

# A Lie-group mean field theory for turbulent channel and pipe flow

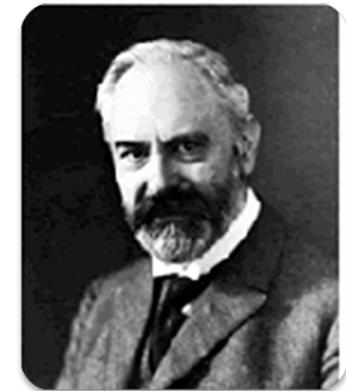
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2011-10-19

In 1904, Prandtl presented the boundary layer concept. This plausibility argument yields a laminar BL solution, but no turbulent BL solution exists.

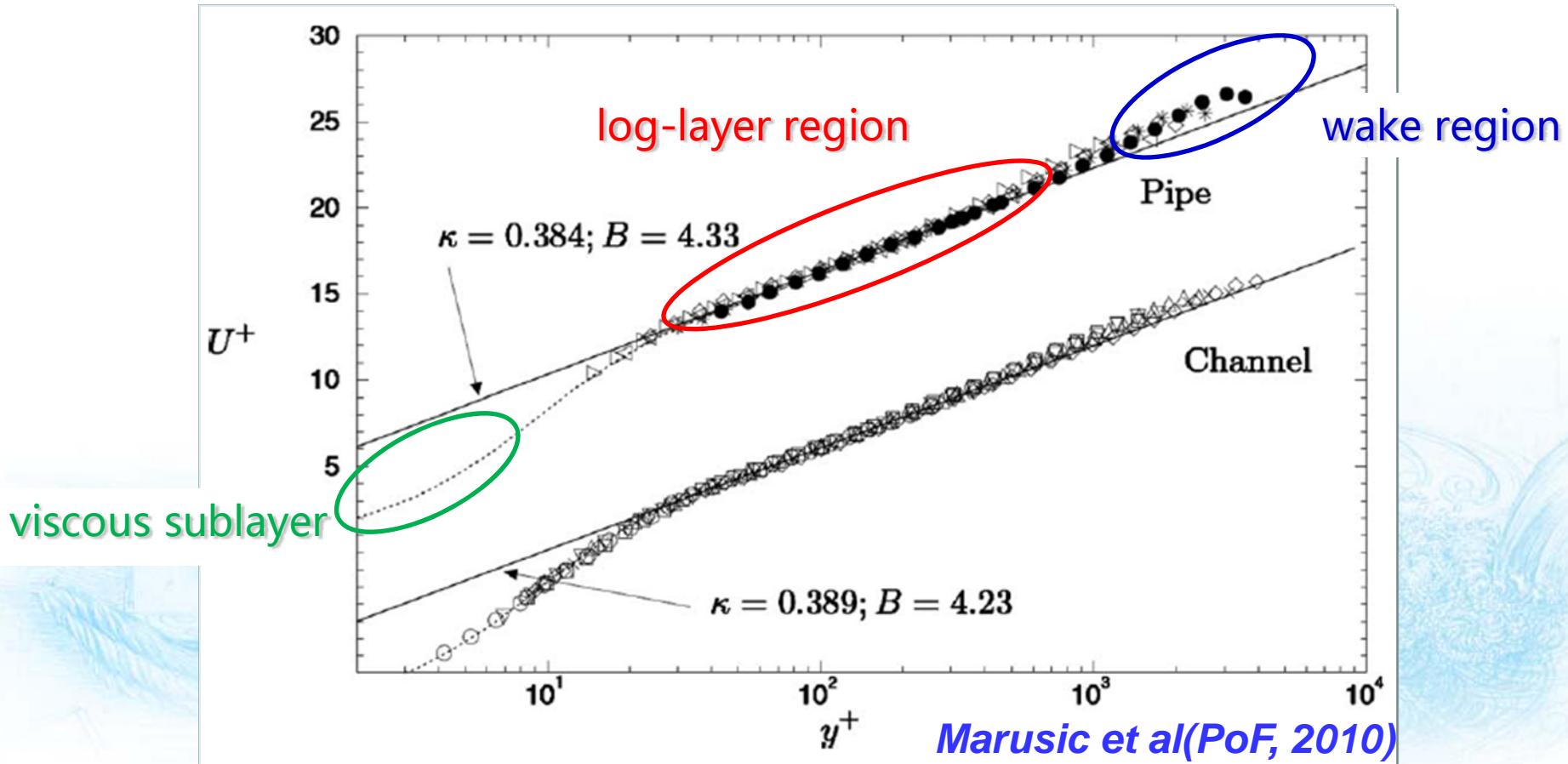
We combine BL concept with Lie-group analysis to obtain an accurate analytic (phenomenological) solution for turbulent channel and pipe.



Ludwig  
Prandtl  
1875 – 1953

## Two theoretical challenges:

- Derive a complete mean velocity profile (MVP) from Eq.?
- The universality of Karman constant?



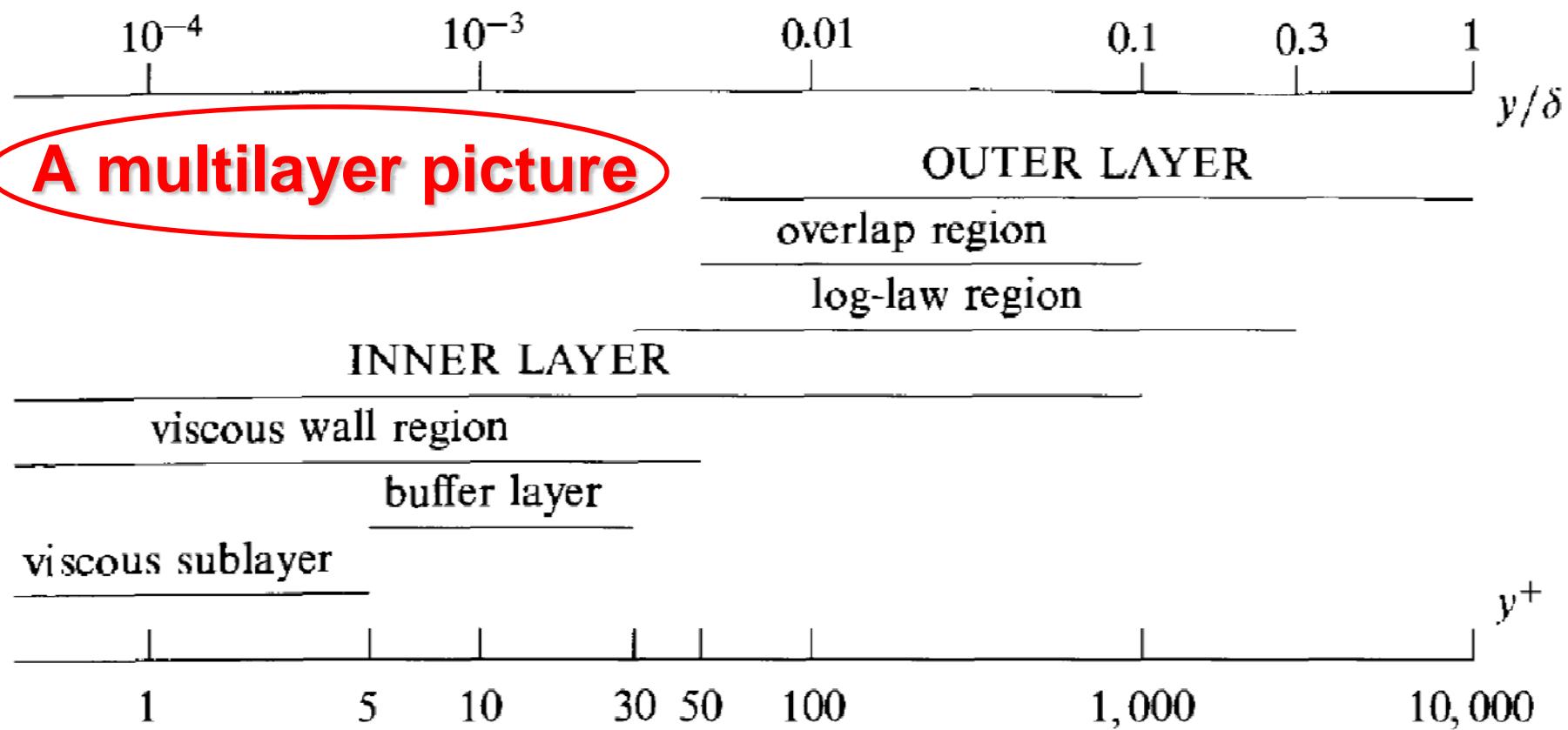


Fig. 7.8. A sketch showing the various wall regions and layers defined in terms of  $y^+ = y/\delta_v$  and  $y/\delta$ , for turbulent channel flow at high Reynolds number ( $Re_\tau = 10^4$ ).

'Turbulent flows' (Pope, 2000)

- Recent effort for modeling MVP:
  - Nickels (JFM, 2004)
  - Monkewitz et al. (PoF, 2007)
  - **Nickels (JFM, 2004)**

$$U^+ = y_c^+ \left[ 1 - \left( 1 + 2(y^+/y_c^+) + \frac{1}{2}(3 - p_x^+ y_c^+)(y^+/y_c^+)^2 - \frac{3}{2}p_x^+ y_c^+ (y^+/y_c^+)^3 \right) e^{-3y^+/y_c^+} \right] \\ + \frac{\sqrt{1 + p_x^+ y_c^+}}{6\kappa_o} \ln \left( \frac{1 + (0.6(y^+/y_c^+))^6}{1 + \eta^6} \right) + b \left( 1 - e^{-\frac{5(\eta^4 + \eta^8)}{1 + 5\eta^3}} \right)$$

A three layers model, no connection to Eq.

- Recent effort for modeling MVP:
  - Nickels (JFM, 2004)
  - Monkewitz et al. (PoF, 2007)

$$\frac{dU_{\text{inner}}^+}{dy^+} = P_{23} + P_{25},$$

$$P_{23} = b_0 \frac{1 + b_1 y^+ + b_2 y^{+2}}{1 + b_1 y^+ + b_2 y^{+2} + \kappa b_0 b_2 y^{+3}},$$

$$P_{25} = (1 - b_0) \frac{1 + h_1 y^+ + h_2 y^{+2}}{1 + h_1 y^+ + h_2 y^{+2} + h_3 y^{+3} + h_4 y^{+4} + h_5 y^{+5}},$$

A purely empirical fitting using Padé approximants

- L'vov, Proccacia, Rodenko (PRL, 2008) – LPR model
  - Using a length function to model effects of fluctuations;
  - Using a wall function and wake function to obtain the entire profile;
  - Fitting parameters:

$$\kappa^{pipe} = 0.415$$

$$\kappa^{CH} = 0.405$$

PRL 100, 054504 (2008)

PHYSICAL REVIEW LETTERS

week ending  
8 FEBRUARY 2008

## Universal Model of Finite Reynolds Number Turbulent Flow in Channels and Pipes

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(Received 30 May 2007; published 8 February 2008)

Our work extends LPR assumptions, by establishing a framework of multi-layer symmetry analysis for length functions in a wide class of flows.

