The tyger phenomenon for the Galerkin-truncated Burgers and Euler equations

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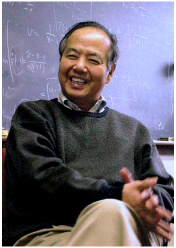
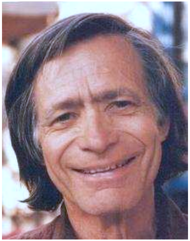
Outline

- Introduction : Statistical Mechanics and Turbulence
- Galerkin Truncation
- The Tyger Phenomenon : 1D Burgers Equation
- The Birth of Tygers : 1D Burgers Equation
- Conclusions and Perspective

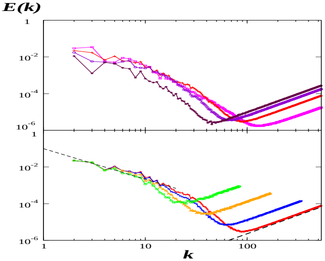
Equilibrium Statistical Mechanics and Turbulence

- Equilibrium statistical mechanics is concerned with conservative Hamiltonian dynamics, Gibbs states, ...
- Turbulence is about dissipative out-of-equilibrium systems.
- In 1952 Hopf and Lee apply equilibrium statistical mechanics to the 3D Euler equation and obtain the equipartition energy spectrum which is very different from the Kolmogorov spectrum.
- In 1967 Kraichnan uses equilibrium statistical mechanics as one of the tools to predict the existence of an inverse energy cascade in 2D turbulence.

Equilibrium Statistical Mechanics and Turbulence

- In 1989 Kraichnan remarks *the truncated Euler system can imitate NS fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wave-number modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared with the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity.*
- In 2005 Cichowlas, Bonaiti, Debbasch, and Brachet discovered long-lasting, partially thermalized, transients similar to high-Reynolds number flow.



The Galerkin-truncated 1D Burgers equation

- The (untruncated) inviscid Burgers equation, written in conservation form, is

$$\partial_t u + \partial_x(u^2/2) = 0; \qquad u(x,0) = u_0(x).$$

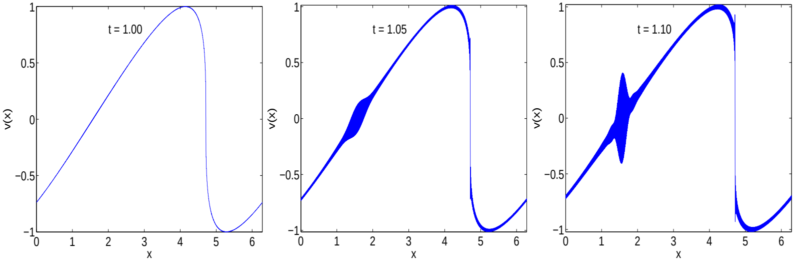
- Let K_G be a positive integer, here called the Galerkin truncation wavenumber, such that the action of the projector $\mathrm{P}_{K_\mathrm{G}}$:

$$\mathrm{P}_{K_\mathrm{G}} u(x) = \sum_{|k| \le K_\mathrm{G}} \mathrm{e}^{\mathrm{i}kx} \hat{u}_k .$$

- The associated Galerkin-truncated (inviscid) Burgers equation

$$\partial_t v + \mathrm{P}_{K_\mathrm{G}} \partial_x(v^2/2) = 0; \qquad v_0 = \mathrm{P}_{K_\mathrm{G}} u_0 .$$

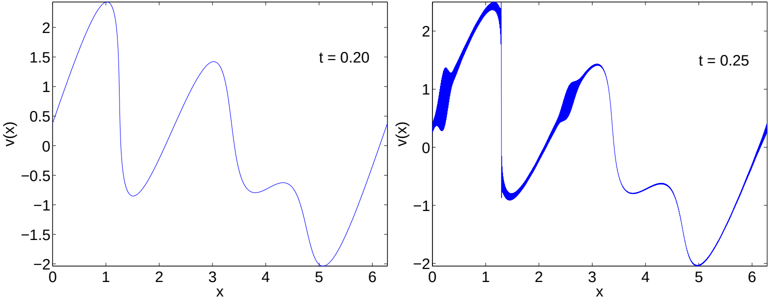
Tygers in the Galerkin-truncated 1D Burgers equation



Growth of a tyger in the solution of the inviscid Burgers equation with initial condition $v_0(x) = \sin(x - \pi/2)$. Galerkin truncation at $K_{\mathrm{G}} = 700$. Number of collocation points $N = 16,384$. Observe that the bulge appears far from the place of birth of the shock.

