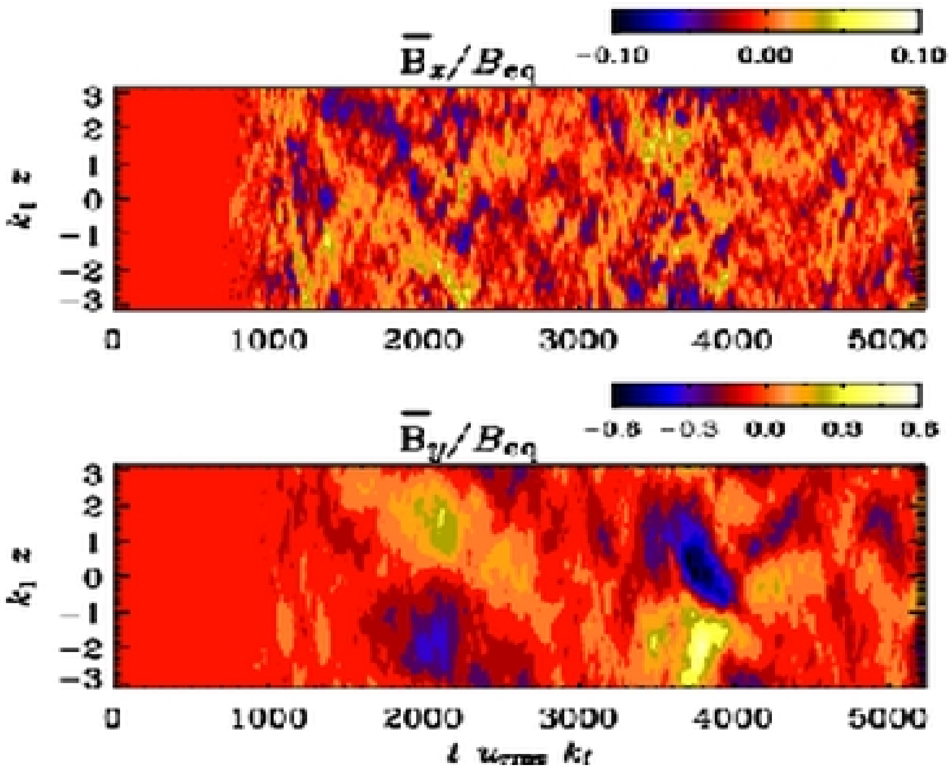
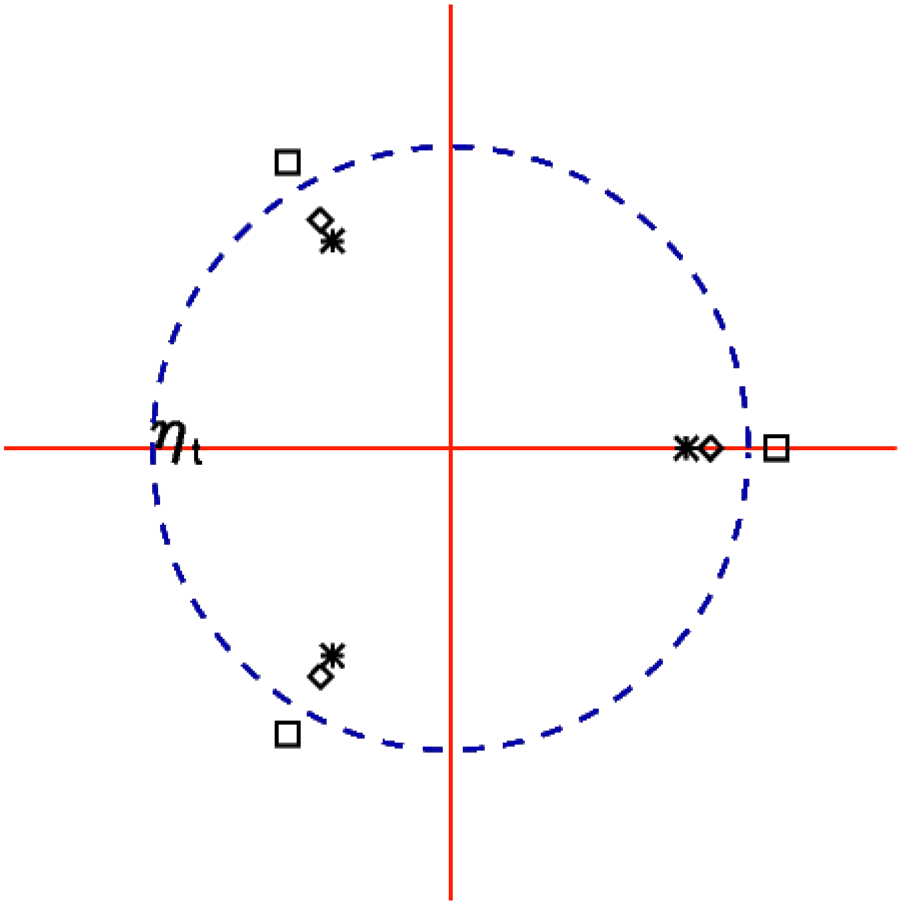
Scaling and intermittency in incoherent α–shear dynamo