# 1. Multi-layer seen in Mixing length

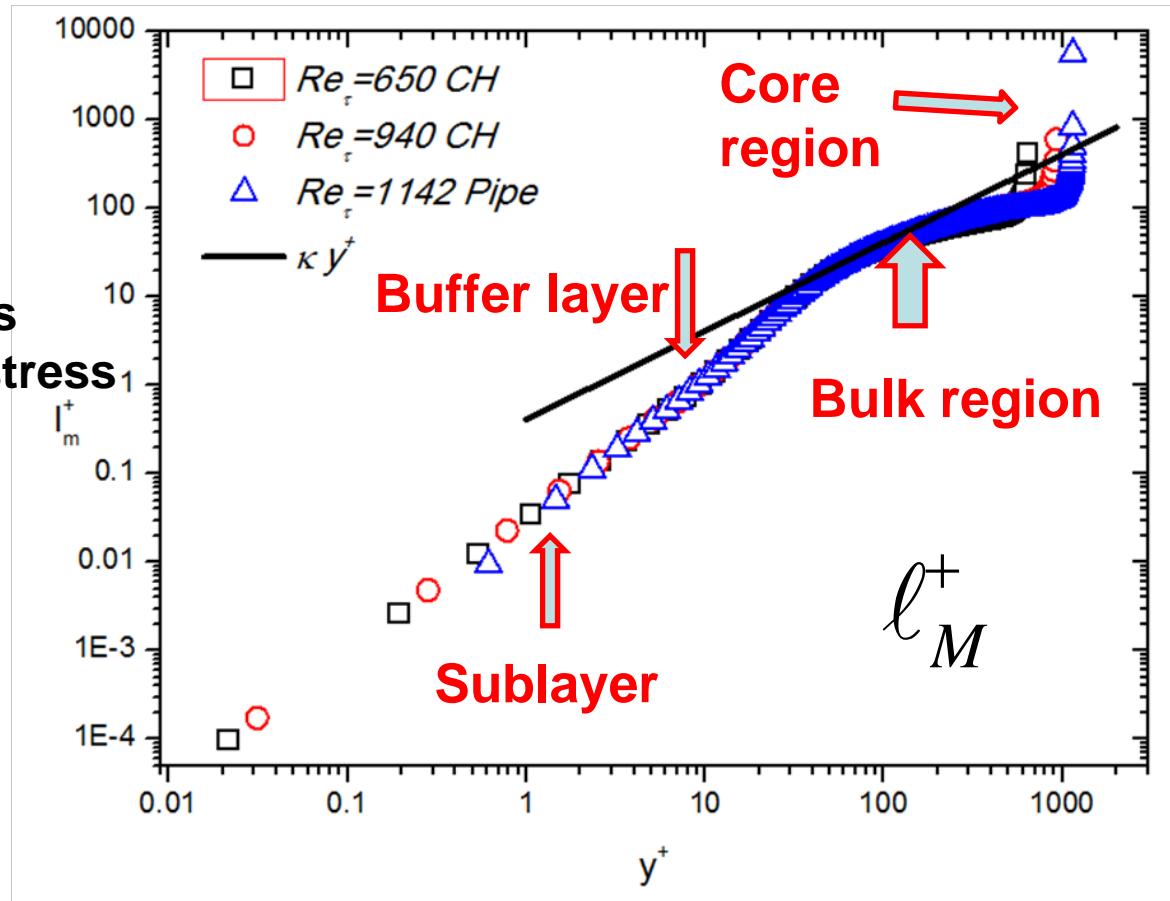
We start with the integration of mean momentum equation (MME).

$$\nu \frac{dU}{dy} - \langle u'v' \rangle = u_\tau^2 (1 - y)$$

$$S^+ = \frac{\nu}{u_\tau^2} \frac{dU}{dy} = \frac{dU^+}{dy^+}$$
 Viscous shear stress

$$W^+ = -\frac{\langle u'v' \rangle}{u_\tau^2}$$
 Reynolds stress

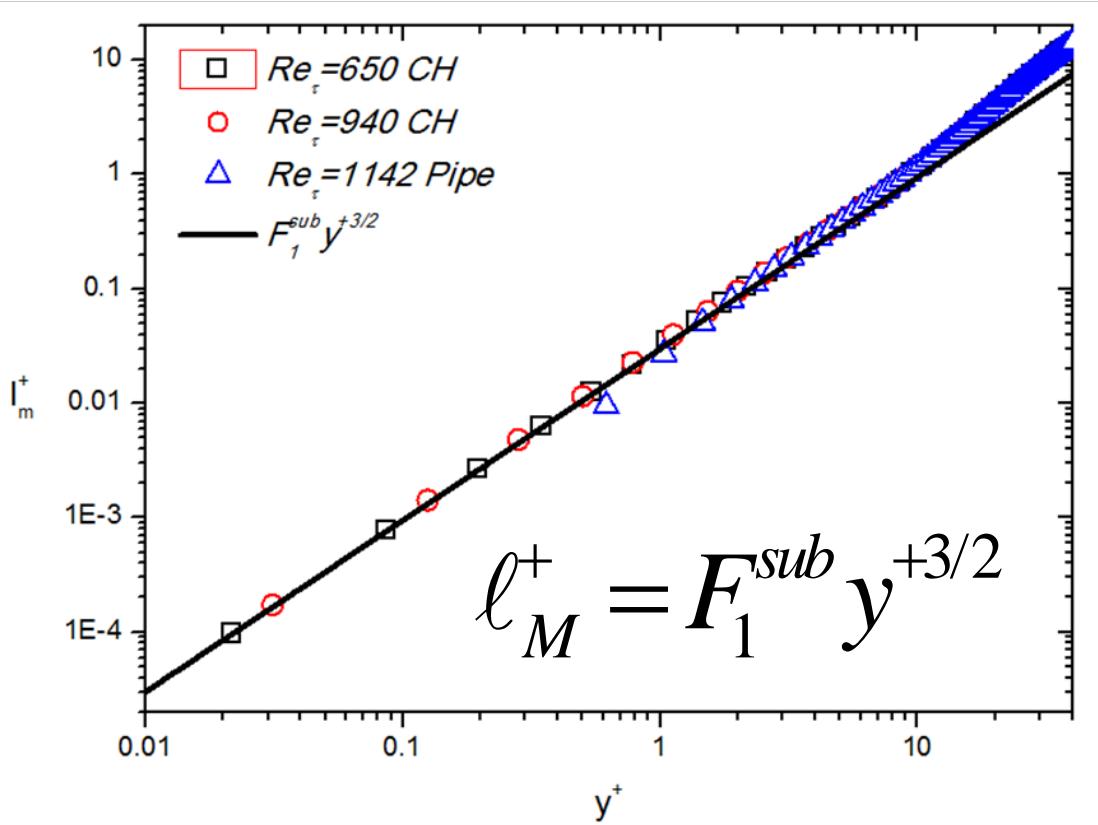
$$\ell_M^+ = \frac{\sqrt{W^+}}{S^+} = f(y^+)$$



# 1. Multi-layer for Mixing length



Power-law in the **viscous sublayer**.



$$S^+ + W^+ = 1 - y^+ / \text{Re}_\tau$$

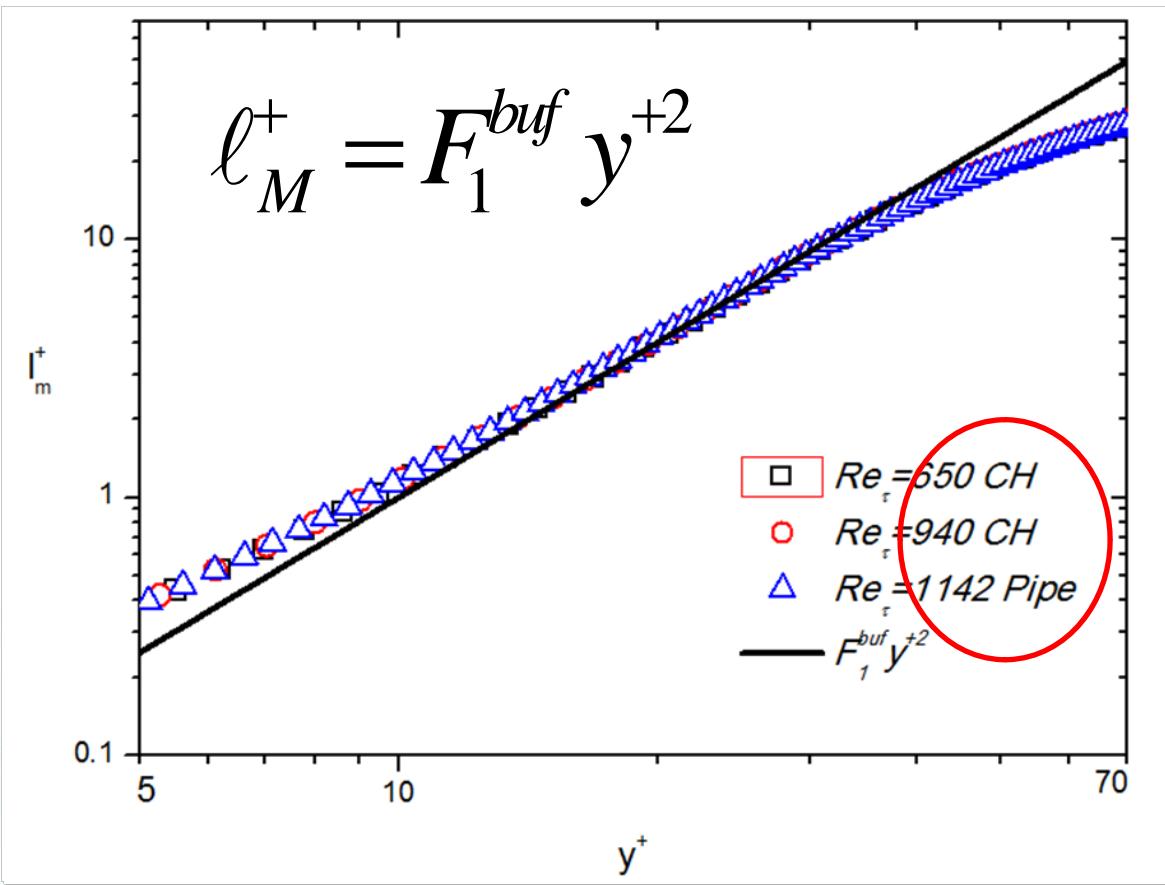
$S^+ \approx 1$  **Viscous shear stress**

$W^+ \propto y^{+3}$  **Reynolds stress**

$$\ell_M^+ = \frac{\sqrt{W^+}}{S^+} \propto y^{+3/2}$$

# 1. Multi-layer for Mixing length

Power-law in the buffer layer (new!).



