Population dynamics in compressible flows

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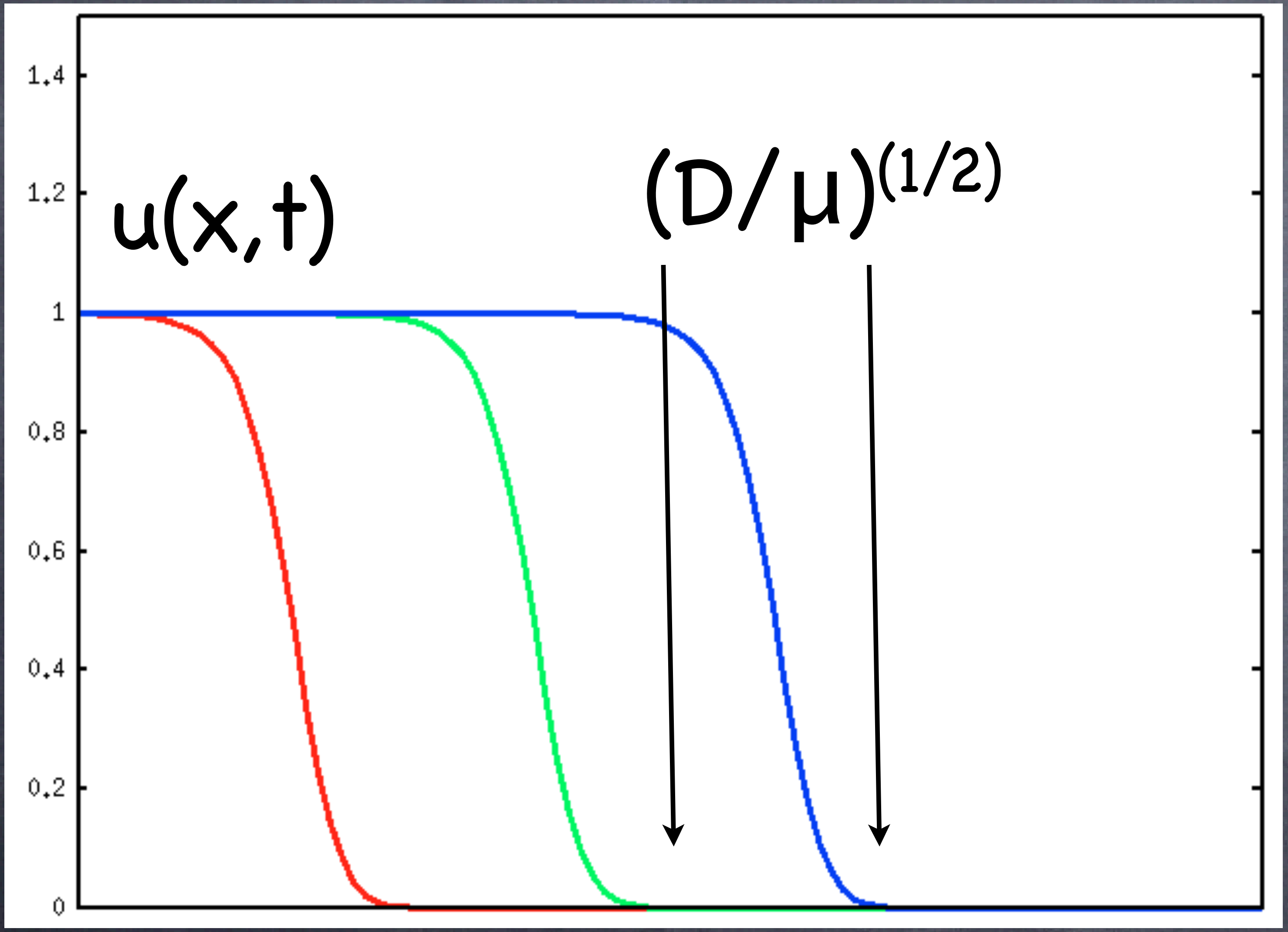
Effect of compressible turbulence in biology

Spreading of bacteria with uniform growing rate μ

Fisher equation $$\partial_t C = D\Delta C + \mu C(1-C)$$



expanding front

$v_f = (D\mu)^{(1/2)}$

Mutation with fixation occur with increasing probability in the tail of the Fisher wave

(Hallatsheck&Nelson 2007)

What happen if we add a turbulent flow?

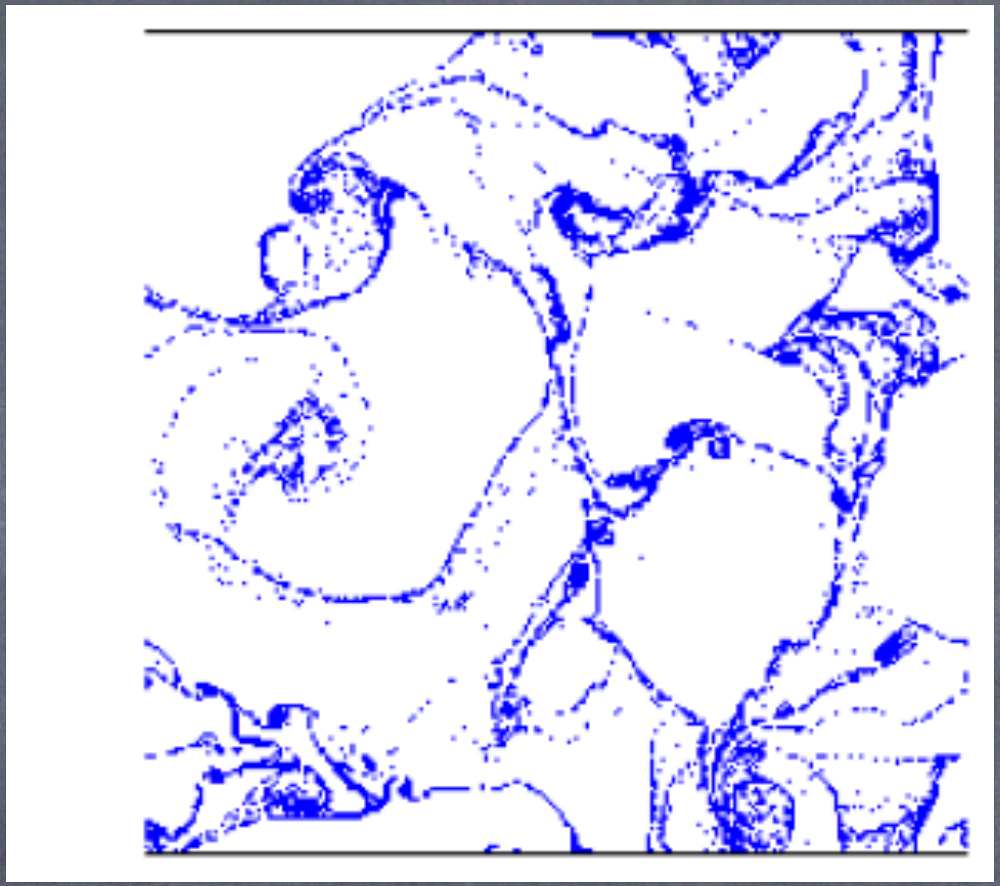
$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}C) = D\nabla^2 C + \mu C(1 - C),$$

if the flow is incompressible (div(v)=0), turbulent increases diffusivity (Richardson).

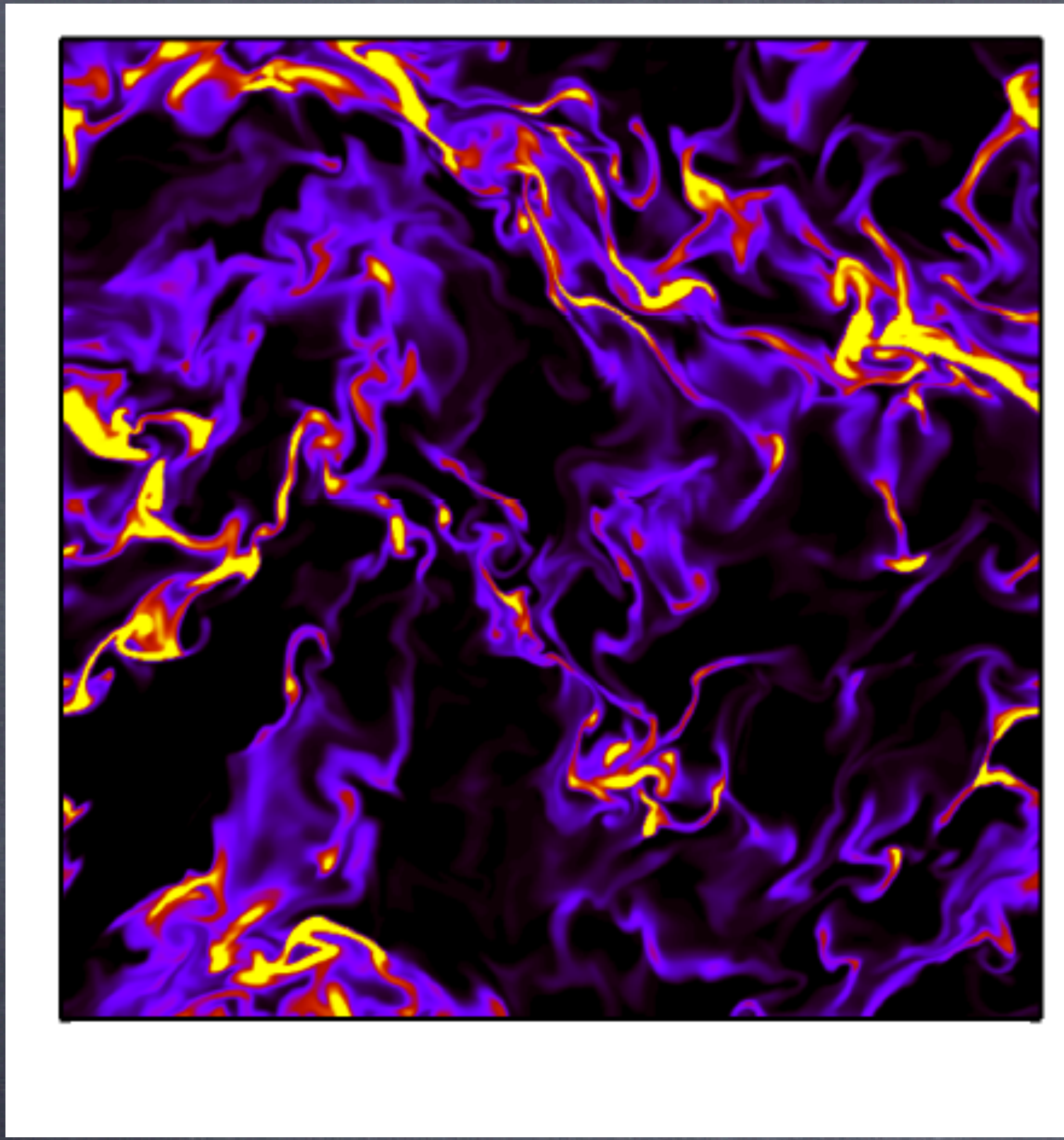
Something new happen when the "bacteria" are constrained to live on a two dimensional surface (i.e. because of buoyancy) or their density is different from the fluid density (inertial particles)