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Introduction:

- In astrophysics: large scale shear is ubiquitous (sun, keplaraian disks ..)
- Most working dynamos are alpha – shear type. Shear is (almost) constant in time, but alpha fluctuates.

Evidence in support of fluctuating alpha

- Simulations with shear and non-helical forcing:

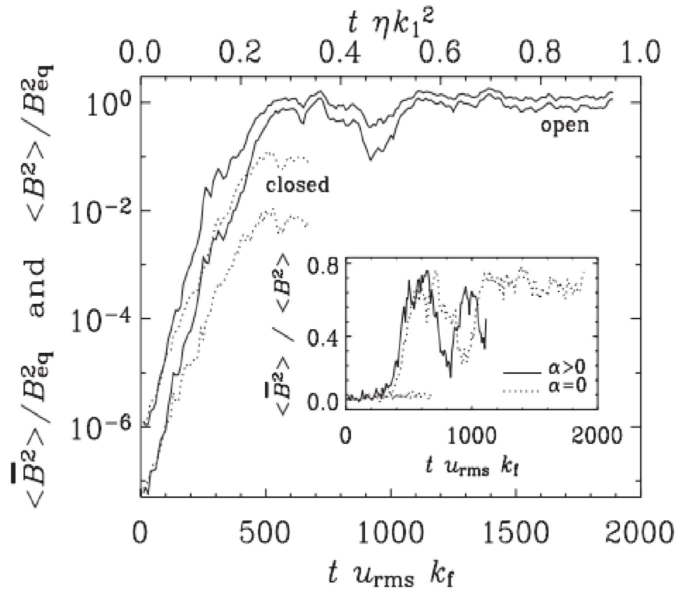
(i) Brandenburg, 2005, Apj **625**, 329.

(ii) B, Radler, Rheinhardt, Kapyla, 2008, ApJ, **676,** 740. (BRRK)

(iii) Yousef, Heinemann, Schekochihin, Kleeorin, Rogachevski, Istakov, Cowley, McWillams, 2008, PRL, **100** 184501 (Y2008)

Summary of DNS results:

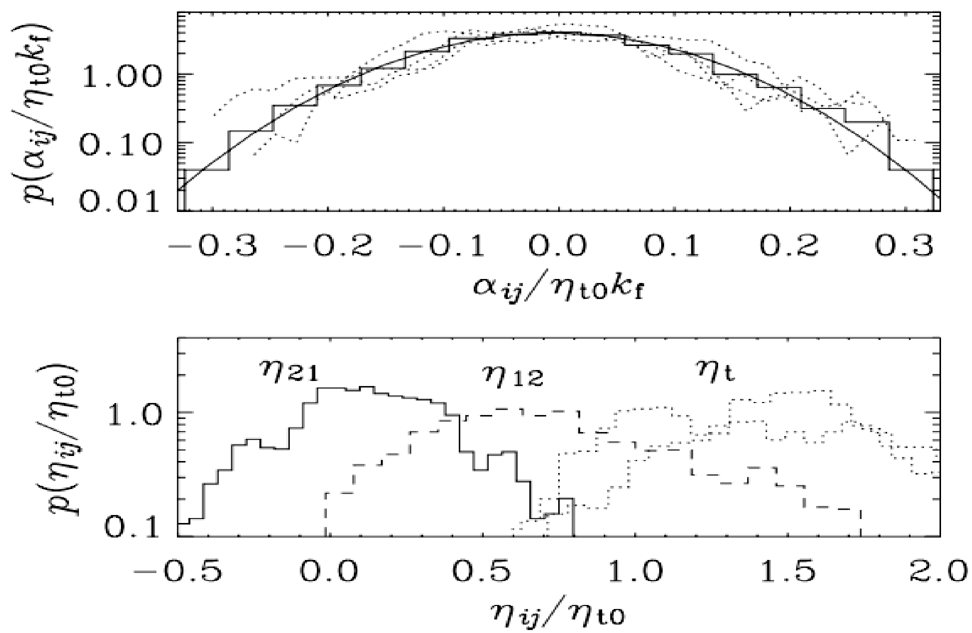
- For an alpha with fluctuations but zero mean, a dynamo develops. (B05, BRRK, Y2008) but for the root-mean square magnetic field.

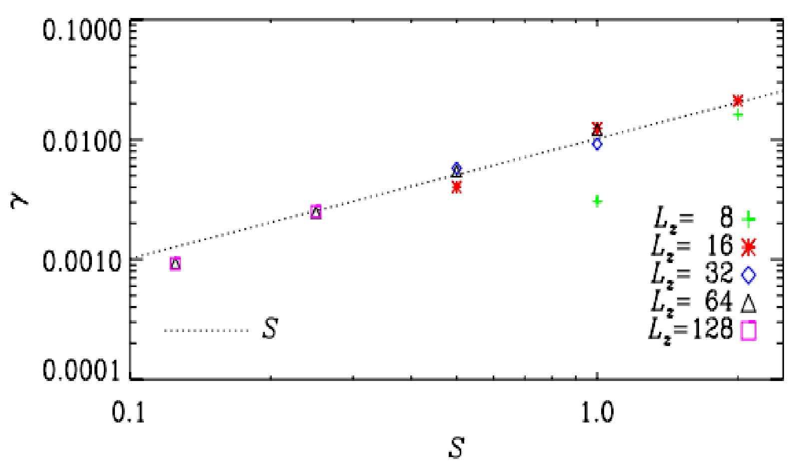


- The fastest growing mode has (Y2008):

$$k^{\rm peak} \sim \sqrt{S} \qquad \gamma \sim S$$

- PDF of alpha is Gaussian (BRRK)





Analytical works:

- Vishniac and Brandenburg 1997, ApJ, **584**, L99 (VB97)
- Sokolov 97 Astronomy Reports **41** 68. (S97)
- Silan'tev 2000, A and A **364** 339.
- Kleeorin and Rogachevskii, 2008 PRE **77**, 036307 (KR08)
- Fedotov, Bashkirtseva, Ryashko 2006, PRE, **73** (FBR06)
- Proctor 2007 MNRAS (P07)
- Heinemann, McWillams, Schekochihin, ArXiv 0810.2225v2 (HMS2011)
- Mitra and Brandenburg, ArXiv 1107.2752 (MB2011)

Remarks on analytical works:

- Two different averages are needed:
  (a) Reynolds averaging (typically over coordinate directions)
  (b) Statistics of alpha.
- Alpha can safely be taken to be a Gaussian.
- The time correlation of alpha is crucial:
  (a) Exact results are possible only if alpha is white-in-time (VB97, MB2011).
  (b) Otherwise results are perturbative in nature (HMS2011, KR2008, P07) or based on arguments (S97).
- Alpha is assumed constant in space (VB97, MB2011, S97)
- Alpha is also a function of space (P07, S00, FBR 2006)
- Formally a problem with multiplicative noise (Kraichnan model of passive scalar advection)

Analytical results:

- Average B does not grow. (VB97, KR08, S97, MB2011, HMS2011).
- But rms B grows (HMS11,MB11) :

$$k^{\mathrm{peak}} \sim \sqrt{S} \qquad \gamma \sim S$$

$$\gamma \sim S^{2/3}$$

- Contradicted by P07 (which also contradicts DNS of Y08)

$$k^{\mathrm{peak}} \sim S \qquad \gamma \sim S^2$$

- The growing field is intermittent (S97, MB11).

Model:

- Constant Shear:.
- Mean field is constructed by averaging over the x-y plane.
- Alpha and eta are 2X2 matrices assumed constant in space.
- Alpha is white-in-time and Gaussian.

$$\overline{\boldsymbol{U}} = Sx\hat{\boldsymbol{e}}_y$$

$$\partial_t \bar{B}_x = -\alpha_{yx}\partial_z \bar{B}_x - \alpha_{yy}\partial_z \bar{B}_y - \eta_{yx}\partial_z^2 \bar{B}_y + \eta_{yy}\partial_z^2 \bar{B}_x,$$

$$\partial_t \bar{B}_y = S\bar{B}_x + \alpha_{xx}\partial_z \bar{B}_x + \alpha_{xy}\partial_z \bar{B}_y - \eta_{xy}\partial_z^2 \bar{B}_x + \eta_{xx}\partial_z^2 \bar{B}_y.$$

$$\langle \alpha_{ij}(t)\alpha_{kl}(t')\rangle = D_{ij}\delta_{ik}\delta_{jl}\delta(t-t').$$

Even simpler model (VB97) :

$$\begin{aligned}\partial_t \bar{B}_x &= \alpha \bar{B}_y - \eta_{\rm t} \bar{B}_x,\\ \partial_t \bar{B}_y &= -S\bar{B}_x - \eta_{\rm t} \bar{B}_y\end{aligned}$$

$$\langle \alpha(t)\alpha(t')\rangle = D\delta(t-t')$$

$$\boldsymbol{C}^1 \equiv (\langle \bar{B}_x\rangle, \langle \bar{B}_y\rangle) \quad \boldsymbol{C}^2 \equiv (\langle \bar{B}_x^2\rangle, \langle \bar{B}_y^2\rangle, \langle \bar{B}_x \bar{B}_y\rangle)$$

$$\partial_t \boldsymbol{C}^p = \mathsf{N}_{\mathsf{p}} \boldsymbol{C}^p$$

Analytical technique :

$$\langle \alpha_{yx} \partial_z \bar{B}_x \rangle = D_{yx} \left\langle \frac{\delta \bar{B}_x}{\delta \alpha_{yx}} \right\rangle = D_{yx} \langle -\partial_z \bar{B}_x \rangle$$

$$\langle F(\boldsymbol{\nu}) \nu_j(t) \rangle = \int dt' \langle \nu_j(t) \nu_k(t') \rangle \left\langle \frac{\delta F}{\delta \nu_k(t')} \right\rangle$$

J. Zinn-Justin 1999 *Quantum Field Theory and Critical Phenomenon*, Oxford University Press.

$$\langle \alpha_{yx}(t) \bar{B}_x \rangle = \int dt' \langle \alpha_{yx}(t) \alpha_{kl}(t') \rangle \left\langle \frac{\delta \bar{B}_x}{\delta \alpha_{kl}(t')} \right\rangle$$

