$
- Scaling symmetries:

$$R^\star_{ij} = e^{2(\overbrace{a_2 - a_3}^{=0})+a_s} R_{ij}; \quad t^\star = e^{\overbrace{a_3}^{=0}} t; \quad \mathbf{r}^\star = e^{\overbrace{a_2}^{=0}} \mathbf{r}; \quad \dots$$

- Invariant solution:

$$R_{ij} = e^{-x_1/a} \tilde{R}_{ij}(r) \quad \Longrightarrow \quad \boxed{\begin{array}{c} \overline{\mathbf{K}} \sim e^{-x_1/a} \\ \ell_t \sim const. \end{array}}$$

# Case2: Wall-bounded shear flows

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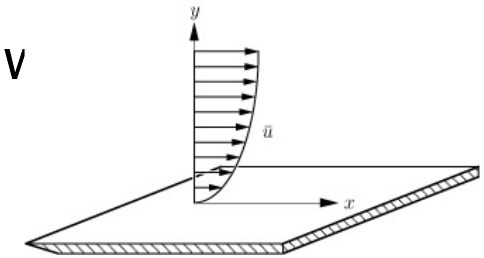
- Fundamental geometrical assumption for the flov

$$\bar{\boldsymbol{U}} = [\bar{U}_1(x_2), 0, 0] \ , \quad R_{ij} = R_{ij}(x_2, \boldsymbol{r}), \cdots$$




- Related symmetries:
  - *Scaling:* $\bar{T}_{s_1-s_3}: \quad x_2^* = e^{a_2} x_2 \, , \quad \bar{U}_1^* = e^{a_2-a_3+a_s} \bar{U}_1 \, ,$

    $$R_{ij}^* = e^{2(a_2-a_3)+a_s} R_{ij} + \ldots \, , \quad \ldots$$

  - *Translation in the function space: (in early work falsely interpreted)*

    $$\bar{T}_{\bar{U}_1}: \quad x_2^* = x_2 \, , \quad \bar{U}_1^* = \bar{U}_1 + a_1 \, , \quad R_{ij}^* = R_{ij} + \ldots \quad , \ldots$$

  - *Translational invariance in $x_2$ – direction:*

    $$\bar{T}_{x_2}: \quad x_2^* = x_2 + a_4 \, , \quad \bar{U}_1^* = \bar{U}_1 \, , \quad R_{ij}^* = R_{ij} \quad , \ldots$$

# Invariant solutions of plane wall-bounded shear flows

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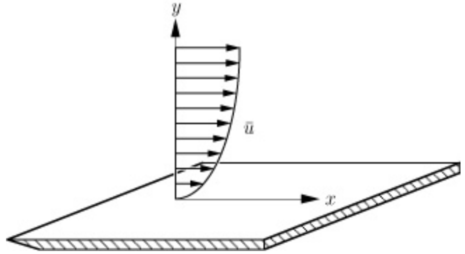
- Invariant (surface) condition

$$\frac{dx_2}{a_2 x_2 + a_4} = \frac{dr_i}{a_2 r_i} = \frac{d\bar{U}_1}{(a_2 - a_3 + a_s)\bar{U}_1 + a_1} = \frac{dR_{[ij]}}{f(R_{ij}, \bar{U}_1, x_2)} = \cdots$$

- Depending on the values of the $a_i$, five different solutions exist for
  - mean velocity (Oberlack JFM 2001)
  - **one- and multi-point correlations**
- For $a_2 - a_3 + a_s = 0$ the von Kármán/Prandtl equation and the classical log-law scaling law is determined

# Logarithmic region I

- Characteristic variable: wall-friction velocity




$$u_\tau \qquad \Rightarrow a_2 - a_3 + a_s = 0$$

- Scale invariance

$$x_2^* = \mathrm{e}^{a_2} x_2 \;, \quad t^* = \mathrm{e}^{a_3} t \;, \quad \bar{U}_1^* = \mathrm{e}^{\overbrace{a_2 - a_3 + a_s}^{=0}} \bar{U}_1 \;, \quad \ldots$$

- Solution

$$\Rightarrow \quad \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}x_2} = \frac{\overbrace{(a_2 - a_3 + a_2)}^{=0} \bar{U}_1 + a_1}{a_2 x_2 + a_4}$$

$$\Rightarrow \quad \bar{U}^+ = \frac{1}{\kappa} \ln\left(x_2^+ + A^+\right) + C$$

# Logarithmic region II

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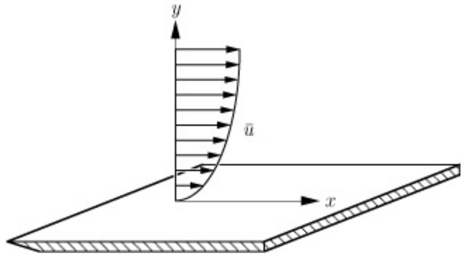
- Invariance condition for the correlations

$$\Rightarrow \quad \frac{\mathrm{d}R_{ij}}{\mathrm{d}x_2} = f(R_{ij}, \bar{U}_1, x_2)$$




- Several cases may be distinguished (here only one)

$$\tilde{r}_k = \frac{r_k}{x_2 + \frac{a_{x_2}}{a_2}},$$

$$R_{ij} = \left(x_2 + \frac{a_{x_2}}{a_2}\right)^{-\frac{a_s}{a_2}} \tilde{R}_{ij}(\tilde{\boldsymbol{r}}) + \frac{a_{R_{ij}}}{a_s} \qquad \text{for } ij \neq 11$$

$$R_{11} = \tilde{R}_{11}(\tilde{\boldsymbol{r}}) \left(x_2 + \frac{a_{x_2}}{a_2}\right)^{-a_s/a_2} + \gamma_1(\tilde{r}_2)$$

$$+ \gamma_2(\tilde{r}_2) \left[\ln\left(x_2 + \frac{a_{x_2}}{a_2}\right) - \frac{a_2}{a_s}\right] - \frac{a_{\bar{U}_1}^2}{a_2^2} \ln^2\left(x_2 + \frac{a_{x_2}}{a_2}\right), \quad \ldots \quad ,$$

# Conclusions

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- Three sets of new symmetries have been derived for the multi-point correlation equations
- The new statistical symmetries generate an enormously extended set of possible invariant solutions (here: decaying turbulence, wall turbulence)



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# Open questions



- Are there more statistical symmetries?
- Do the new scaling laws really fit the data?
- How do we determine the values for the parameters (if not just fitted)
- How do layers of scaling laws match e.g. in near-wall flows?

# Thank you for your attention!