div(v)≠0

If lagrangian particles particles are constrained on a two dimensional surface then the flow is compressible.

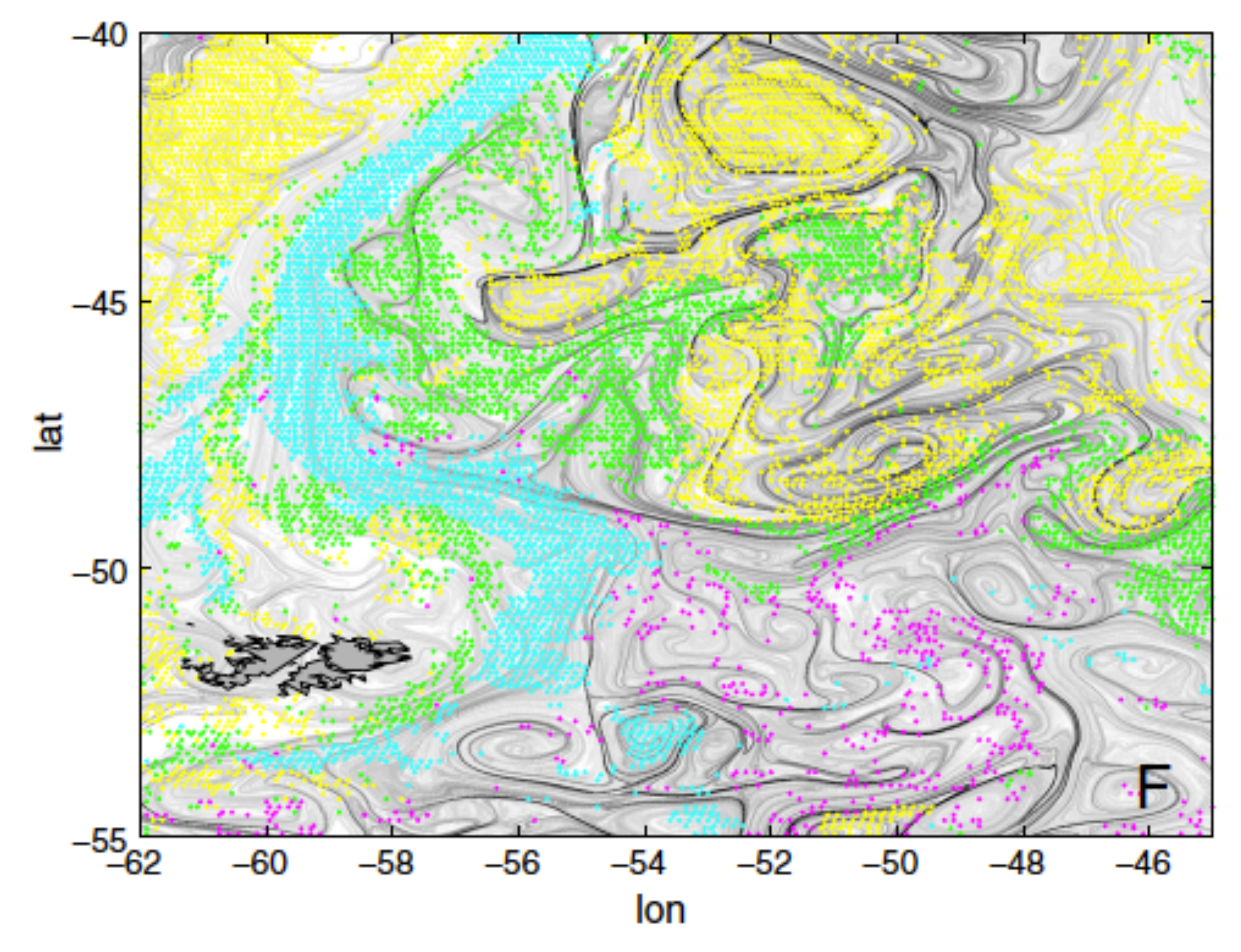


Boffetta et.al. 2005



Perlaker, RB, Nelson, Toschi, PRL 2010

see the movie



D'ovidio, Del Monte, Alvain, Dandonneau, Levy, 2010

A simple one dimensional case: $u(x) = -\Gamma(x - x_0)$, $\mu=0$

$$\frac{dx}{dt} = -\Gamma(x - x_0) + \sqrt{2D}\eta(t)$$

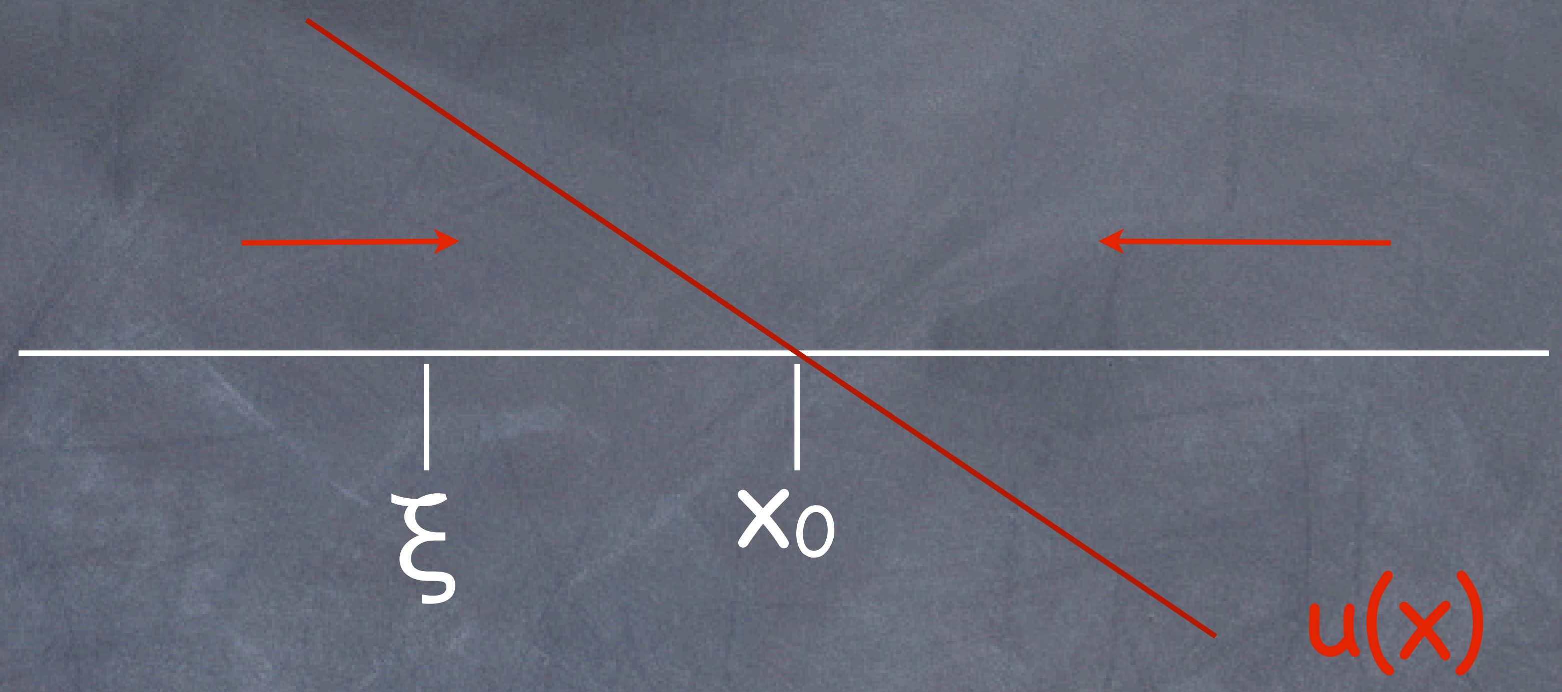
Two length scales: ξ_t and ξ_l

$$\xi_t = \frac{v_*}{\Gamma}$$ Turbulence correlation length

$$\xi_l = \sqrt{\frac{D}{\Gamma}}$$ Localization length

$$\sqrt{\frac{D}{\Gamma}} \leq \frac{v_*}{\Gamma} \rightarrow \frac{v_*^2}{D\Gamma} > 1$$