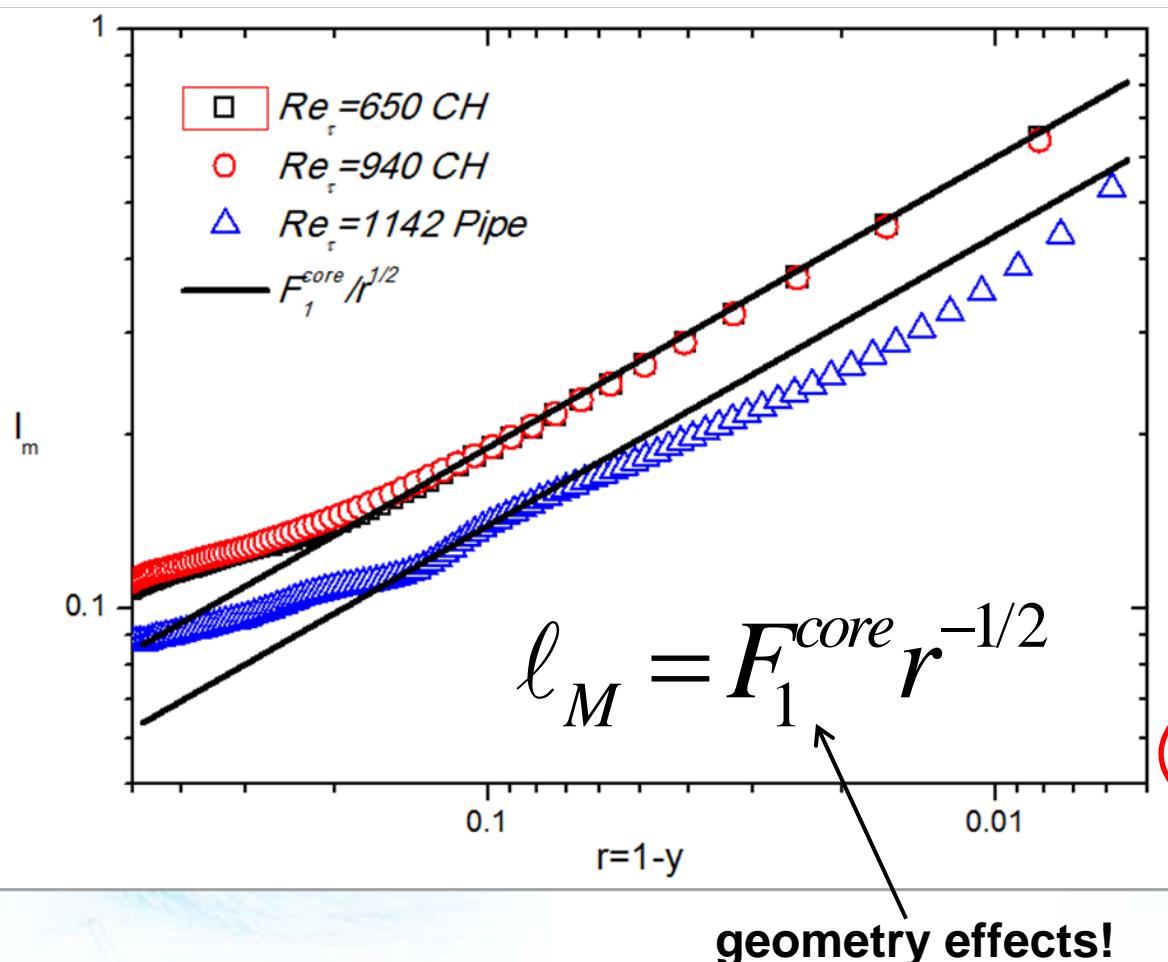
$$S^+ + W^+ = 1 - y^+ / Re_\tau$$

$$\ell_M^+ = \frac{\sqrt{W^+}}{S^+} \propto y^{+2}$$

Why square?  
No answer yet.  
Coherent vortex  
structure near the wall.

# 1. Multi-layer for Mixing length

## Power-law in the core layer.



$$S^+ + W^+ = 1 - y = r$$

$S^+ \propto r$       **Central symmetry**

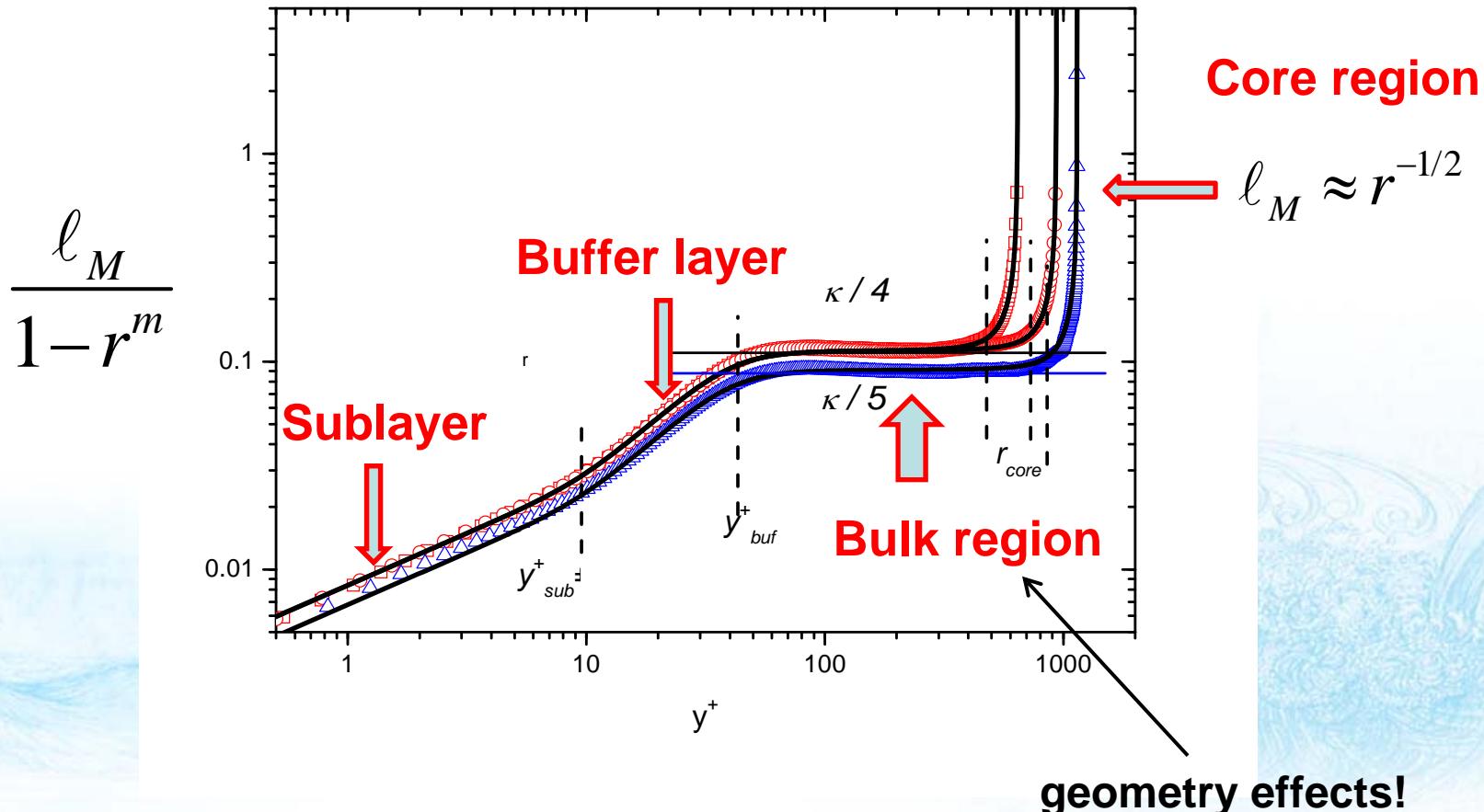
$$(U_c^+ - U^+ \propto r^2)$$

$W^+ \approx r$       **Vanishing viscous effect**

$$\boxed{\ell_M = \ell_M^+ / \text{Re}_\tau \propto r^{-1/2}}$$

# 1. Multi-layer for Mixing length

Discover a **bulk flow**,  $\ell_M \approx 1 - r^4$  channel flow and  
 $\ell_M \approx 1 - r^5$  for pipe flow (new!)



## 2. Lie-group explanation for Mixing length



They must be a set of multi-layer Lie-group similarity solutions!

Step1: symmetry transformation (Cantwell, Introduction to symmetry analysis, 2000)

Outer MME :

$$\frac{-1}{\text{Re}_\tau} \frac{d^2 U^+}{dr^2} + 2\ell_M^2 \frac{dU^+}{dr} \frac{d^2 U^+}{dr^2} + 2\ell_M \dot{\ell}_M \left( \frac{dU^+}{dr} \right)^2 = 1 \quad \dot{\phi} = \frac{d\phi}{dr}$$

Boundary condition:  $U^+(0) - U_c^+ = 0$ ,  $\dot{U}^+(0) = 0$ ,  $\ell_M(0) = \infty$ ,  $\dot{\ell}_M(0) = \infty$

Symmetry transformation:

$$r^* = e^\varepsilon r, \quad \ell_M^* = e^{\alpha\varepsilon} \ell_M, \quad \dot{\ell}_M^* = e^{(\alpha-1)\varepsilon} \dot{\ell}_M$$
$$U_c^+ - U^{+*} = e^{(\alpha-3/2)\varepsilon} (U_c^+ - U^+), \quad \text{Re}_\tau^* = e^{-(1/2+\alpha)\varepsilon} \text{Re}_\tau$$

## 2. Lie-group explanation for Mixing length



### Step 2: Invariant surface for similarity solution

**Characteristic equations:**

$$\frac{dr}{r} = \frac{d\ell_M}{\alpha\ell_M} = \frac{d\dot{\ell}_M}{(\alpha-1)\dot{\ell}_M} = \frac{d\text{Re}_\tau}{-(1/2+\alpha)\text{Re}_\tau}$$
$$= \frac{dU^+}{(\alpha-3/2)(U_c - U^+)} = \frac{d\dot{U}^+}{(\alpha-5/2)\dot{U}^+} = \frac{d\ddot{U}^+}{(\alpha-7/2)\ddot{U}^+}$$

**group invariants:**

$$F_1 = \ell_M / r^\alpha, \quad F_2 = \dot{\ell}_M / r^{\alpha-1}, \quad H = \text{Re}_\tau r^{1/2+\alpha}$$
$$G_1 = (U_c^+ - U^+) / r^{3/2-\alpha}, \quad G_2 = \dot{U}^+ / r^{5/2-\beta}, \quad G_3 = \ddot{U}^+ / r^{7/2-\alpha}$$

**Invariant-surface:**

$$\Psi(F_1, F_2, G_1, G_2, G_3, H) = 0$$

## 2. Lie-group explanation for Mixing length



### Step 3: Similarity solution

Outer MME in invariants:  $-G_3 / H + 2F_1^2 G_2 G_3 + 2F_1 F_2 G_2^2 = 1$

Two possible local similarity solutions for  $\text{Im}$ :

**Power-law:**  $F_1 = \text{const.}$   $\ell_M = F_1 r^\alpha$   
 $F_2 = \text{const.}$   $\alpha = F_2 / F_1$

**Defect power-law:**  $F_1 \neq \text{const.}$   $\ell_M = c + (F_2 / \alpha) r^\alpha = \frac{\kappa}{m} (1 - r^m)$   
 $F_2 = \text{const.}$

They describe empirical (DNS) solutions shown above.