Tygers only at regions of positive strain

$$u_0(x) = \sin(x) + \sin(2x + 0.9) + \sin(3x)$$

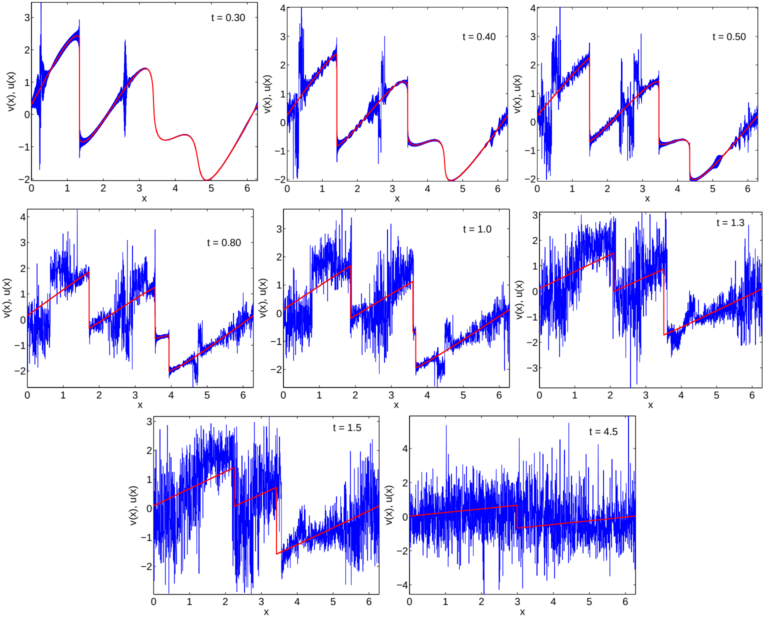


Three-mode initial condition. Tygers appear at the points having the same velocity as the shock and positive strain.

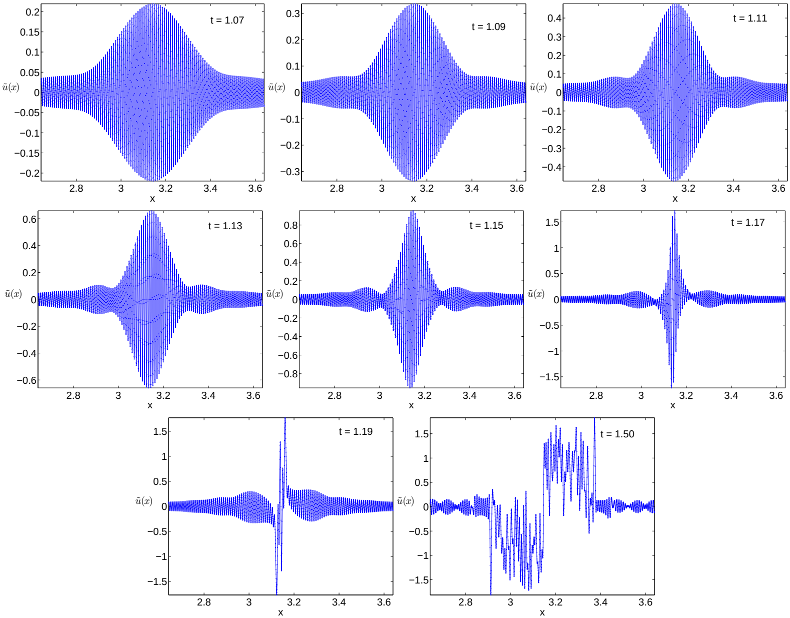
Phenomenological Explanation

- A localized strong nonlinearity, such as is present at a preshock or a shock, acts as a source of a *truncation wave*.
- Away from the source this *truncation wave* is mostly a plane wave with wavenumber close to K_{G}.
- The radiation of truncation waves begins only at or close to the time of formation of a preshock.
- Resonant interactions are confined to particles such that $\tau \Delta v \equiv \tau |v - v_{\mathrm{s}}| \lesssim \lambda_{\mathrm{G}}$.
- If τ is small the region of resonance will be confined to a small neighborhood of widths $\sim {K_{\mathrm{G}}}^{-1/3}$ around the point of resonance.
- In a region of negative strain a wave of wavenumber close to K_{G} will be squeezed and thus disappearing beyond the *truncation horizon* which acts as a kind of black hole.