The case with $\mu>0$.



expanding front velocity = compressible flow

$v_f=(D\mu)^{(1/2)} = \Gamma \xi$

$\rightarrow \xi = (D\mu)^{(1/2)}/ \Gamma$

the effect of compressibility is relevant if $\xi < (D/\mu)^{1/2}$

$\rightarrow \Gamma > \mu$

A more general estimate

$$\frac{\partial P}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} P) = D \nabla^2 P$$

$$\frac{1}{2}\langle P^2(\nabla \cdot \mathbf{u})\rangle = -D\langle(\nabla P)^2\rangle,$$

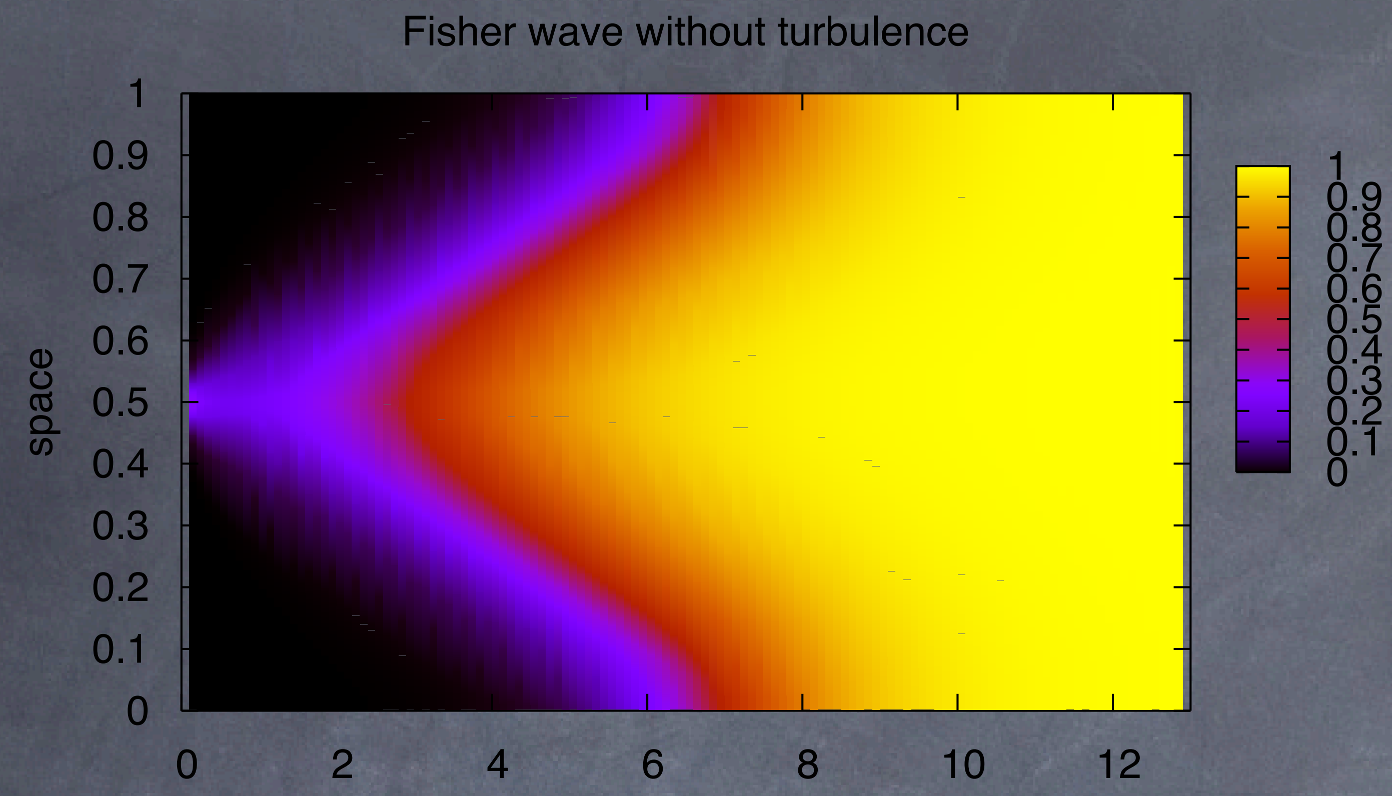
$$\kappa \equiv \langle(\nabla \cdot \mathbf{u})^2\rangle / \langle(\nabla \mathbf{u})^2\rangle,$$

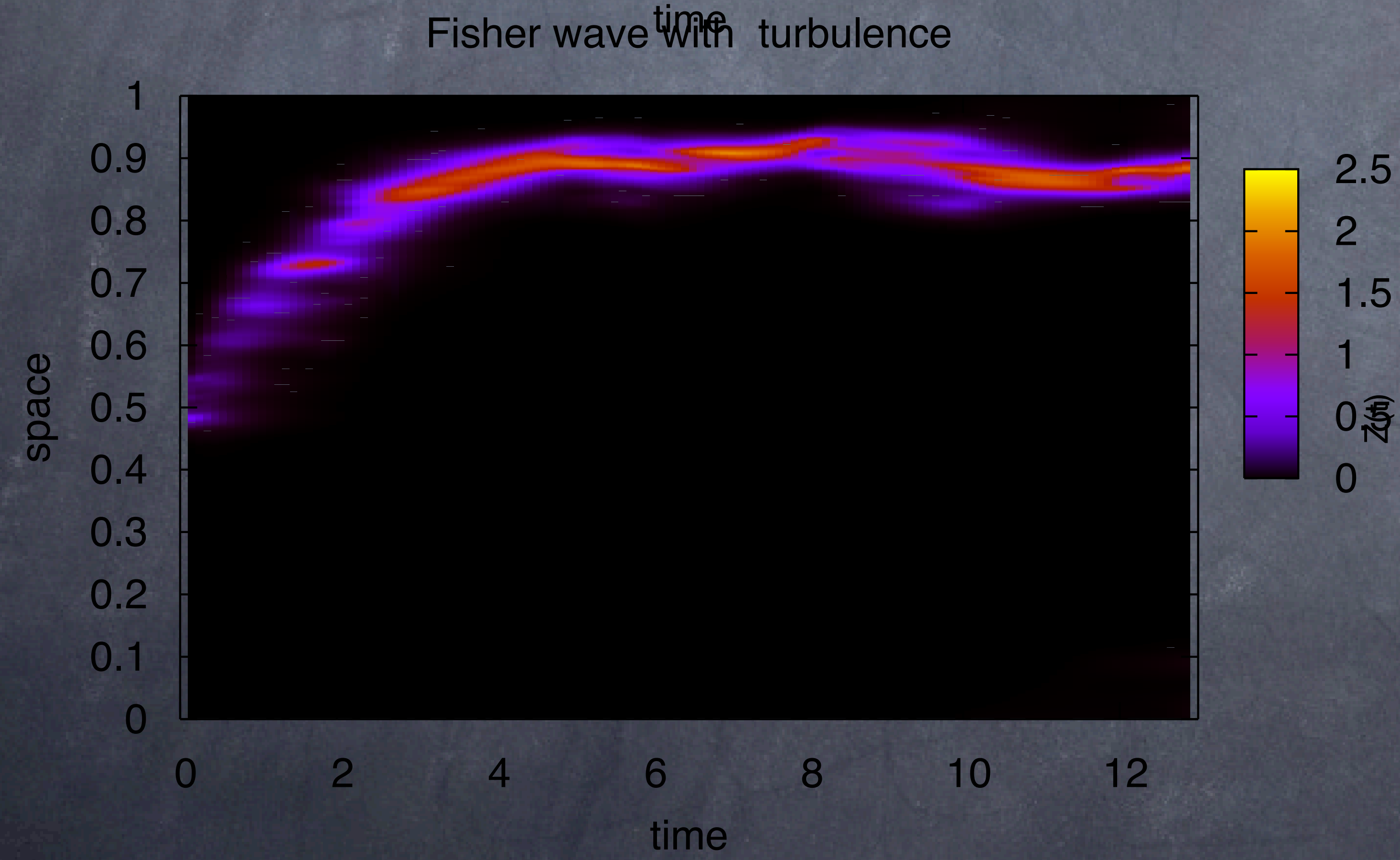
$$\langle P^2(\nabla \cdot \mathbf{u})\rangle = -A_1 \langle P^2\rangle \langle(\nabla \cdot \mathbf{u})^2\rangle^{1/2}$$