## 2. Lie-group explanation for Mixing length



Step 3: Similarity (invariant) solution

Outer MME in invariants:  $-G_3 / H + 2F_1^2 G_2 G_3 + 2F_1 F_2 G_2^2 = 1$

**Defect power-law:**  $F_1 \neq \text{const.}$   $\rightarrow \ell_M = c + (F_2 / \alpha)r^\alpha$   
 $F_2 = \text{const.}$

let  $\ell_M(1) = 0 \rightarrow \ell_M^{\text{bulk}} = -F_2 / m(1 - r^m) = \ell_0(1 - r^m)$

near wall  $\ell_M^{\text{bulk}} \approx -F_2 y$  hence  $K = -F_2$

Our bulk flow solution naturally yields the log-layer, and therefore, Karman constant is a bulk flow Lie-group invariant!

## 2. Lie-group explanation for Mixing length



Step 3: Similarity (invariant) solution

Outer MME in invariants:  $-G_3 / H + 2F_1^2 G_2 G_3 + 2F_1 F_2 G_2^2 = 1$

**Defect power-law:**  $F_1 \neq \text{const.}$        $\rightarrow \ell_M = c + (F_2 / \alpha)r^\alpha = \ell_0(1 - r^m)$   
 $F_2 = \text{const.}$

A side note: Un-closure of MME turns to our favor, we freely postulate a form for  $F_1$  and  $F_2$ , solve the original MME for  $G_2$ ,  $G_3$ , and then outer MME can always be satisfied. In this sense, we find an analytic expression as a solution.

Note also that not all invariant function is actual empirical solution. But we have a viable candidate, which is shown to be actually good.

## 2. Lie-group explanation for Mixing length



### Step 4: Matching solution (new!)

#### 4.1 Transition ansatz (a specific invariant surface)

$$\gamma(r) \equiv \frac{F_2}{F_1} = \frac{d \ln \ell_M}{d \ln r}$$

$$\frac{\gamma(r) - \gamma^{bulk}}{\gamma^{core} - \gamma^{bulk}} = \left( \frac{F_1^{core}}{F_1(r; \gamma^{core})} \right)^{n_{b-c}}$$

from bulk to core:  $\gamma^{bulk} = 0$ ,  $\gamma^{core} = -1/2$ ,  $n^{b-c} = 4$

#### 4.2 Composite invariant solution

$$\ell_M^{(bulk-core)} = \frac{F_1^{core}}{\sqrt{r_{core}}} \left( 1 + \left( \frac{r}{r_{core}} \right)^{-n_{b-c}/2} \right)^{1/n_{b-c}}$$

#### 4.3 Multiplicative rule $\phi^{I-II} = \phi^I \times \phi^{II} / \phi^{common}$

$$\ell_M^{Outer} = \frac{\kappa}{m Z_{core}} (1 - r^m) \left( 1 + \left( \frac{r}{r_{core}} \right)^{-2} \right)^{1/4}$$

$$Z_{core} = \left( 1 + r_{core}^2 \right)^{1/4}$$

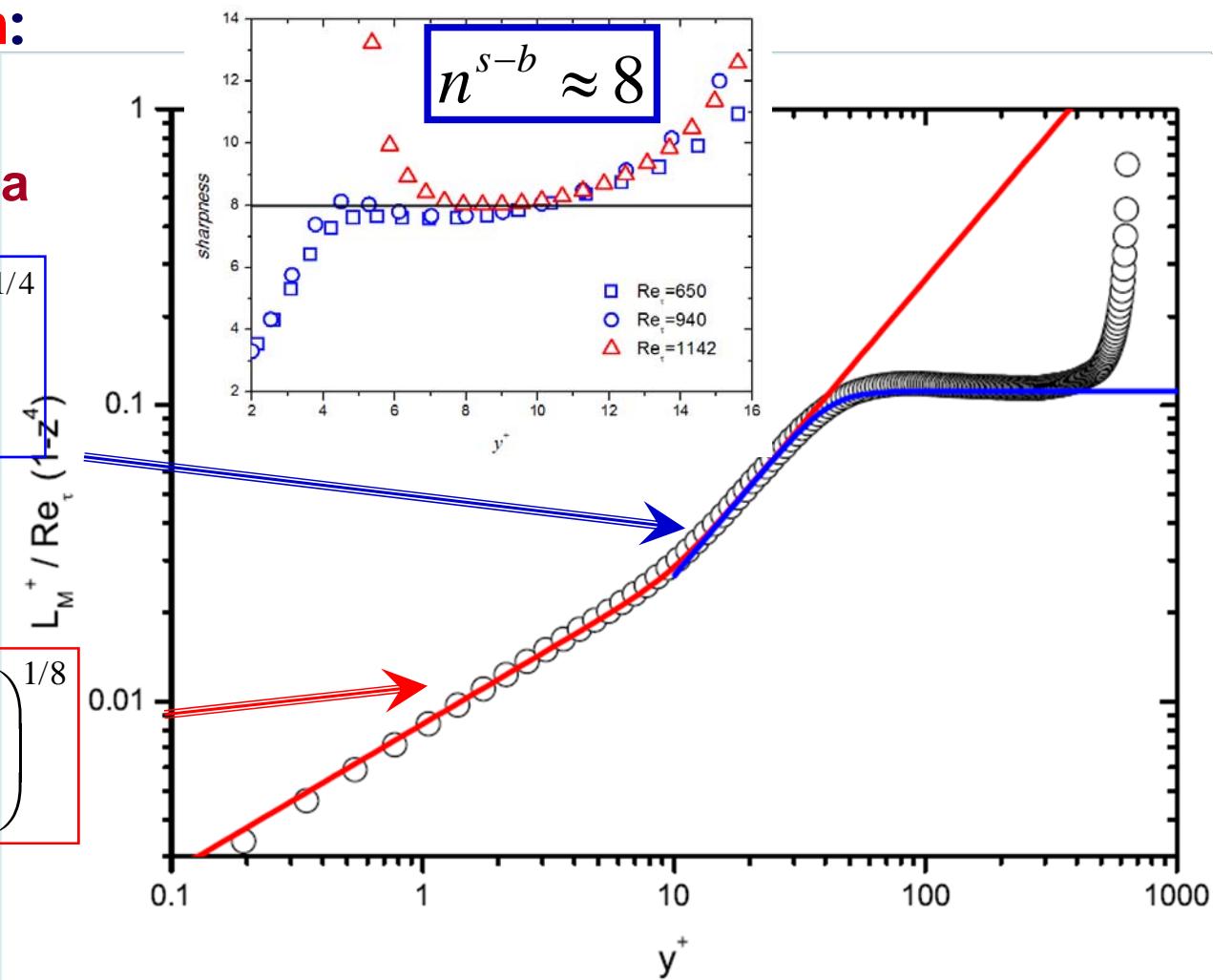
## 2. Lie-group explanation for Mixing length

- DNS verification of the transition ansatz and composite invariant solution:

Channel flow DNS data

$$\ell_M^+ \approx \left( \frac{y^+}{y_{sub}^+} \right)^2 \left( 1 + \left( \frac{y^+}{y_{buf}^+} \right)^4 \right)^{-1/4}$$

$$\ell_M^+ \approx \left( \frac{y^+}{y_{sub}^+} \right)^{3/2} \left( 1 + \left( \frac{y^+}{y_{sub}^+} \right)^4 \right)^{1/8}$$



## 2. A summary

Layers	Layer Invariant	Layer Solution	Composite Solution
Sub Layer	$F_1(\ell_M^+) = \rho / y_{sub}^{+3/2}$	$\ell_M^+ = \rho \left( \frac{y^+}{y_{sub}^+} \right)^{3/2}$	$\ell_M^{+(sub-buf)} = \rho \left( \frac{y^+}{y_{sub}^+} \right)^{3/2} \left( 1 + \left( \frac{y^+}{y_{sub}^+} \right)^{n_{r\rightarrow}} \right)^{1/n_{r\rightarrow}}$
Buffer Layer	$F_1(\ell_M^+) = \rho / y_{sub}^{+2}$	$\ell_M^+ = \rho \left( \frac{y^+}{y_{sub}^+} \right)^2$	$\ell_M^{+(buf-log)} = \rho \left( \frac{y^+}{y_{sub}^+} \right)^2 \left( 1 + \left( \frac{y^+}{y_{buf}^+} \right)^{n_{r\rightarrow}} \right)^{-1/n_{r\rightarrow}}$
Log Layer	$F_1(\ell_M^+) = \kappa$	$\ell_M^+ = \kappa y^+$	$\ell_M^{bulk} = \frac{\kappa}{m} (1 - r^m)$
Bulk	$F_2(\ell_M) = -\kappa$	$\ell_M = \frac{\kappa}{m} (1 - r^m)$	$\ell_M^{(bulk-core)} = \frac{\kappa}{m Z_{core}} \left( 1 + \left( \frac{r}{r_{core}} \right)^{-n_{r\rightarrow}/2} \right)^{1/n_{r\rightarrow}}$
Core Layer	$F_1(\ell_M) = \frac{\kappa r_{core}^{1/2}}{m Z_{core}}$	$\ell_M = \frac{\kappa}{m Z_{core}} \left( \frac{r}{r_{core}} \right)^{-1/2}$	