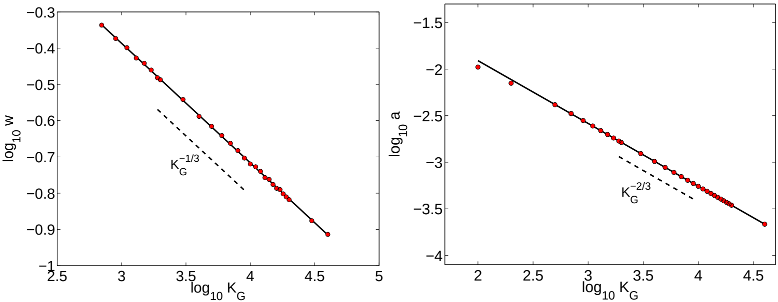
From tygers to thermalization



From tygers to thermalization



Scaling properties of the early tygers

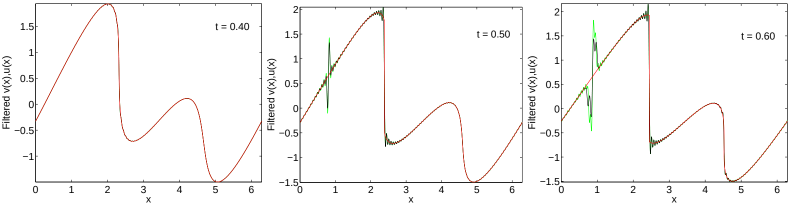


$$
\begin{aligned}
\text{width} \quad &\propto \quad K_{\mathrm{G}}{}^{-1/3} \qquad \text{(using phase mixing arguments)} \\
\text{amplitude} \quad &\propto \quad K_{\mathrm{G}}{}^{-2/3} \qquad \text{(using energy conservation arguments)}
\end{aligned}
$$

Scaling properties of the early tygers

- Scaling of the tyger widths :
  - By the time $t_\star$, truncation is significant only for a lapse of time $O({K_{\rm G}}^{-2/3})$.
  - The phase mixing argument tells us that the coherent build up of a tyger will affect only those locations whose velocity differs from that at resonance by an amount $\Delta v \lesssim \frac{2\pi}{{K_{\rm G}}^{-2/3} K_{\rm G}} \propto {K_{\rm G}}^{-1/3}$.
  - Since at such times, the velocity v of the truncated solution is expected to stay close to the velocity u of the untruncated solution and the latter varies linearly with x near the resonance point, the width of the $t_\star$ tyger is itself proportional to ${K_{\rm G}}^{-1/3}$.
- Scaling of the tyger amplitudes :
  - The Galerkin-truncated Burgers equation conserves energy.
  - The apparent energy loss due to truncation $\sim \int_0^{\lambda_{\rm G}} x^{2/3} dx \sim {K_{\rm G}}^{-5/3}$.
  - Conservation demands that this energy-loss is transferred to the tygers which gives the tyger-amplitude scaling as $\propto {K_{\rm G}}^{-2/3}$.
  - The above argument is appealing but not rigorous.

Weak solutions?



Plots of solution of the Galerkin-truncated Burgers equation, with $K_G = 5,461$ (green) and $K_G = 21,845$ (black), low-pass filtered at wavenumber $K = 100$, at various times. Initial condition $v_0(x) = \sin(x) + \sin(2x - 0.741)$. The untruncated solution is shown in red.

Birth of tygers : Systematic theory

- Define discrepancy $\tilde{u} \equiv v - u$ to obtain

$$\partial_t \tilde{u} + \mathrm{P}_{\kappa_{\mathrm{G}}} \partial_x \left(u\tilde{u} + \frac{\tilde{u}^2}{2} \right) = \left(\mathrm{I} - \mathrm{P}_{\kappa_{\mathrm{G}}} \right) \partial_x \frac{u^2}{2}, \quad \tilde{u}(0) = 0.$$

- Decompose $u = u^< + u^>$, where $u^< \equiv \mathrm{P}_{\kappa_{\mathrm{G}}} u$ and $u^> \equiv (\mathrm{I} - \mathrm{P}_{\kappa_{\mathrm{G}}})u$.
- Similarly the perturbation $u' \equiv \mathrm{P}_{\kappa_{\mathrm{G}}} \tilde{u}$.
- Finally we obtain :

$$\partial_t u' + \mathrm{P}_{\kappa_{\mathrm{G}}} \partial_x \left(u^< u' + \frac{(u')^2}{2} \right) = \mathrm{P}_{\kappa_{\mathrm{G}}} \partial_x \left(u^< u^> + \frac{(u^>)^2}{2} \right).$$