$$\langle(\nabla P)^2\rangle = A_2 \frac{\langle P^2\rangle}{\xi^2}$$

$$\langle(\nabla \mathbf{u})^2\rangle = \epsilon/\nu$$

$$\xi^2 = \frac{2 A_2 D \sqrt{\nu}}{A_1 \sqrt{\kappa \epsilon}}.$$

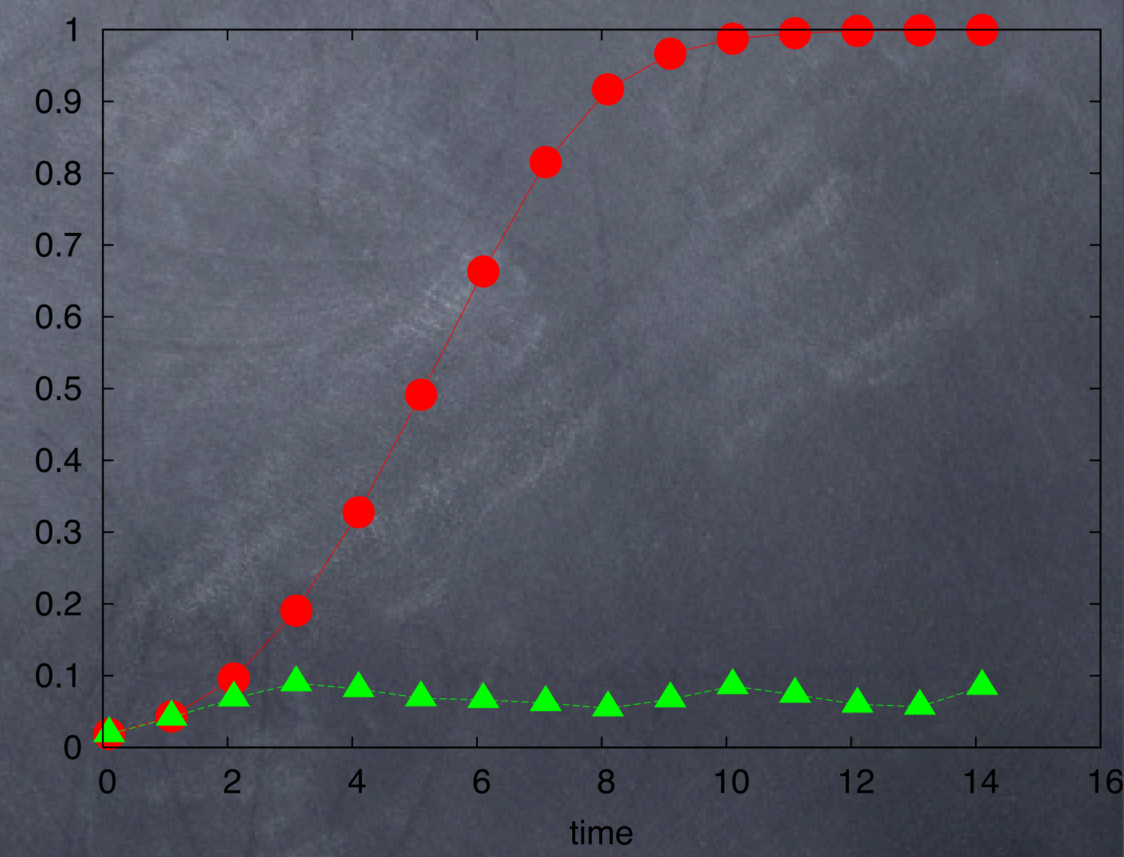




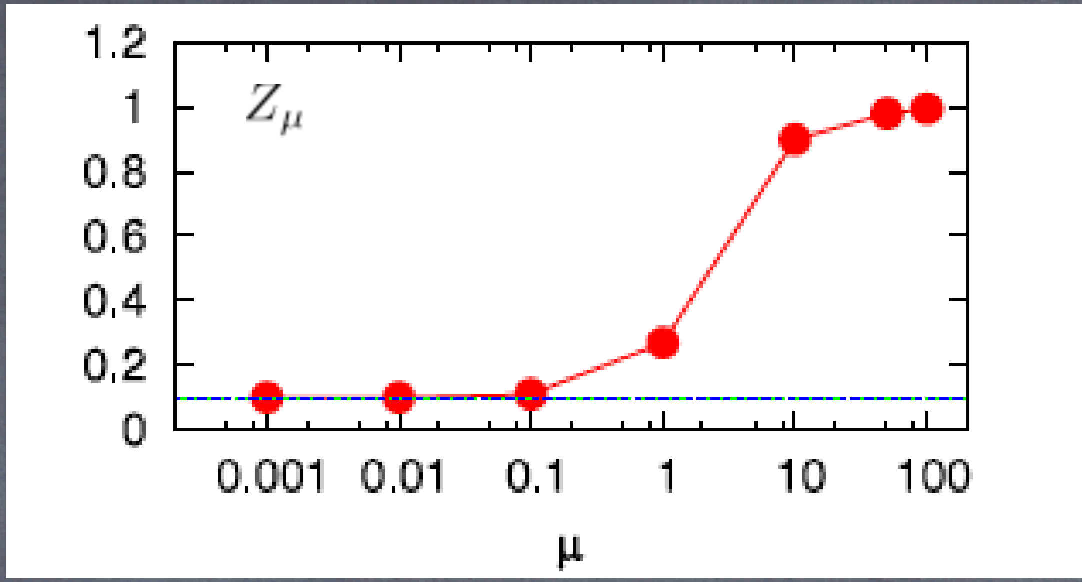
The one dimensional case with turbulence (from shell'model)

$$Z \equiv \int dx c(x,t)$$

carrying capacity



RB, Nelson 2009



The limit $\mu \to 0$

$$\frac{1}{\int dx c_0(x,t)^2}$$

Let $c_0(x,t)$ be the solution for $\mu = 0$

We assume that in the limit $\mu \to 0$ $c_\mu(x,t) = Z_\mu c_0(x,t)$

$$\int c_0(x,t)dx = 1 \to \int c_\mu dx = Z_\mu \int c_0 dx = Z_\mu$$

$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} C) = D\nabla^2 C + \mu C(1-C),$$

$\langle c \rangle - \langle c^2 \rangle = 0$

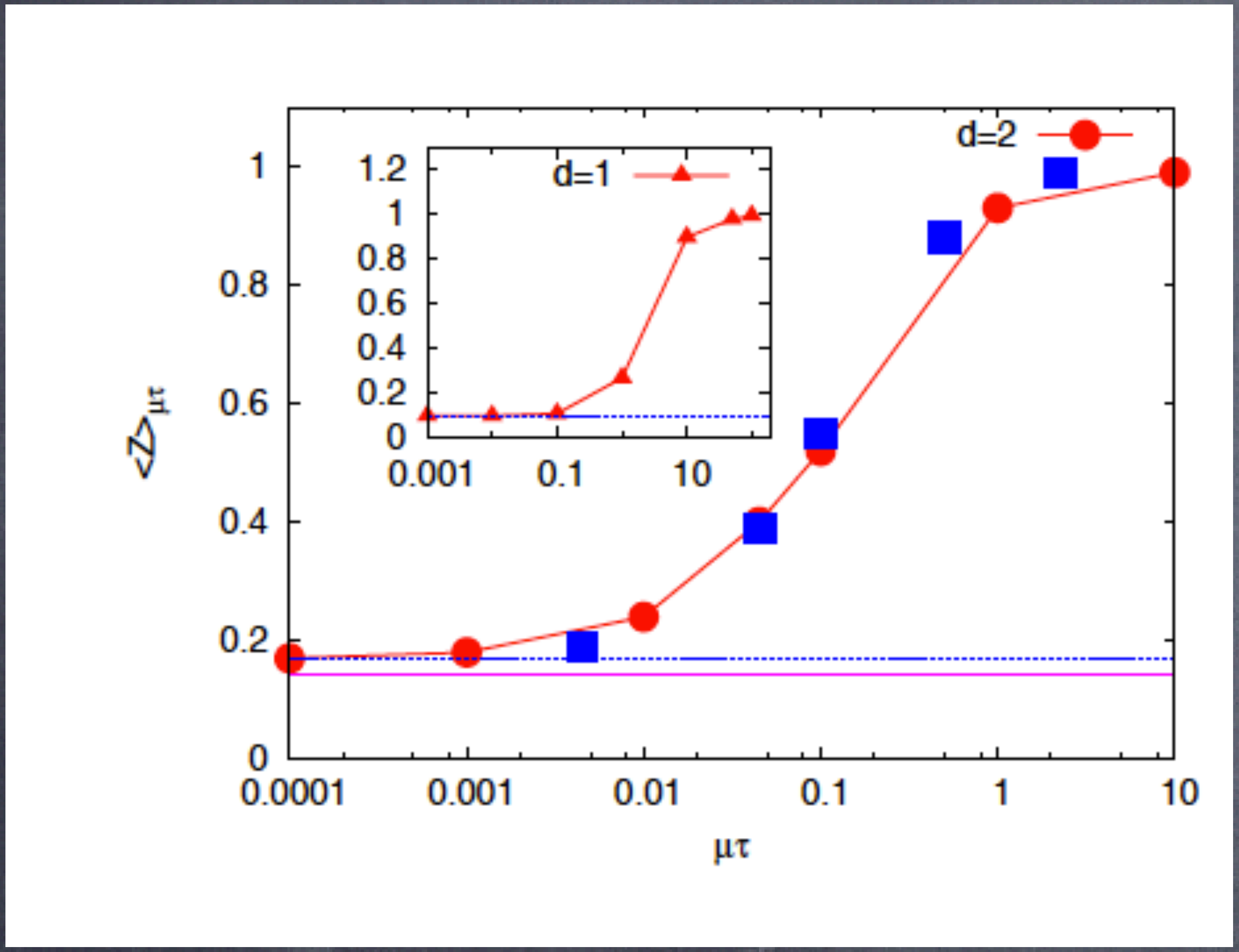
$$Z_\mu - Z_\mu^2 \int dx c_0^2(x,t) = 0 \to Z_\mu = \frac{1}{\int dx c_0(x,t)^2}$$

This is a generic result which can be generalized in more than one dimension.