Functional derivatives :

$$\frac{\delta \bar{B}_x}{\delta \alpha_{yx}} = -\partial_z \bar{B}_x, \qquad \frac{\delta \bar{B}_x}{\delta \alpha_{yy}} = -\partial_z \bar{B}_y,$$

$$\frac{\delta \bar{B}_y}{\delta \alpha_{xx}} = \partial_z \bar{B}_x, \qquad \frac{\delta \bar{B}_y}{\delta \alpha_{xy}} = \partial_z \bar{B}_y.$$

Eigenvalue problem (VB97) :

$$\partial_t \boldsymbol{C}^p = \mathsf{N}_\mathsf{p} \boldsymbol{C}^p$$

$$\mathsf{N}_1 = \begin{bmatrix} -\eta_\mathrm{t} & 0 \\ -S & -\eta_\mathrm{t} \end{bmatrix} \qquad \mathsf{N}_2 = \begin{bmatrix} -2\eta_\mathrm{t} & 2D & 0 \\ 0 & -2\eta_\mathrm{t} & -2S \\ -S & 0 & -2\eta_\mathrm{t} \end{bmatrix}$$

$$\boldsymbol{C}^p \sim \exp(p\gamma_p t)$$

- The eigenvalues are solutions of cubic equations !

$$\gamma_1 = -\eta_\mathrm{t}, \quad \gamma_2 = -\eta_\mathrm{t} + \left(\frac{4DS^2}{8}\right)^{1/3} \sim S^{2/3}$$

Eigenvalue problem: verion 2

$$\partial_t \boldsymbol{C}^p = \mathsf{N_p} \boldsymbol{C}^p \qquad \boldsymbol{C}^p \sim \exp(p\gamma_p t)$$

$$\mathsf{N_1} = \begin{bmatrix} -k^2(\eta_{yy} + D_{yx}) & k^2\eta_{yx} \\ S + k^2\eta_{xy} & -k^2(\eta_{xx} + D_{xy}) \end{bmatrix}$$

$$\begin{bmatrix} -2k^2\eta_{yy} & 2k^2 D_{yy} & k^2\eta_{yx} \\ 2k^2 D_{xx} & -2k^2\eta_{xx} & S + k^2\eta_{xy} \\ 2(S + k^2\eta_{xy}) & 2k^2\eta_{yx} & -k^2(D_{yx} + D_{xy} + 2\eta_{xx}) \end{bmatrix}$$

- The eigenvalues are <u>again</u> solutions of cubic equations !

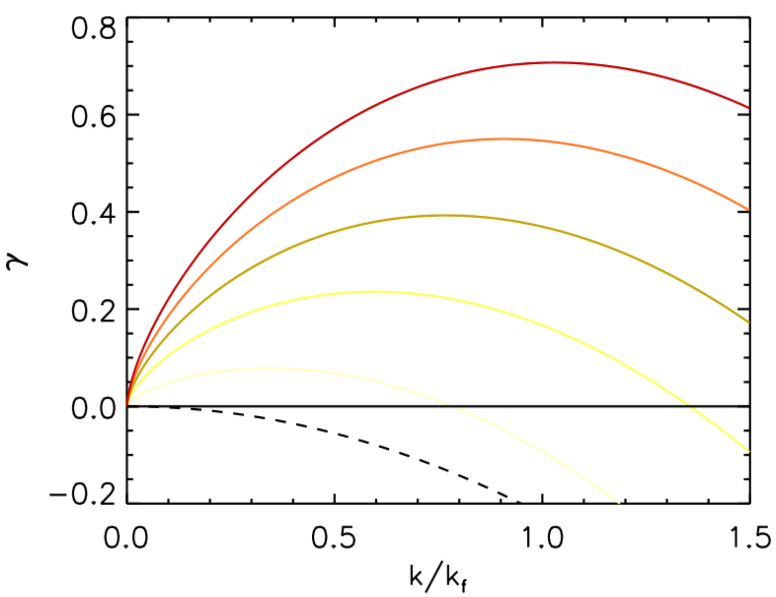
Properties of the dynamo:

- Dispersion relation:

$$\xi^3 - 4k^2 D_{yy} \left[S^2 + k^2 D_{xx} \xi \right] = 0,$$

$$\gamma = -k^2 \eta_{\rm t} + \left(\frac{1}{2} k^2 D_{yy} S^2 \right)^{1/3} (1, \omega, \omega^2),$$

- Scaling:



$$k^{\rm peak} = |S|^{1/2} \left(\frac{D_{yy}}{54 \eta_{\rm t}^3} \right)^{1/4}$$

$$\gamma = \frac{2^{1/3}}{6} \left(1 - \frac{2^{1/6}}{\sqrt{3}} \right) \left(\frac{D_{yy}}{\eta_{\rm t}} \right)^{1/2} |S|.$$

Interpretion of DNS results:

- How does the growth-rate of rms magnetic field scales with shear in your DNS ?
- This model has no oscillatory solutions.
- In DNS, Alpha is *never* delta-correlated in time. $D \approx \alpha_{\rm rms}^2 \tau_{\rm cor},$
- Time correlations of alpha might allow for oscillatory solutions but not likely to change the scaling behaviour.
- For low shear you might get incoherent alpha-square dynamo.

$$\frac{4k^8 D_{xx}^3 D_{yy}}{S^4} > 1,$$

or $\sqrt{2}k^2 D_{xx}/|S| > 1$ for $D_{xx} = D_{yy}$.

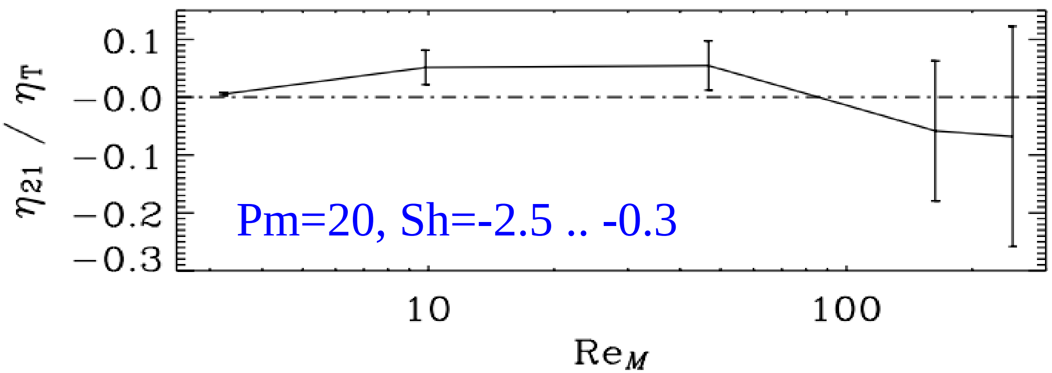
Competition with shear-current effect.