### 3. Solution for entire profile:



- From mixing length to mean velocity profile (MVP)

$$\ell_M^+ = \rho \left( \frac{y^+}{y_{sub}^+} \right)^{3/2} \left( 1 + \left( \frac{y^+}{y_{sub}^+} \right)^4 \right)^{1/8} \left( 1 + \left( \frac{y^+}{y_{buf}^+} \right)^4 \right)^{-1/4} \frac{1 - r^5}{5(1 - r)} \left( 1 + \left( \frac{r}{r_{core}} \right)^{-2} \right)^{1/4}$$

$$S^+ = \frac{-1 + \sqrt{4rl_m^2 + 1}}{2l_m^2}$$

**Mean shear**

$$r = 1 - y = 1 - \frac{y^+}{\text{Re}_\tau}$$

$$\rho = \frac{\kappa y_{sub}^{+2}}{y_{buf}^+}$$

$$U^+(y^+) = \int_0^{y^+} S^+ dy^+$$

**Mean velocity profile**

**Friction factor**

$$\overline{U}_{Pipe}^+ \equiv \frac{1}{\pi R^2} \int_0^R U^+ 2\pi r dr \quad \text{Average velocity (pipe)}$$

$$f = 4C_f = \frac{8}{\overline{U}^{+2}}$$

$$\overline{U}_{CH}^+ \equiv \int_0^1 U^+ dy$$

**Average velocity (channel)**

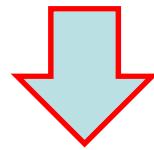
## 4. Determination of parameters

The theoretical MVP for the bulk and core region:  $200 < y^+ < \text{Re}_\tau$

$$U^{+Theory}(r) = U^+(0) - \frac{1}{\kappa} f(r, r_{core})$$

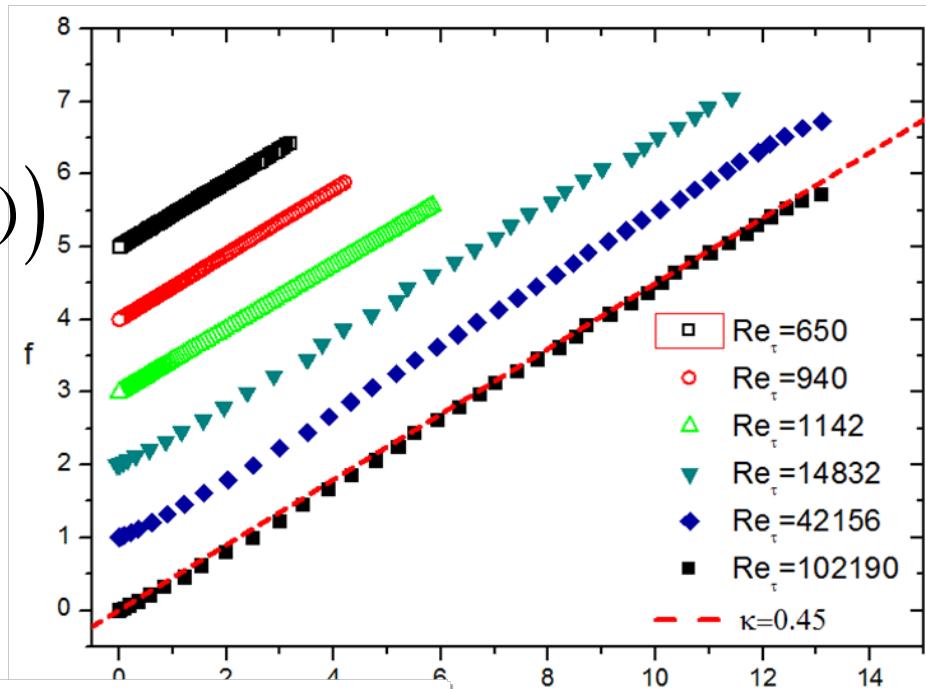
$$f(r, r_{core}) = m Z_{core} \int_0^r \frac{r' dr'}{(1 - r'^m)(r'^2 + r_{core}^2)^{1/4}}$$

A least-squares problem **Karman constant**, knowing  $f$ .



$$f(r, r_{core}) = \kappa \left( U^+(0) - U^+(r) \right)$$

$$\kappa \approx 0.45 = 9 / 20$$



$$U_d^+(r) = U^+(0) - U^{+EXP}(r)$$



## 4. Determination of parameters

Karman constant is a bulk flow property!

Many data:

CH: Ret=650 (DNS)

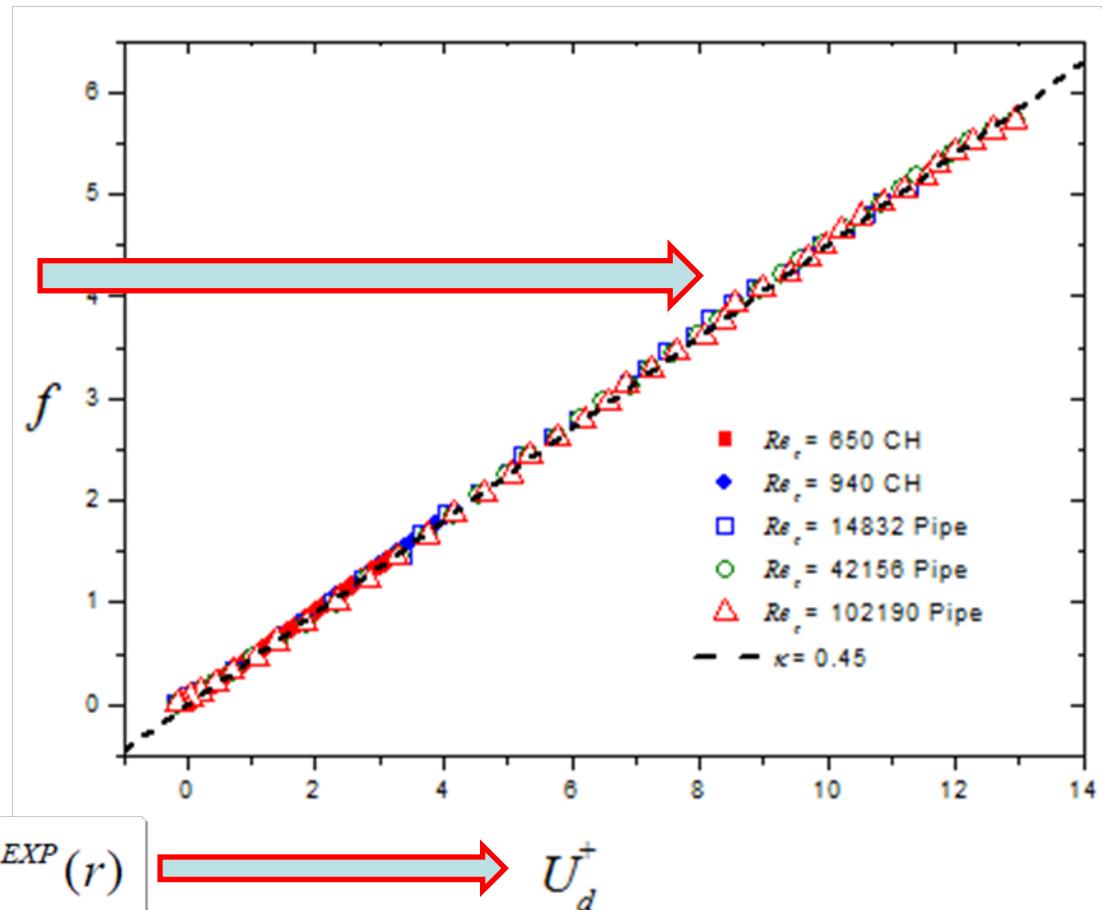
CH: Ret=940 (DNS)

Pipe: Ret=14832 (Exp)

Pipe: Ret=42156 (Exp)

Pipe: Ret=102190 (Exp)

$$\kappa \approx 0.45 = 9 / 20$$



$$U_d^+(r) = U^+(0) - U^{+EXP}(r) \rightarrow U_d^+$$

## 4. Determination of parameters

The theoretical MVP for the bulk and core region:  $200 < y^+ < \text{Re}_\tau$

$$U^{+Theory}(r) = U^+(0) - \frac{1}{K} f(r, r_{core})$$

$$f(r, r_{core}) = m Z_{core} \int_0^r \frac{r' dr'}{(1 - r'^m)(r'^2 + r_{core}^2)^{1/4}}$$

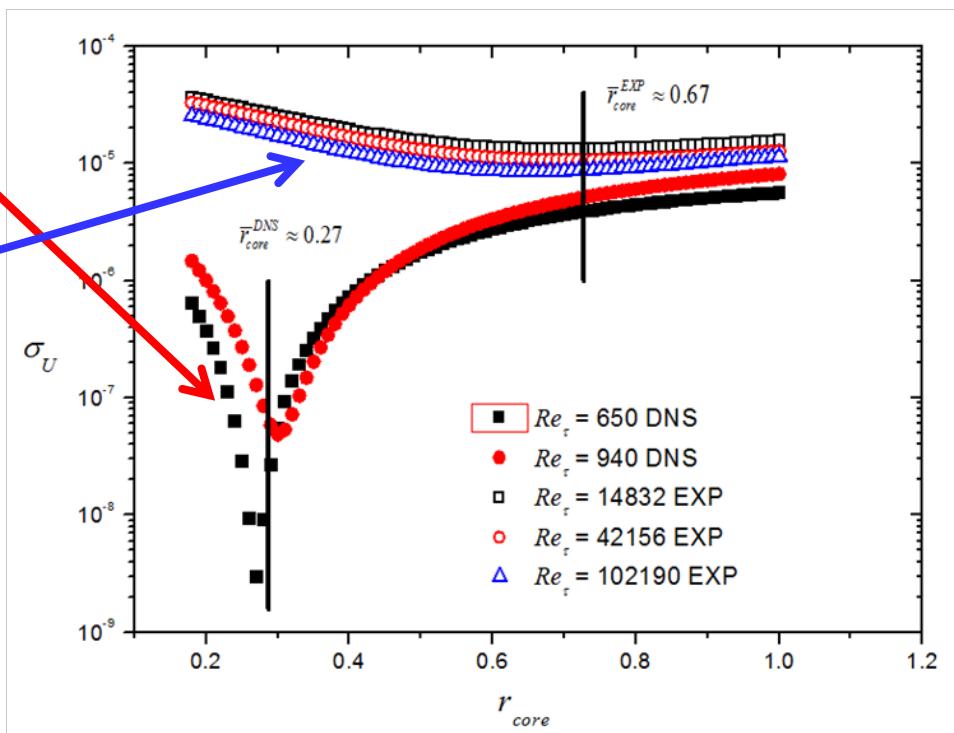
Using an error function to determine  $r_{core}$ .