Two dimensional case with surface flows

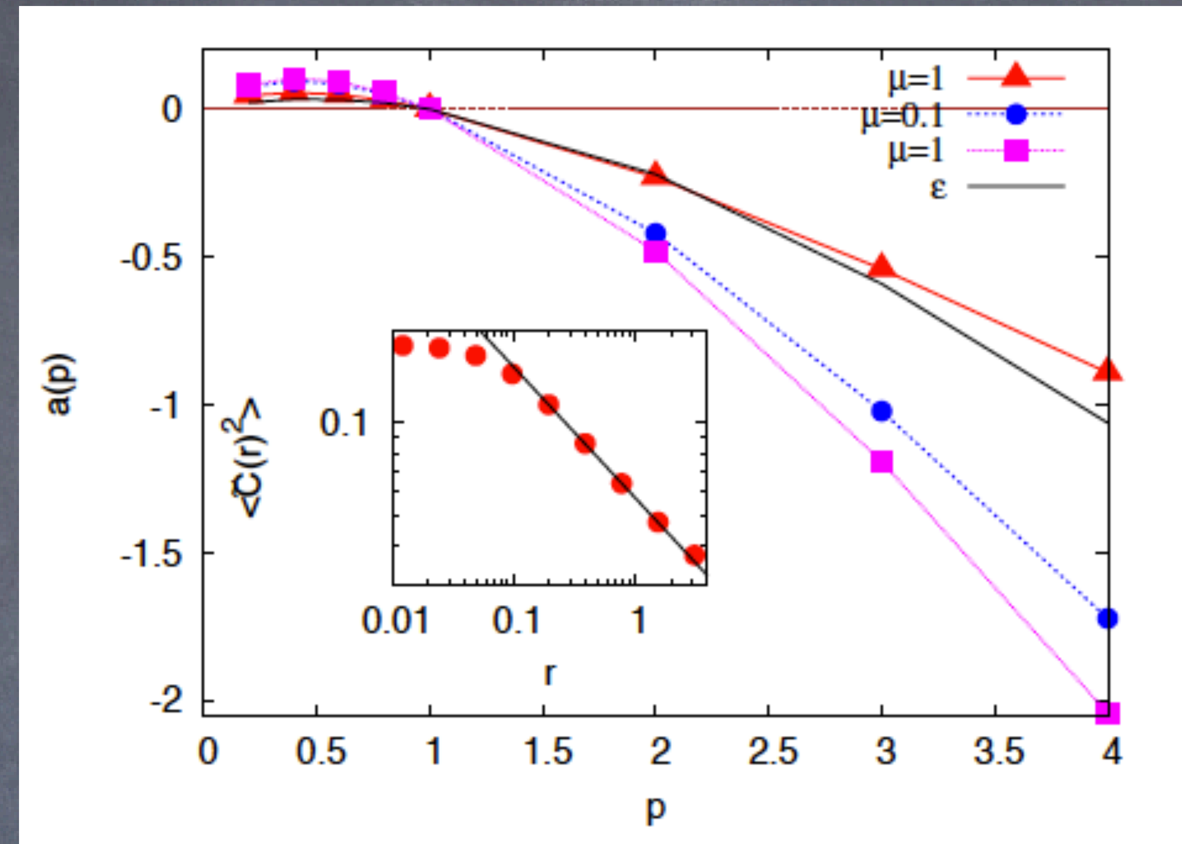
P.Perlaker, RB, D.R. Nelson,F. Toschi PRL 2010

$$Z(t) = \frac{1}{L^2} \int dxdyC(\boldsymbol{x};t),$$



Theoretical limit

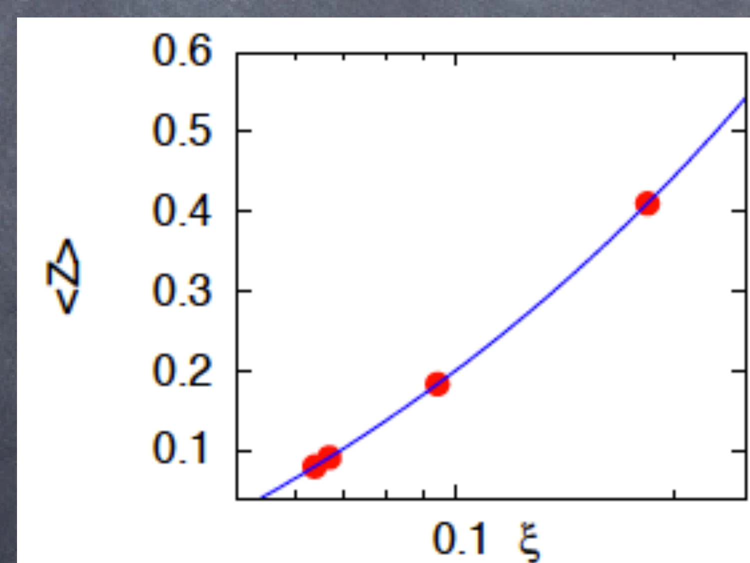
Multifractal analysis



$$\tilde{C}_\mu(r) \equiv \frac{1}{r^2} \int_{B(r)} C(x,y,t)dxdy$$

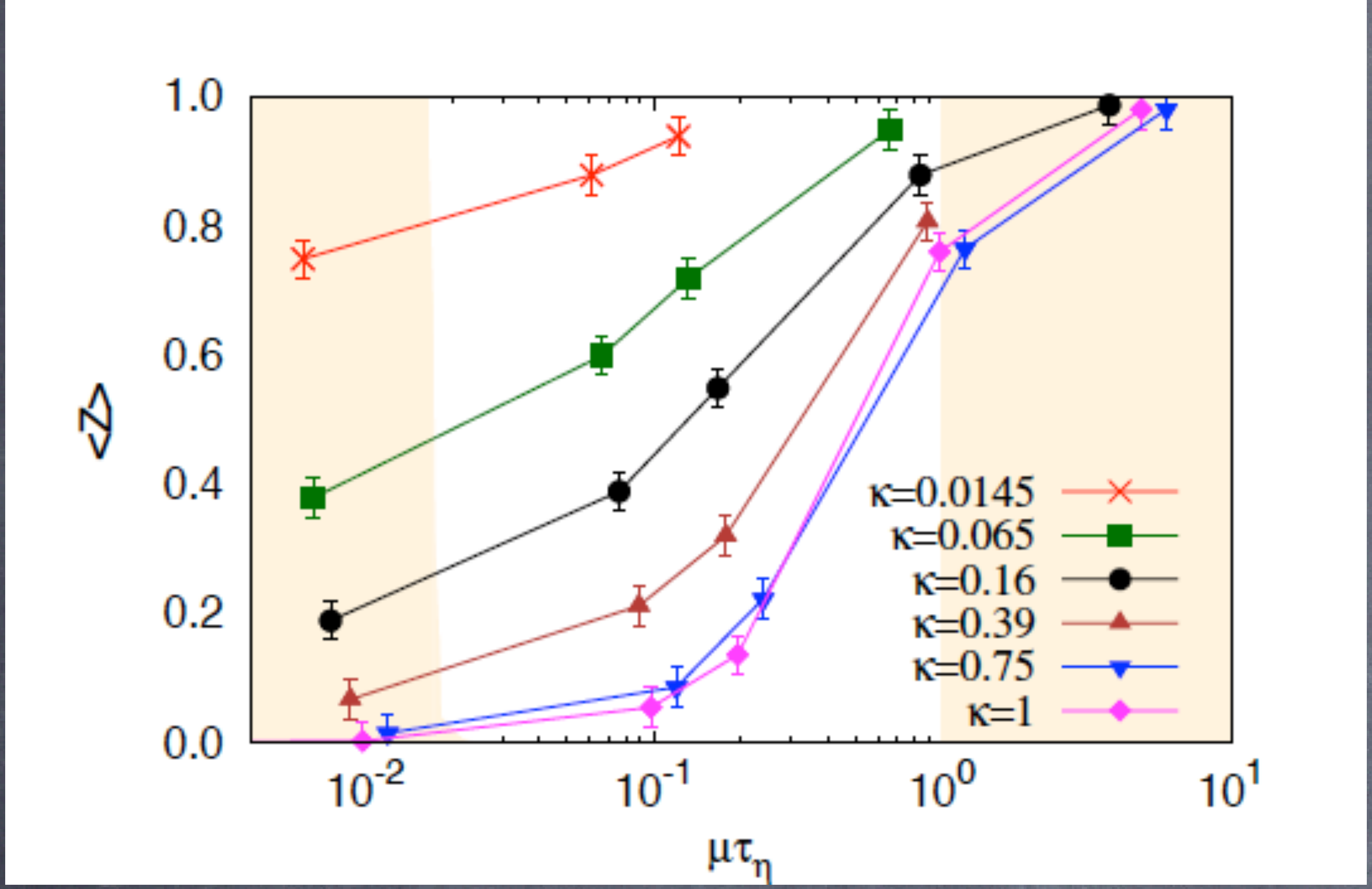
$$\langle P^2(x,t)\rangle \sim \xi^{a(2)}$$

$$\langle Z\rangle \sim \xi^{-a(2)}$$



$$\xi^2 = \frac{2A_2 D\sqrt{\nu}}{A_1\sqrt{\kappa\epsilon}}.$$

Carrying capacity as a function of compressibility



$$\kappa \equiv \langle(\nabla \cdot \mathbf{u})^2\rangle / \langle(\nabla \mathbf{u})^2\rangle,$$