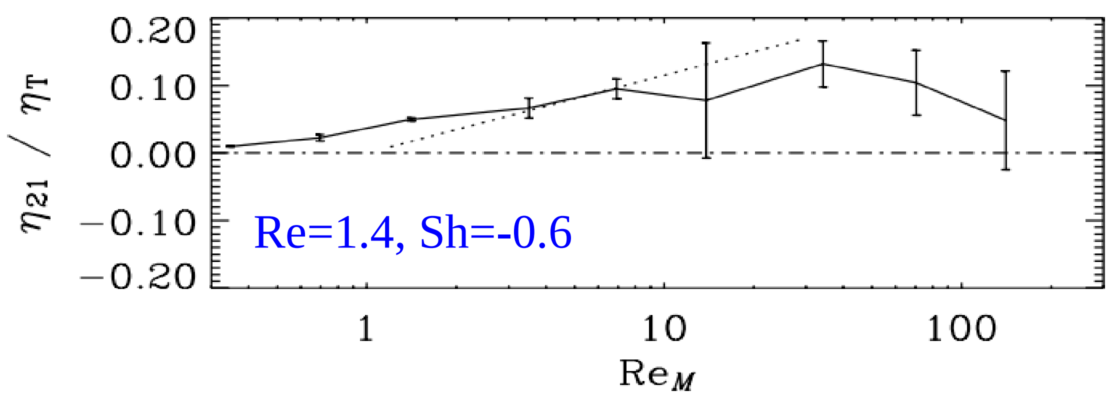
- The shear-current effect works via an off-diagonal component of the turbulent magnetic diffusivity tensor. For the DNS of BRRK, must have been negative.

$$D_{\eta S} \equiv \frac{S\,\eta_{21}}{(\eta_t k)^2} > 1 \qquad \eta_{21} \sim -S$$

$$\gamma = \frac{S\,\eta_{21}}{4\,\eta_t} \sim S^2$$

$$k^{peak} = \frac{\sqrt{S\,\eta_{21}}}{2\,\eta_t} \sim S$$





- Needed is η_{21} that is negative but independent of S for large enough S

Higher order moments

$$\begin{aligned} \boldsymbol{C}^3 &\equiv (\langle \bar{B}_x^3 \rangle, \langle \bar{B}_x^2 \bar{B}_y \rangle, \langle \bar{B}_x \bar{B}_y^2 \rangle, \langle \bar{B}_y^3 \rangle), \\ \boldsymbol{C}^4 &\equiv (\langle \bar{B}_x^4 \rangle, \langle \bar{B}_x^3 \bar{B}_y \rangle, \langle \bar{B}_x^2 \bar{B}_y^2 \rangle, \langle \bar{B}_x \bar{B}_y^3 \rangle, \langle \bar{B}_y^4 \rangle) \end{aligned}$$

$$\partial_t \boldsymbol{C}^p = \mathsf{N}_\mathsf{p} \boldsymbol{C}^p$$

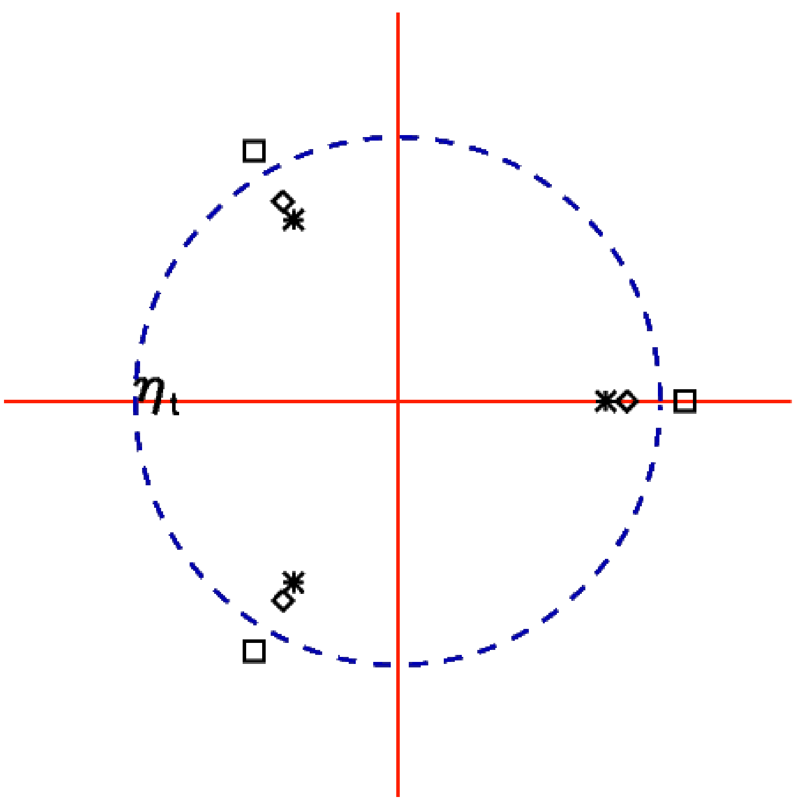
$$\mathsf{N}_3 = \begin{bmatrix} -3\eta_\mathrm{t} & 0 & 6D & 0 \\ -S & -3\eta_\mathrm{t} & 0 & D \\ 0 & -2S & -3\eta_\mathrm{t} & 0 \\ 0 & 0 & -3S & -3\eta_\mathrm{t} \end{bmatrix}$$