Birth of tygers : Systematic theory

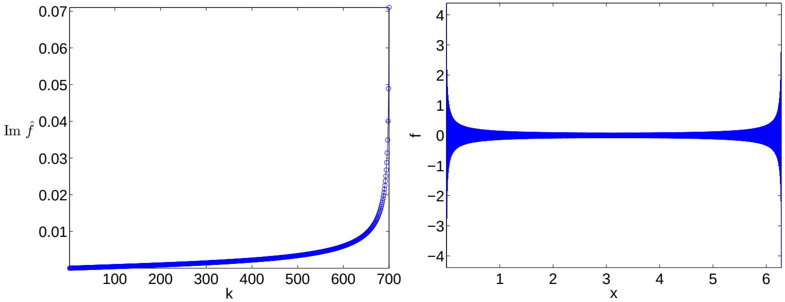
- Strategy :
  1. The term $(u')^2$ is discarded;
  2. The perturbation u' is set to zero at time t_{G};
  3. The untruncated solution is frozen to its $t_\star$ value.
- With the three approximations the temporal dynamics of the perturbation near $t_\star$ is

$$\frac{d}{d\tau}\hat{u}'_k = \sum_{k'=-K_{\mathrm{G}}}^{K_{\mathrm{G}}} A_{kk'}\,\hat{u}'_{k'} + \hat{f}_k\,, \qquad \hat{u}'_k(0) = 0,$$

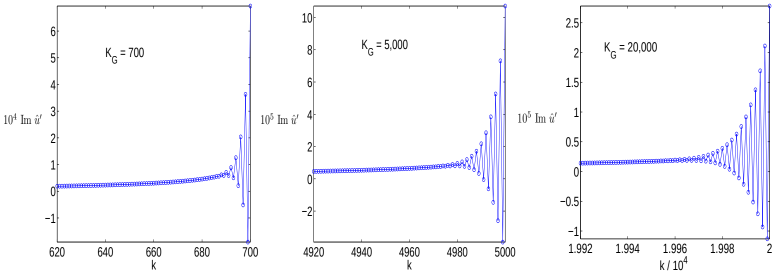
$$A_{kk'} \equiv -\mathrm{i}k\,\hat{u}^{<}_{\star,\,k-k'}\,,$$

$$\hat{f}_k \equiv \mathrm{i}k \sum_{p+q=k} (\hat{u}^{<}_{\star p}\,\hat{u}^{>}_{\star q} + \frac{1}{2}\hat{u}^{>}_{\star p}\,\hat{u}^{>}_{\star q}).$$

The beating input

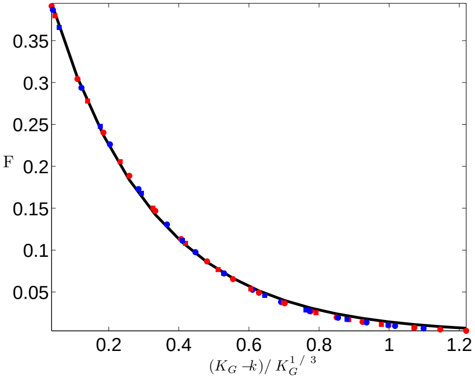


Fourier space solution of the perturbation



The boundary layer in Fourier space near K_{G}. Shown are the imaginary parts of $\hat{u}'(t_\star)$ for three values of K_{G}. The origin is at the preshock. The even-odd oscillations indicate that most of the activity is at the tyger, a distance π away.

Scaling function for the boundary layer



The envelopes of the various boundary layers shown in earlier (with preshock contributions subtracted out), collapsed into a single curve after rescaling. Red circles: $K_{\mathrm{G}} = 20,000$, blue circles: $K_{\mathrm{G}} = 15,000$, red squares: $K_{\mathrm{G}} = 10,000$, blue squares: $K_{\mathrm{G}} = 5,000$. The thick black line is the exponential fit.

Conclusions and Perspective

- Tygers provide a clue as to the onset of thermalization.
- We do not have a complete understanding of the phenomenon.
- Tygers do not modify shock dynamics but modify the flow elsewhere because the tygers induce Reynolds stresses on scales much larger than the Galerkin wavelength; hence the weak limit of the Galerkin-truncated solution as $K_{\mathrm{G}} \to \infty$ is NOT the inviscid limit of the untruncated solution.
- There is good evidence that the key phenomena associated to tygers are also present in the two-dimensional incompressible Euler equation (and also perhaps in three dimensions).
- It is clear that complex-space singularities approaching the real domain within one Galerkin wavelength are the triggering factor in both the 2D Euler and the 1D Burgers case.
- Can we "purge tygers away" and thereby obtain a subgrid-scale method which describes the inviscid-limit solution right down to the Galerkin wavelength?

Thank you, Uriel!!

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful symmetry?

William Blake, from *The Tyger*