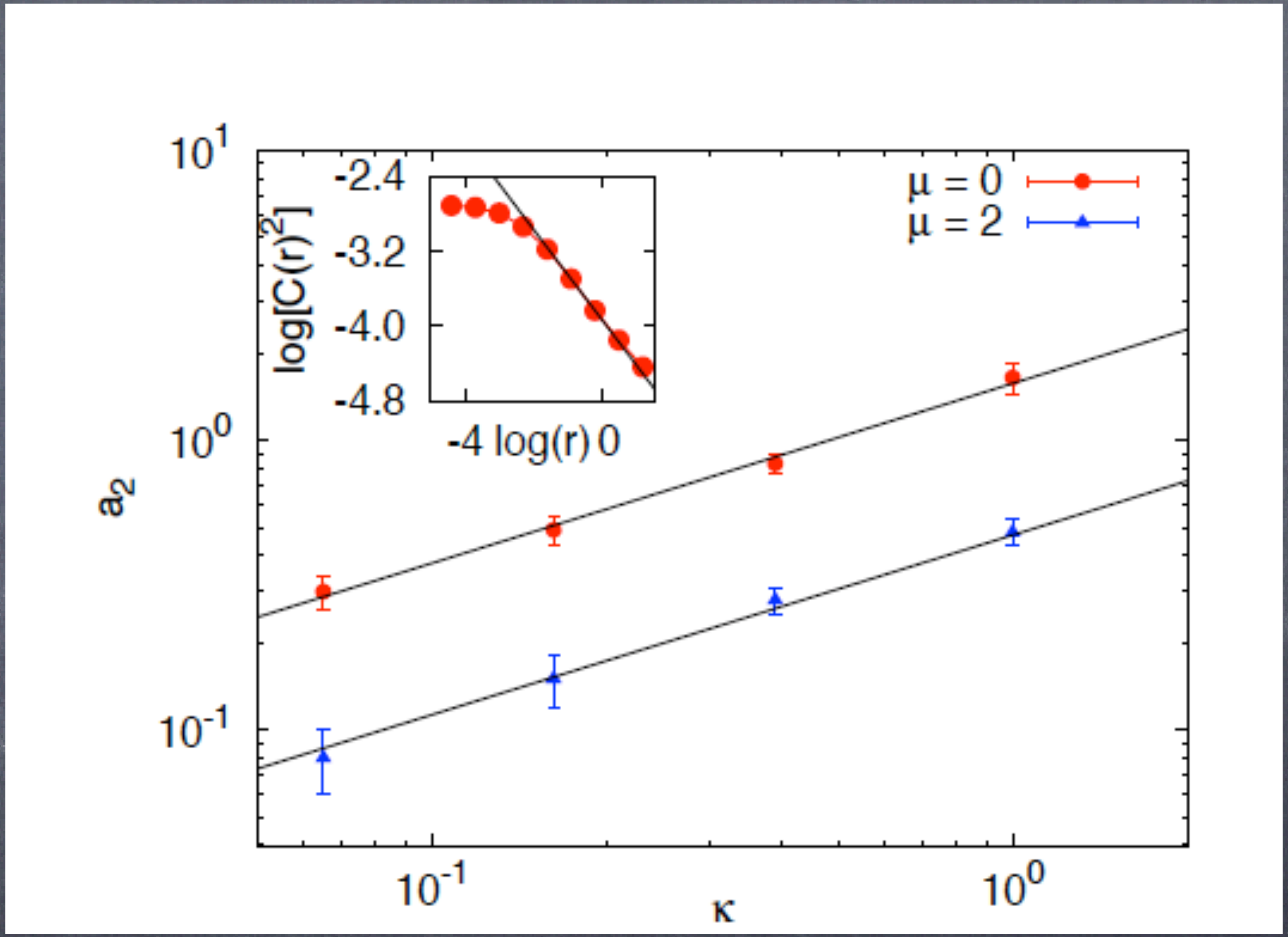
$$\boldsymbol{u} \equiv \sqrt{2}[\boldsymbol{u}^c \sin(\theta) + \boldsymbol{u}^i \cos(\theta)]$$

Dimensional analysis:

$a_2 = f(\mu\tau_\eta, \kappa)$

relevant characteristic time scale $1/\langle(\mathrm{div}\ \mathbf{v})2\rangle^{1/2} = \tau_\eta/\kappa^{1/2}$