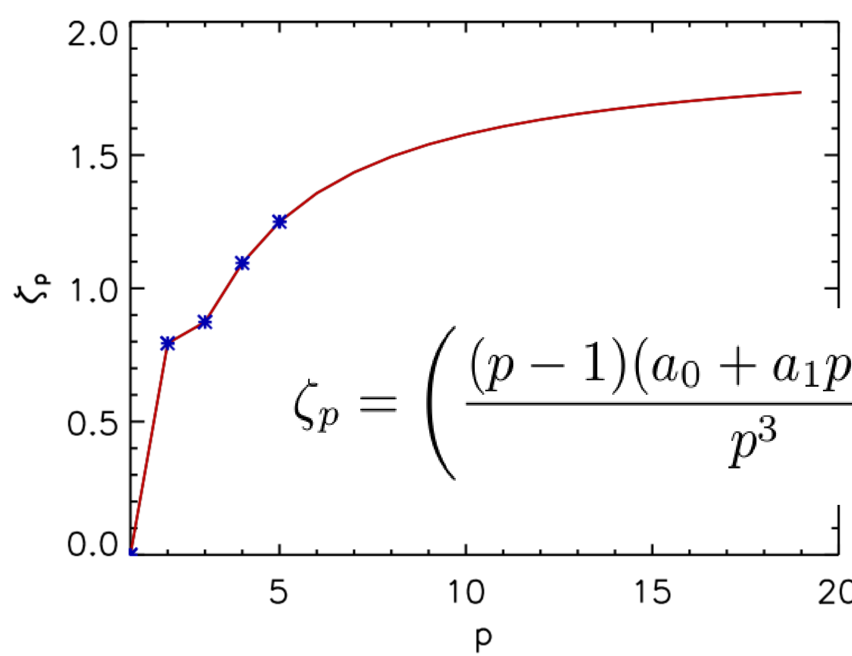
$$\mathsf{N}_4 = \begin{bmatrix} -4\eta_\mathrm{t} & 0 & 12D & 0 & 0 \\ -S & -4\eta_\mathrm{t} & 0 & 6D & 0 \\ 0 & -2S & -4\eta_\mathrm{t} & 0 & 2D \\ 0 & 0 & -3S & -3\eta_\mathrm{t} & 0 \\ 0 & 0 & 0 & -4S & -4\eta_\mathrm{t} \end{bmatrix}$$

Intermittency:

$$\gamma_1 = -\eta_t, \quad \gamma_2 = -\eta_t + \left(\frac{4DS^2}{8}\right)^{1/3} \sim S^{2/3}$$

$$\gamma_3 = -\eta_t + \left(\frac{18DS^2}{27}\right)^{1/3}, \quad \gamma_4 = -\eta_t + \left(\frac{84DS^2}{64}\right)^{1/3}. \qquad \zeta_p = \frac{\gamma_p + \eta_t}{(DS^2)^{1/3}}$$





$$\zeta_p = \left(\frac{(p-1)(a_0 + a_1 p + a_2 p^2)}{p^3}\right)^{1/3}.$$