DNS data (no noise)

Exp data (with noise)

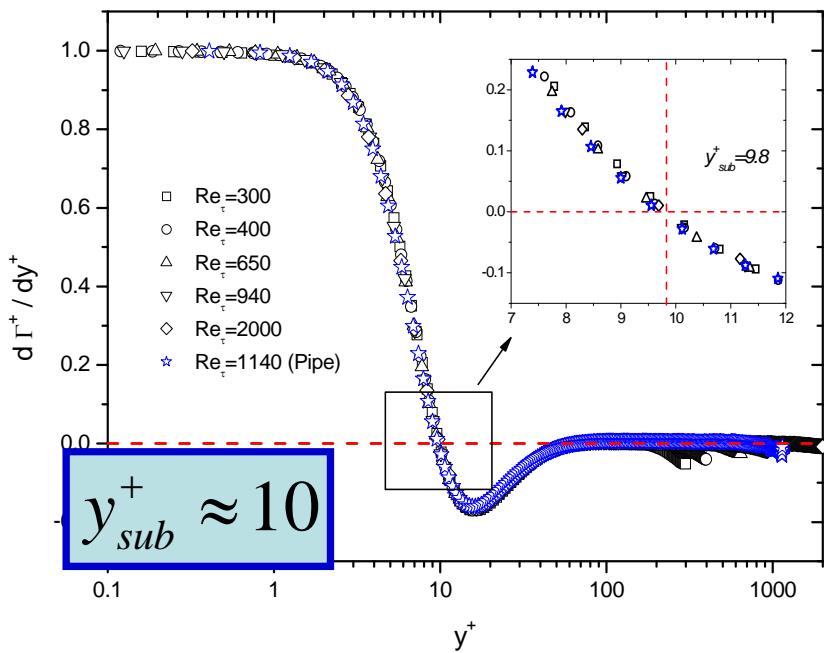
$$r_{core}^{CH} \approx 0.27$$

$$r_{core}^{Pipe} \approx 0.67$$



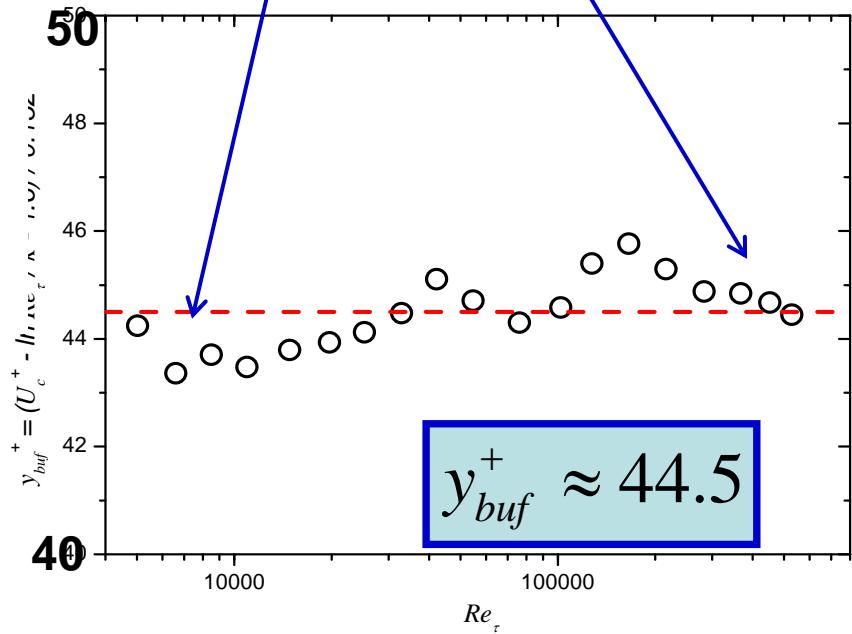
## 4. Determination of parameters

$$\Gamma = y^+ dU^+ / dy^+$$



Princeton pipe data

$$U_c^{+Pipe} \approx \ln(Re_\tau) / \kappa + 1.60 + 0.152 y_{buf}^+$$



## 5. Predictions for pipe (Princeton data):



$$\ell_M^+ (y^+, r) = 1.07 \left( \frac{y^+}{10} \right)^{3/2} \left( 1 + \left( \frac{y^+}{10} \right)^4 \right)^{1/8} \left( 1 + \left( \frac{y^+}{44.5} \right)^4 \right)^{-1/4} \frac{1 - r^5}{5(1 - r)} \left( 1 + \left( \frac{r}{0.67} \right)^{-2} \right)^{1/4}$$

here  $r = 1 - y^+ / \text{Re}_\tau$

**Mean shear**

**MVP**

**Theory**

**Average  
velocity**

**Friction  
coefficient**

$$S^+ = (-1 + \sqrt{4r\ell_M^{+2} + 1}) / (2\ell_M^{+2})$$

$$U^+(y^+) = \int_0^{y^+} S^+(y') dy'$$

$$\overline{U^+}_{\text{Pipe}} = 2 \int U^+ r dr$$

$$f = 8 / \overline{U^+}^2$$

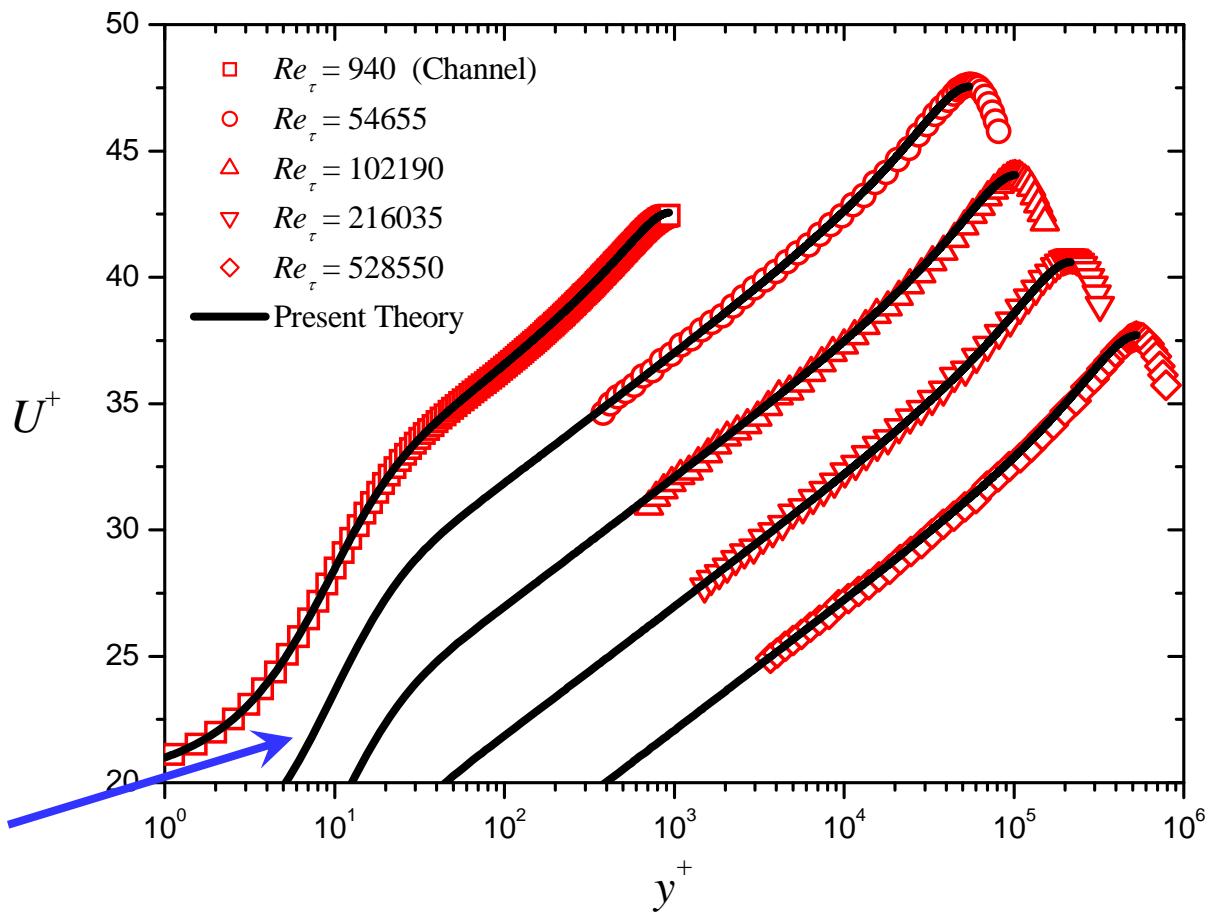
## 5. Predictions for pipe (Princeton data):



$$\kappa = 0.45$$

MVP:

Theory  
(lines)



## 5. Predictions for pipe (Princeton data):



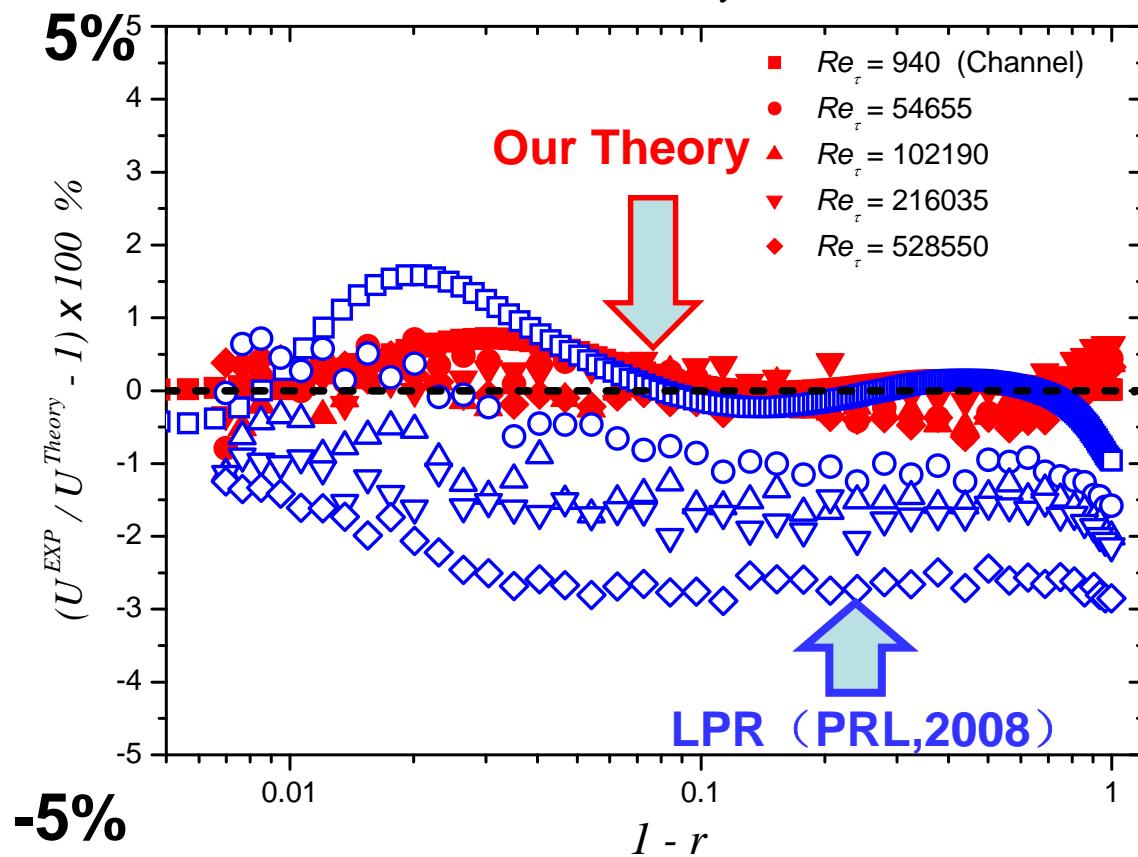
$$\kappa = 0.45$$

MVP:

Relative errors at all measured points:

Reynolds number range:

$$Re_\tau : 600 - 528550$$



## 5. Predictions for pipe (Princeton data):



### Centerline velocity

$$U_c^{+Pipe} \approx \frac{20}{9} \ln(\text{Re}_\tau) + 8.36$$

$$U_c^{+CH} \approx \frac{20}{9} \ln(\text{Re}_\tau) + 7.24$$

### Average velocity

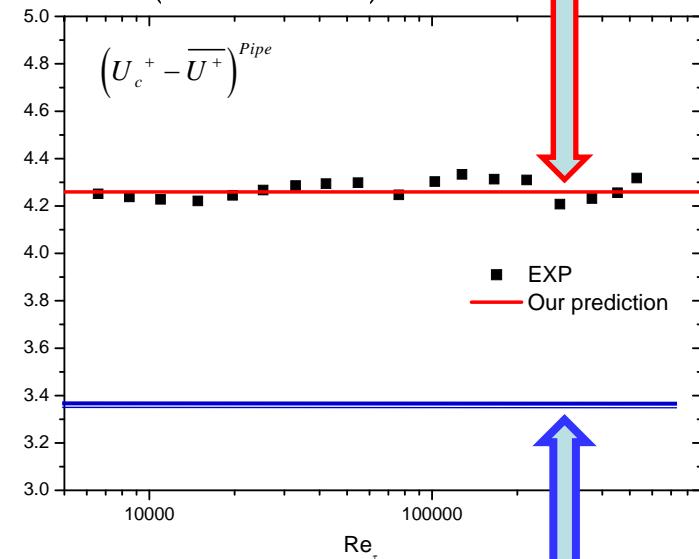
$$\bar{U}^{Pipe} \approx \frac{20}{9} \ln(\text{Re}_\tau) + 4.10$$

$$\bar{U}^{CH} \approx \frac{20}{9} \ln(\text{Re}_\tau) + 4.76$$

### Princeton pipe data

Our theory:

$$(U_c^+ - \bar{U}^+)^{Pipe} \approx 4.26$$



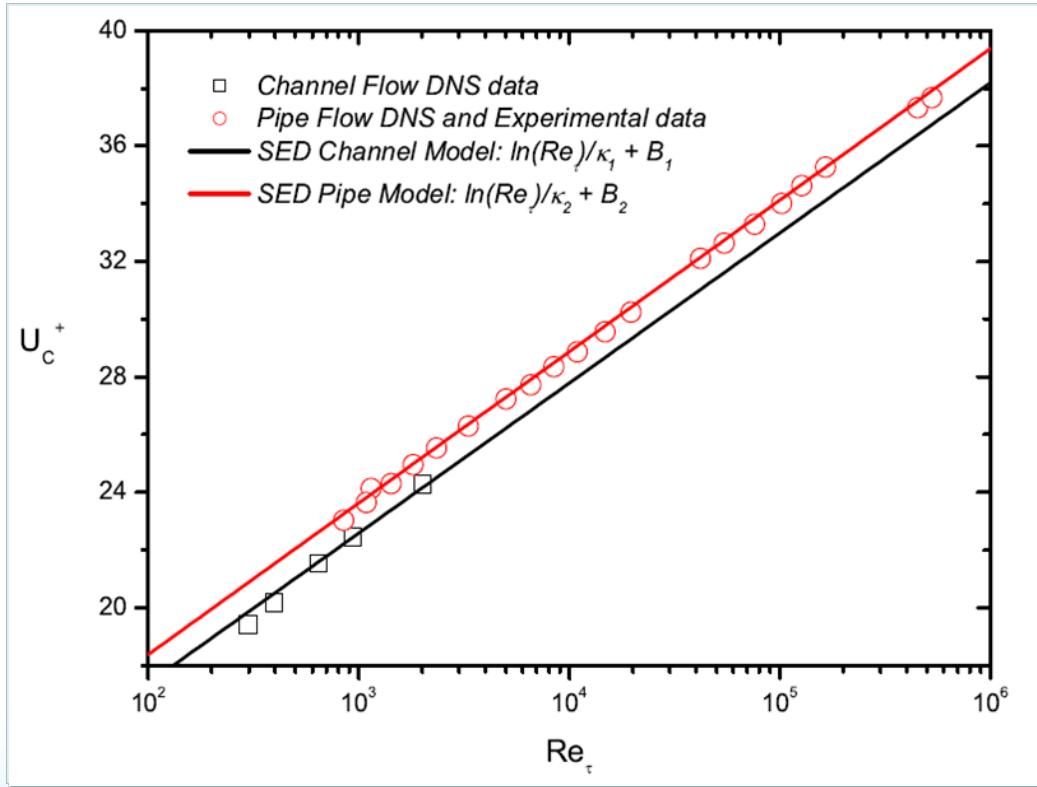
textbook (Pope, 2000)

$$(U_c^+ - \bar{U}^+)^{Pipe} \approx \frac{3}{2\kappa} \approx 3.33$$

## 5. Predictions for pipe (Princeton data):



At  
Same  
Re!



- Centerline velocity difference between CH and Pipe

$$U_{center-pipe} - U_{center-channel} \approx \frac{1}{K} \left( \int_{r_{core}}^1 \frac{5\sqrt{r}}{1-r^5} dr - \int_{r_{core}}^1 \frac{4\sqrt{r}}{1-r^4} dr + \frac{1}{2} r_{core}^{\frac{3}{2}} \right) \approx 1.1$$

## 5. Predictions for pipe (Princeton data):



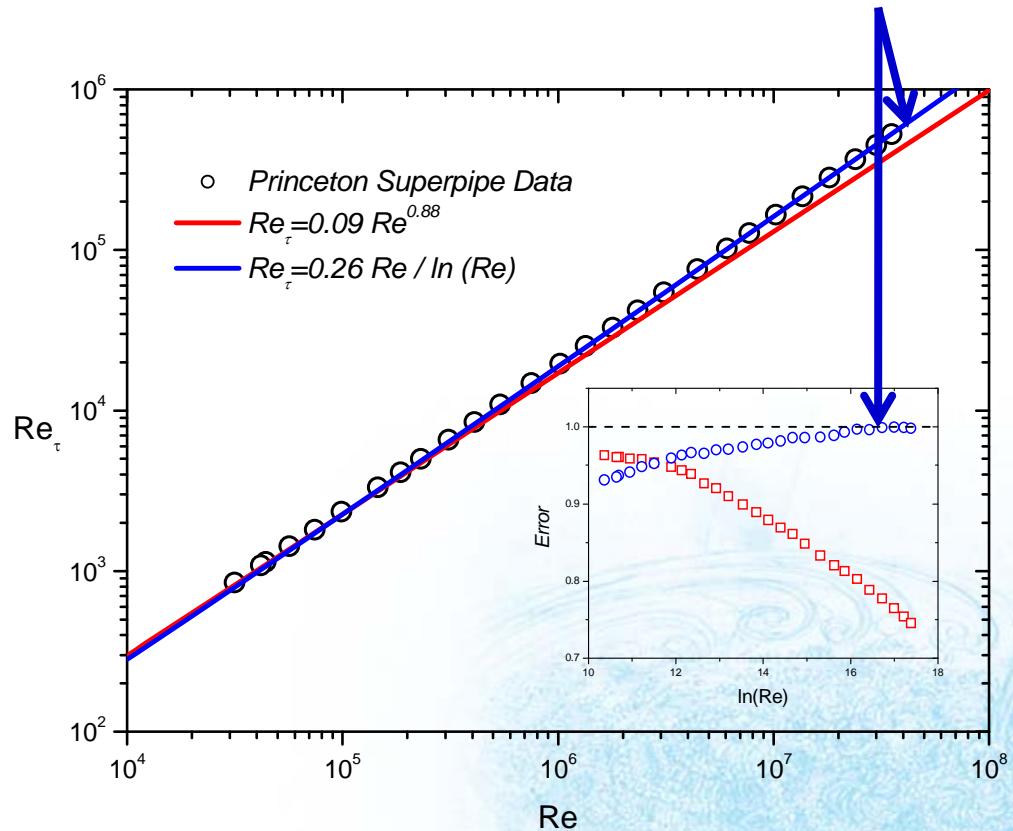
### Friction Reynolds number vs Bulk Reynolds number

textbook (Pope, 2000)