$$a_2 \propto \sqrt{\kappa}/(\mu\tau_\eta + \beta)$$

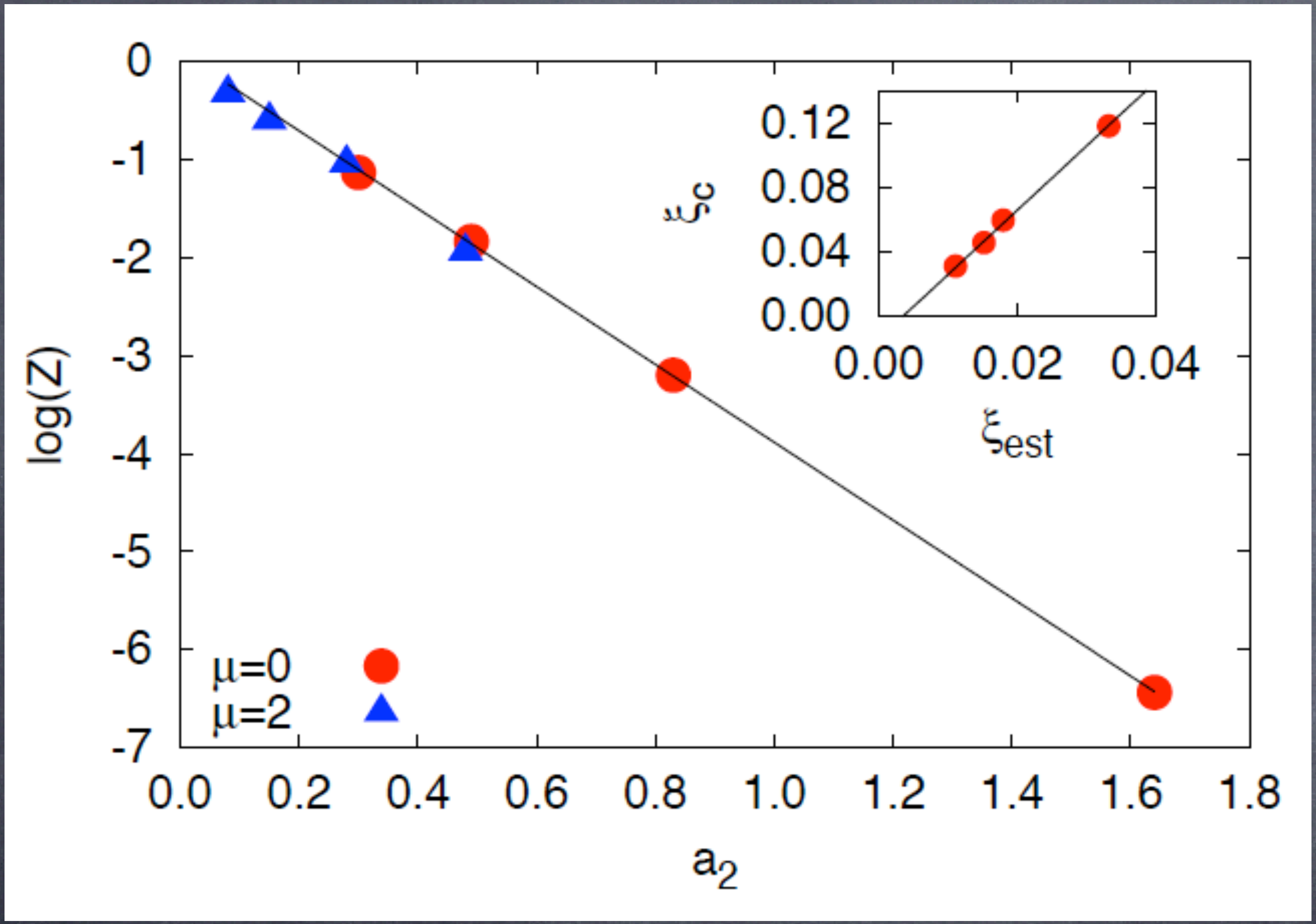


$$a_2(\kappa, \mu\tau_\eta) = a(\mu\tau_\eta)\kappa^\gamma$$

$\gamma \approx 0.62$

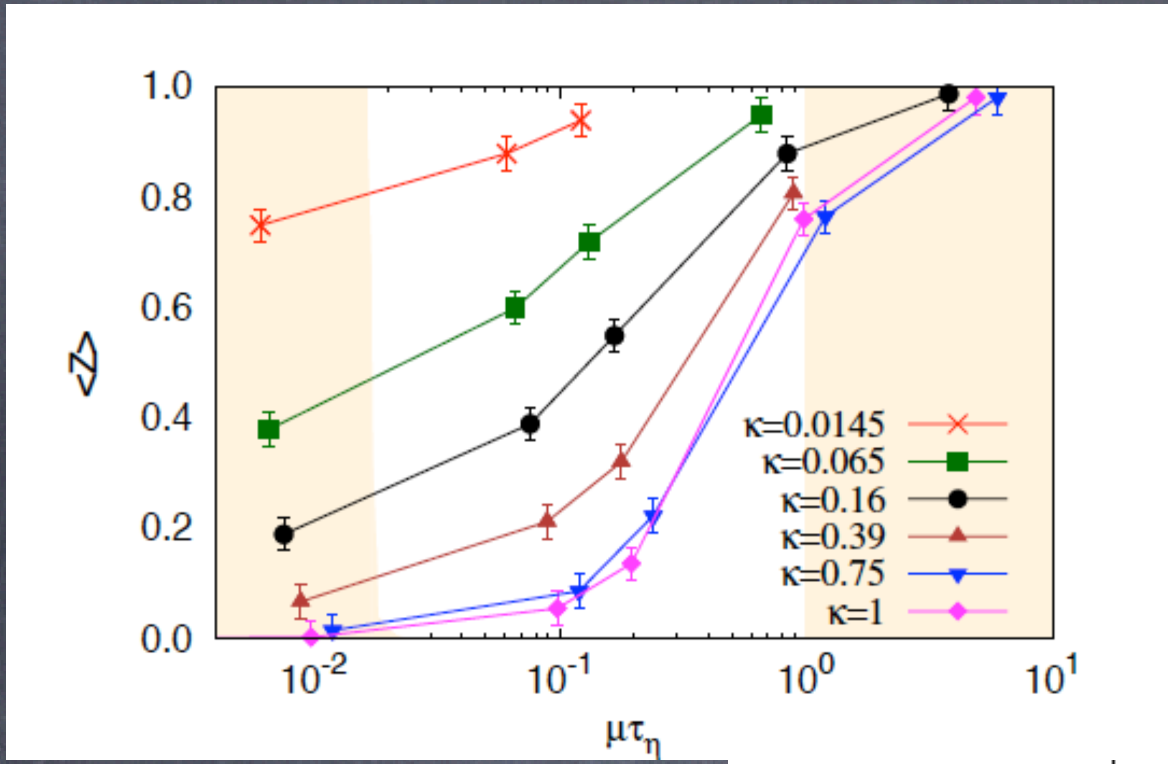
$$Z(\mu, \tau_\eta, \kappa) = \xi_{est}^{a_2(\kappa, \mu\tau_\eta)}$$

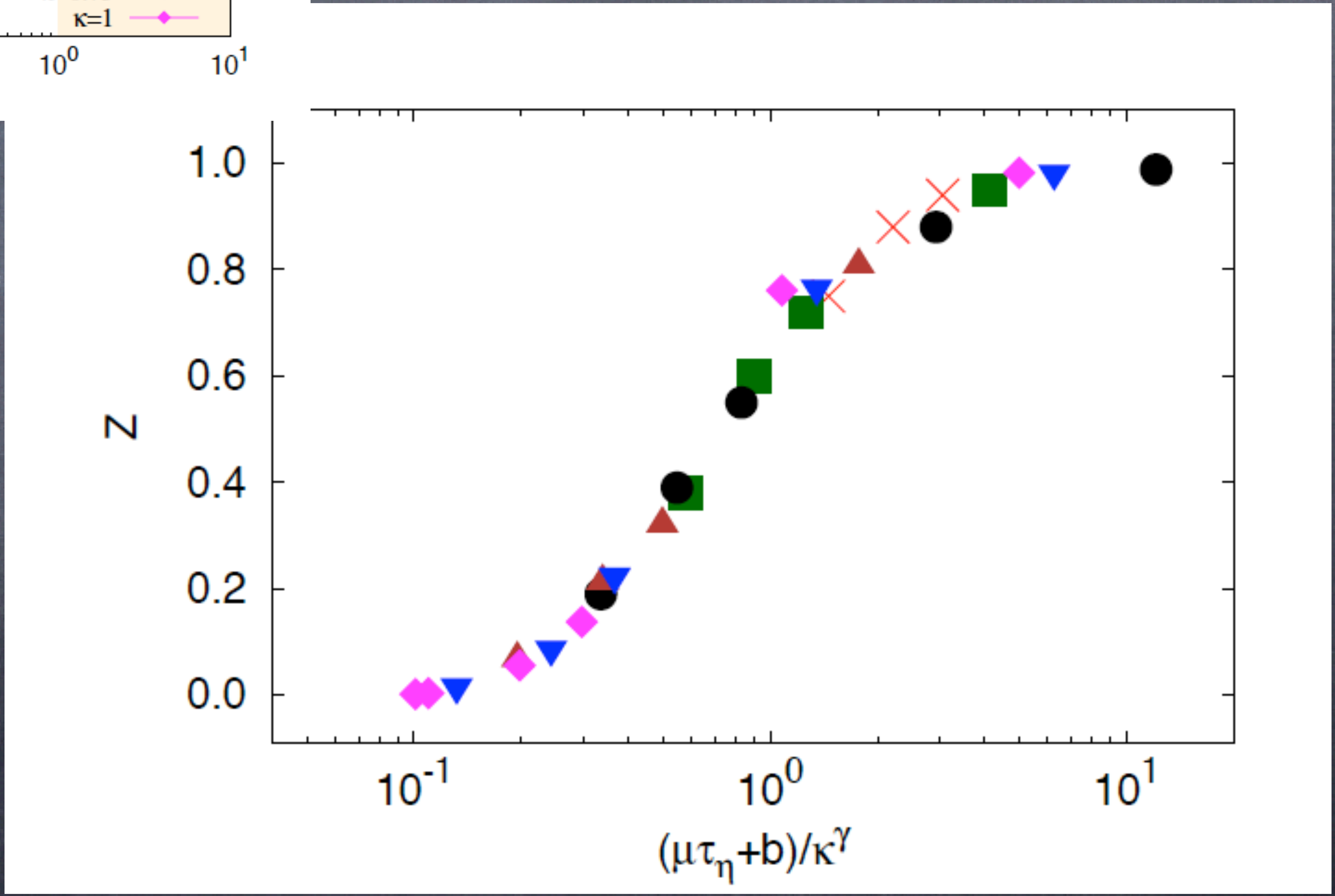
$$\xi_c^2 = \frac{\langle c^2 \rangle}{\langle (\nabla c)^2 \rangle}.$$



$$a_2 = a\ \kappa^\gamma / (\mu \tau_\eta + b)$$

"Universal" curve for different compressibility





Two species A and B

$$
\begin{aligned}
W_A(+1, n_A, n_B) &= \mu n_A \\
W_A(-1, n_A, n_B) &= \tilde{\mu} n_A (n_A + n_B) \\
W_B(+1, n_A, n_B) &= \mu n_B \\
W_B(-1, n_A, n_B) &= \tilde{\mu} n_B (n_A + n_B)
\end{aligned}
$$

birth process

death process

N = number of individuals

$$\tilde{\mu}/N = \mu.$$

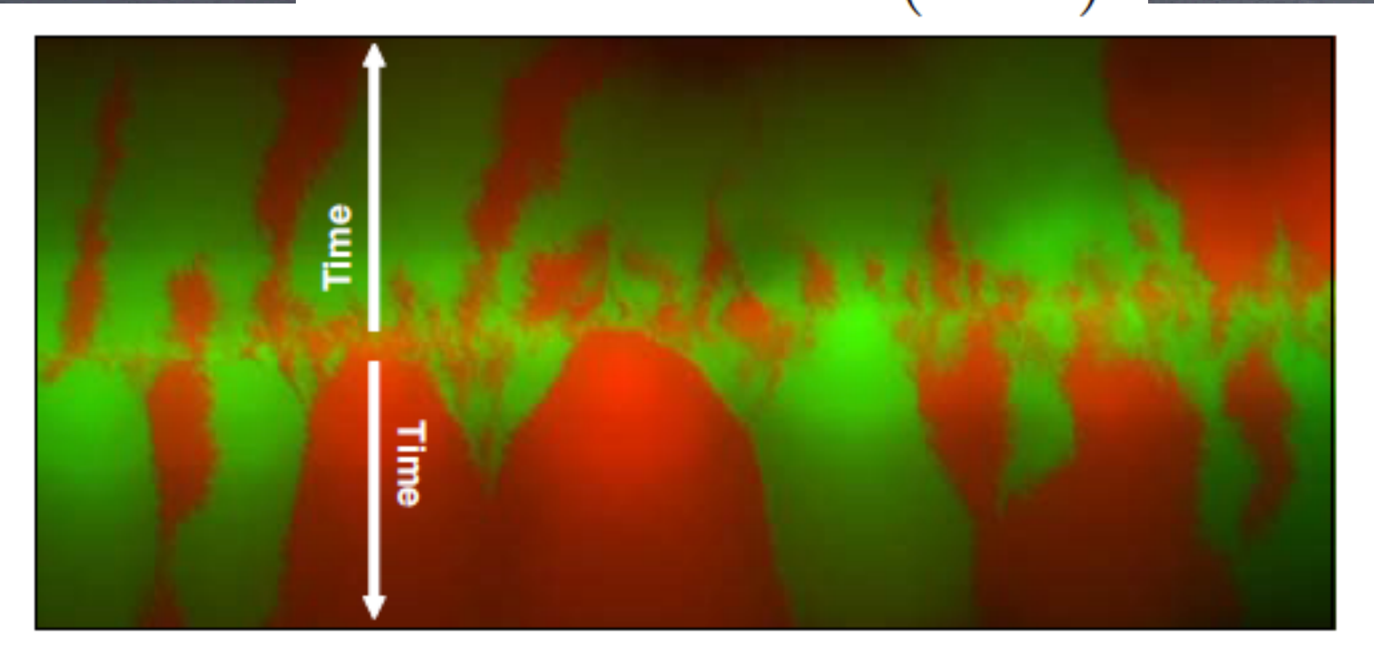
$$
\begin{aligned}
\frac{dc_A}{dt} &= \mu c_A (1 - c_A - c_B) + \sqrt{\frac{\mu c_A (1 + c_A + c_B)]}{N}}\xi \\
\frac{dc_B}{dt} &= \mu c_B (1 - c_A - c_B) + \sqrt{\frac{\mu c_B (1 + c_A + c_B)}{N}}\xi'.
\end{aligned}
$$