$$\text{Old: } Re_\tau \approx 0.09 Re^{0.88}$$

$$\text{New: } Re_\tau \approx 0.26 Re / \ln Re$$

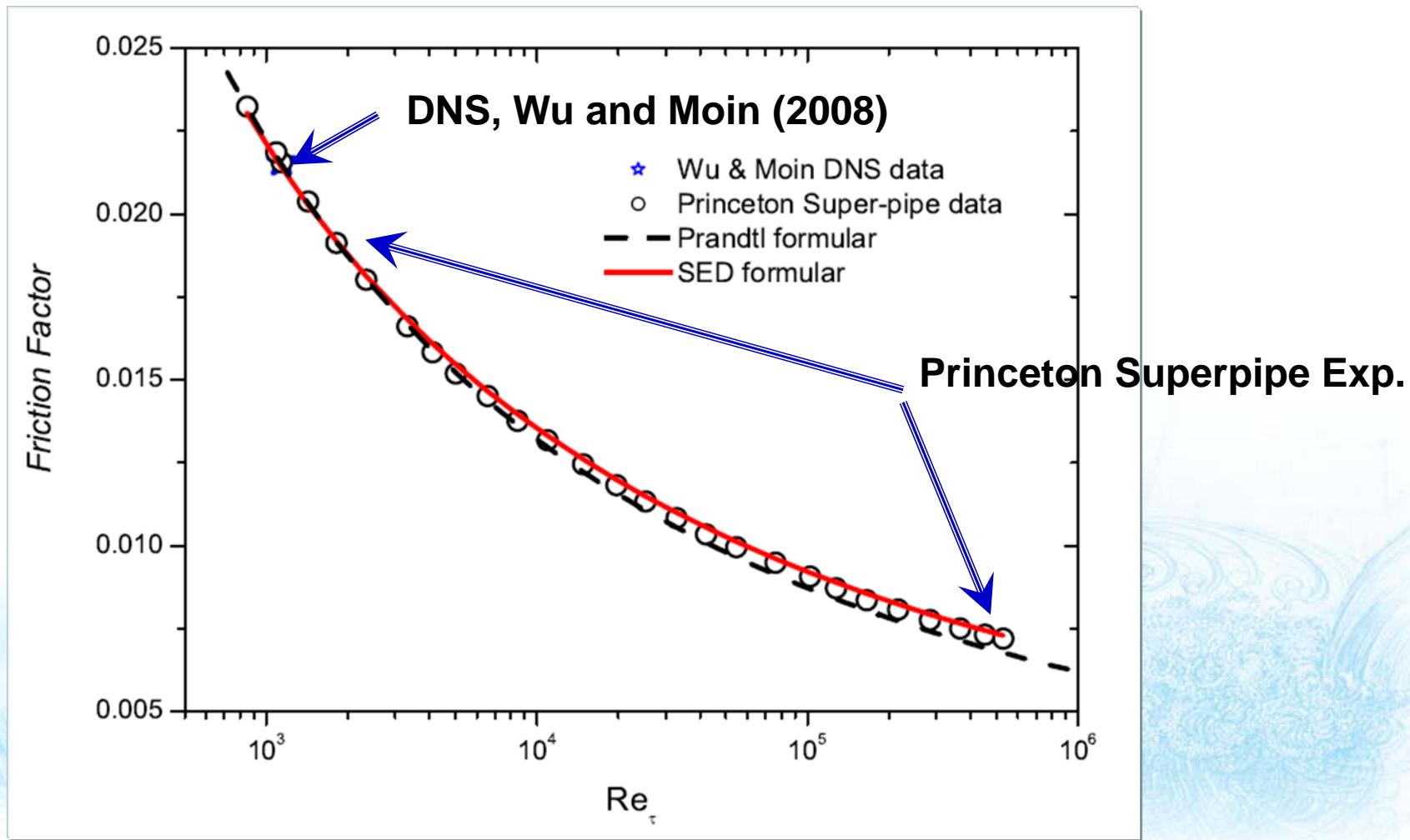
25% error corrected!



## 5. Predictions for pipe (Princeton data):



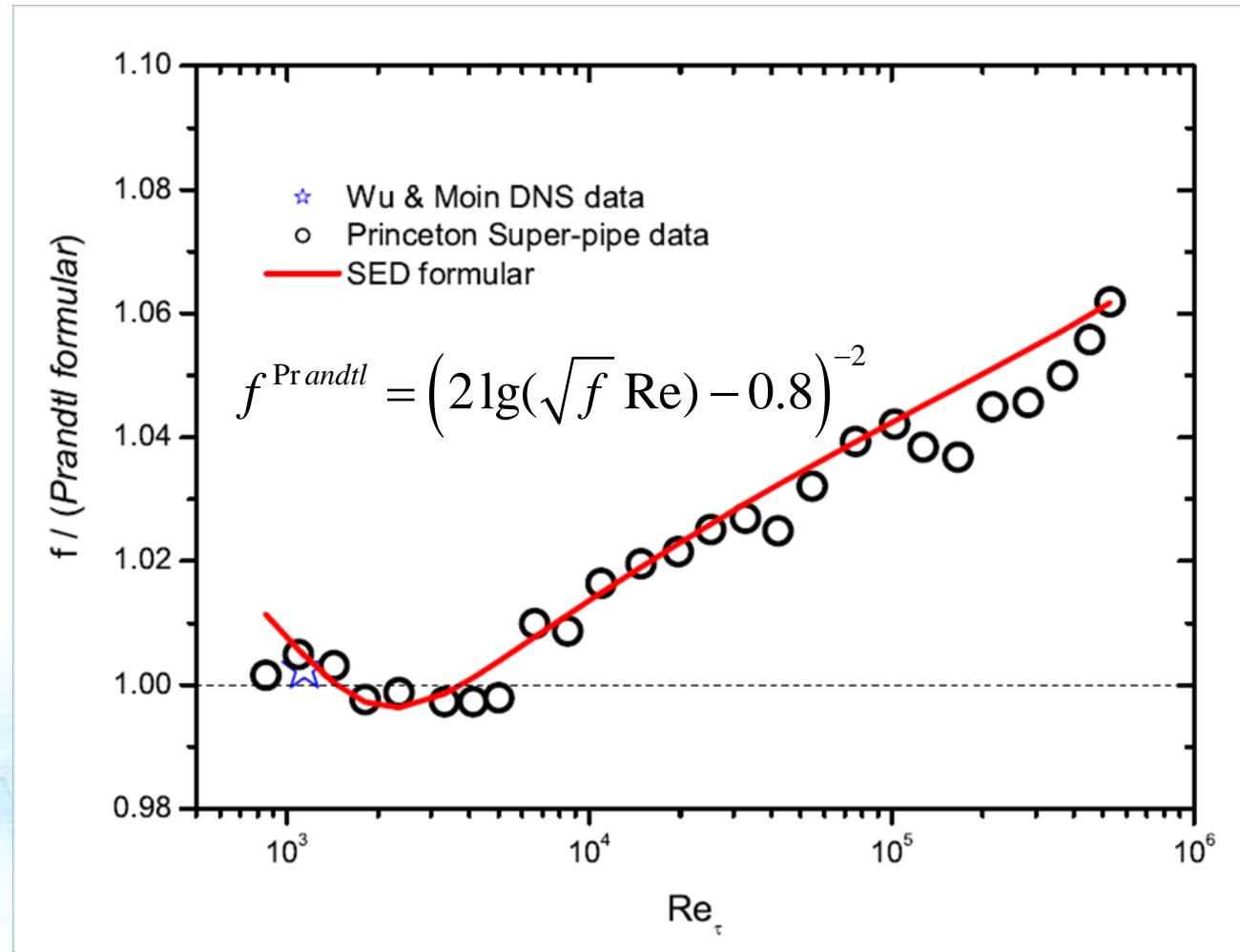
### • Friction coefficient for pipe



## 5. Predictions for pipe (Princeton data):



- Friction coefficient for pipe (compensated plot)



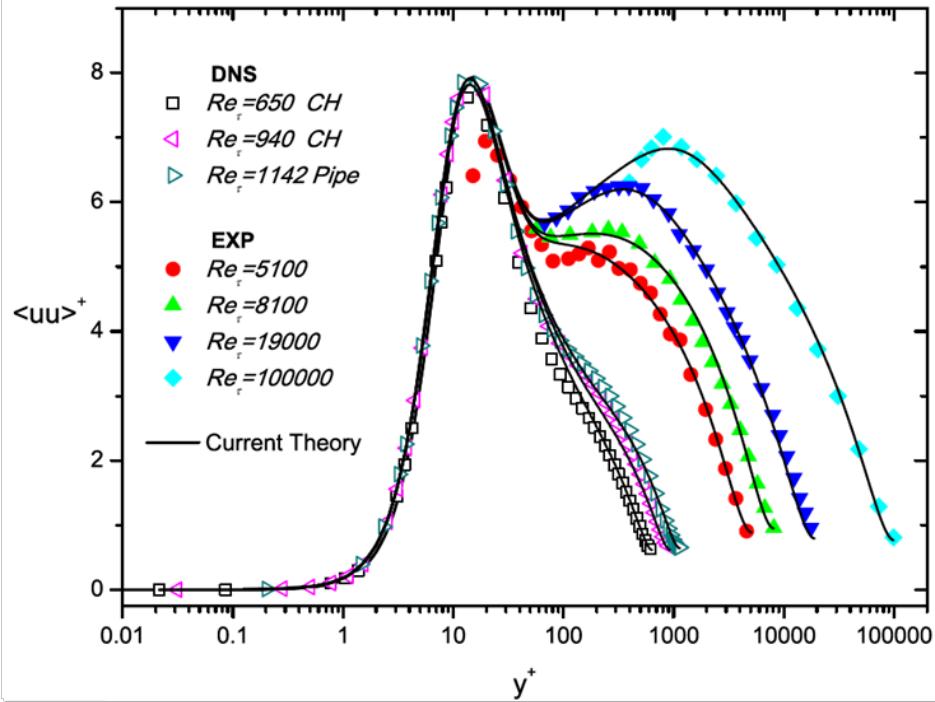
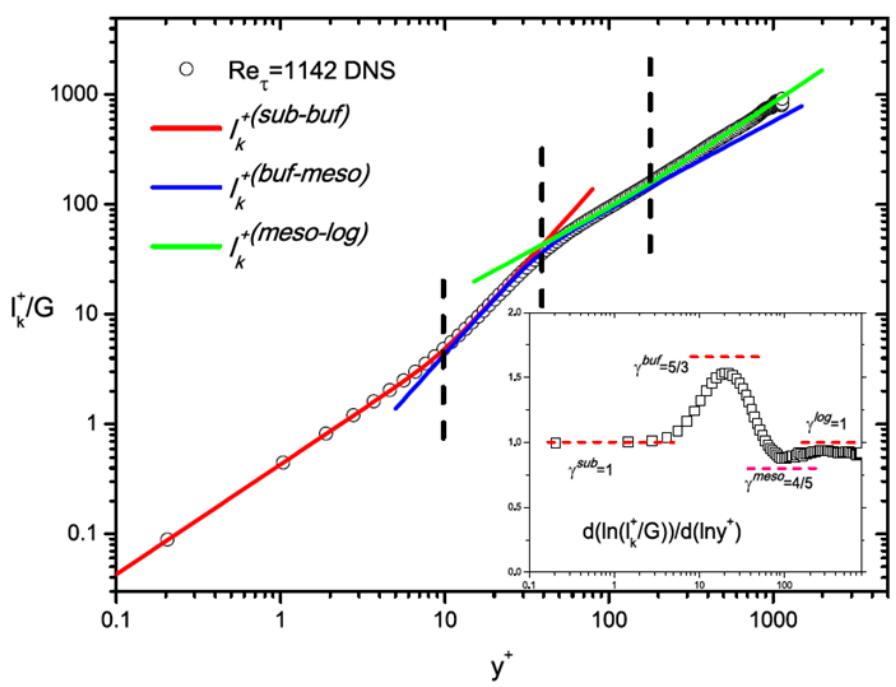
We present a multi-layer BL theory for MVP:

- Similarity solution for mixing length (order function).
- The physical constants are determined, rather fitted.
- The framework can be extended to other mean-field quantities, such as kinetic energy; and to other flows, such as rough pipe, compressible channel, incompressible and compressible TBL, Rayleigh-Benard convection, etc, because multi-layer structure and Lie-group dilation invariance are general.

- Streamwise turbulent kinetic energy**

$$\ell_k^+ = \sqrt{\langle uu \rangle^+} / S^+$$

$$\langle uu \rangle^+ = \left( \ell_k^+ (-1 + \sqrt{4r\ell_M^{+2} + 1}) / (2\ell_M^{+2}) \right)^2$$



*Thanks for comments!*



*Congratulations to Vriel !*