↓

$$
\begin{aligned}
\partial_t c_A &= -\partial_x (v c_A) + D \partial_x^2 c_A + \mu c_A (1 - c_A - c_B) + \sigma_{AB} \xi \\
\partial_t c_B &= -\partial_x (v c_B) + D \partial_x^2 c_B + \mu c_B (1 - c_A - c_B) + \sigma_{BA} \xi'
\end{aligned}
$$

for $v=0$ and $c_A+c_B=1$ the model known as stepping stone model

Hallatschek *et al.* (2007).



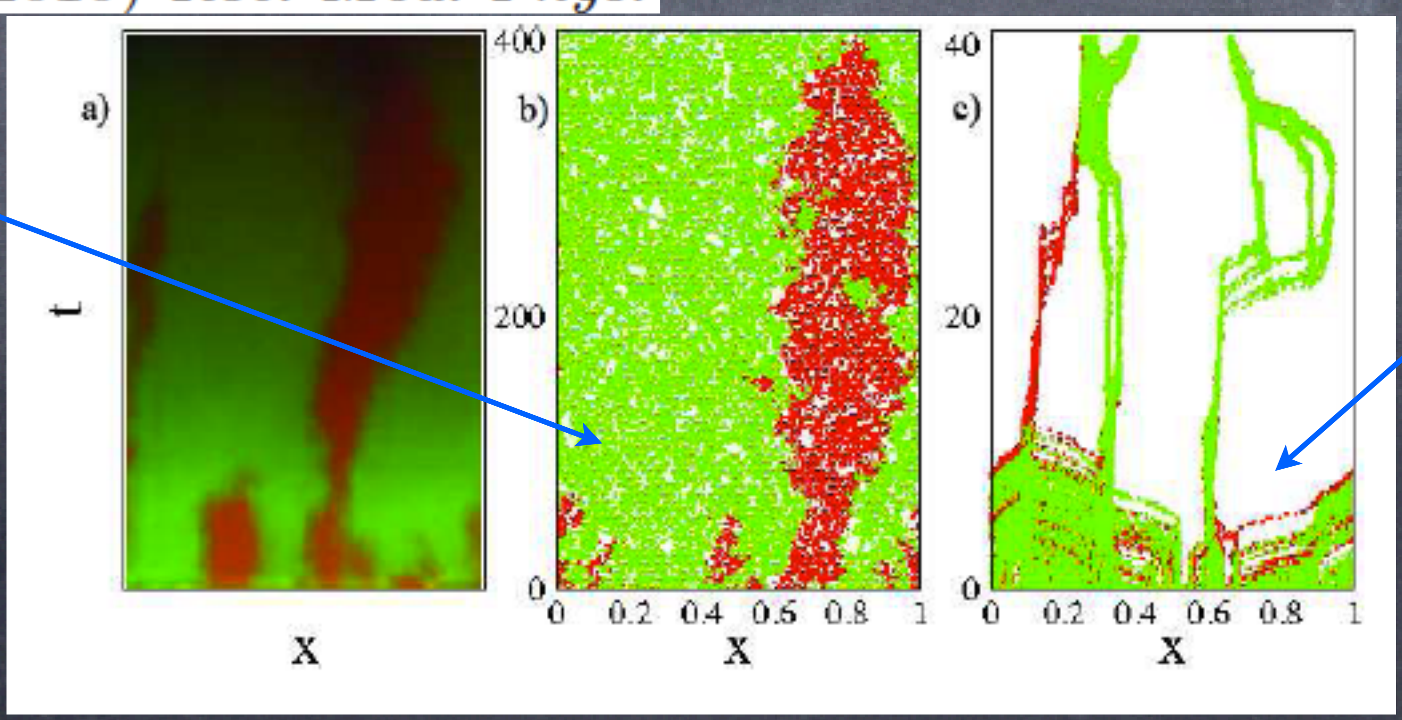
One dimension

With no velocity (v=0), spontaneous segregation is observed experimentally.

The dynamics can be explained by the stepping stone model which predicts segregation on a time scale $\tau_s \approx N^2 D$. Global fixation (only one species alive) on time scale $\tau_f \approx L^2/D$.

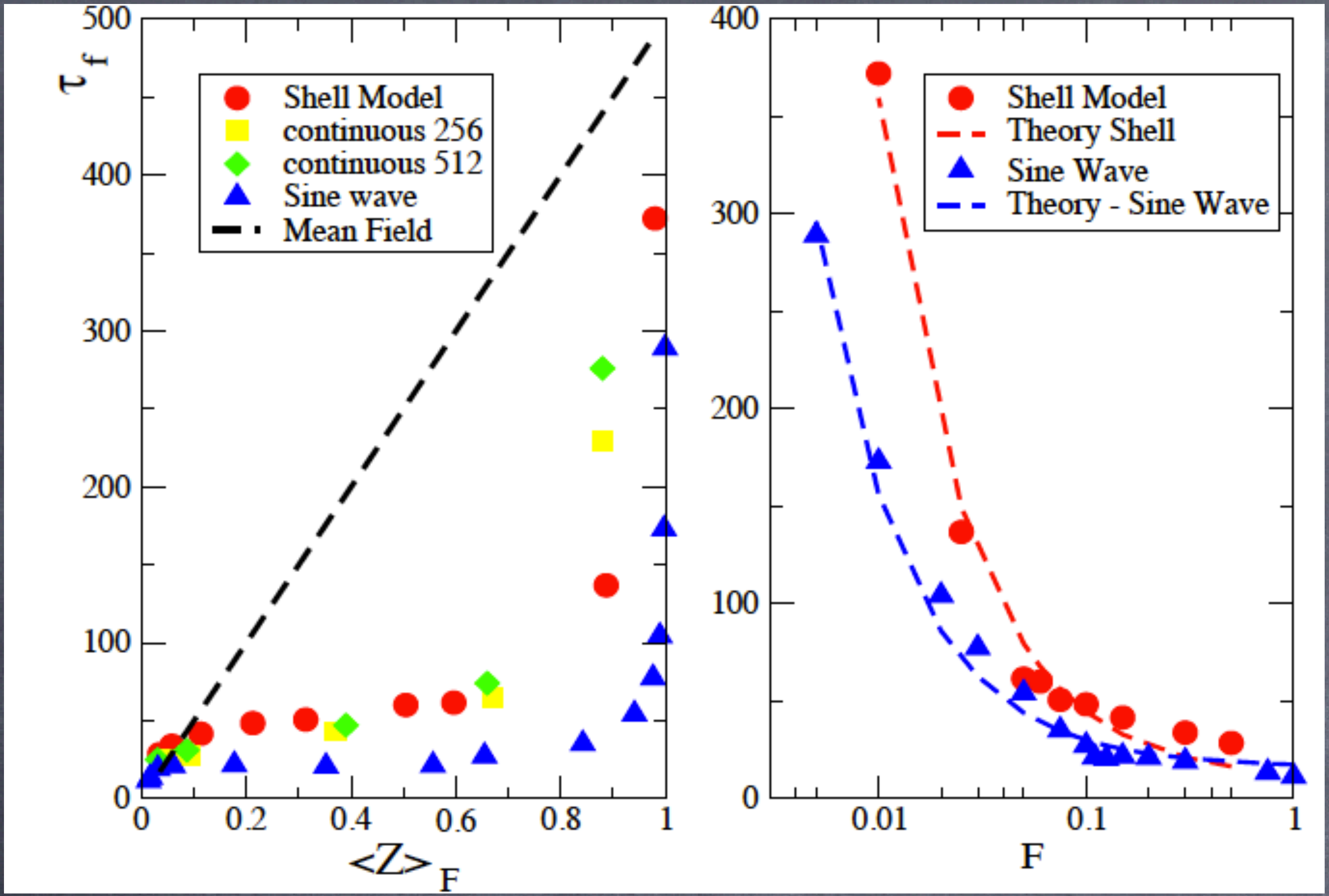
Korolev K et al. (2010) *Rev. Mod. Phys.*

no flows



with turbulence

S Pigolotti, R.B, D.R. Nelson, M. Jensen PRL submitted 2011



S Pigolotti, R.B, D.R. Nelson, M. Jensen PRL submitted 2011

Global fixation time as a function of the carrying capacity

$\tau_f \cong 1/\Gamma$ where Γ is the velocity gradient