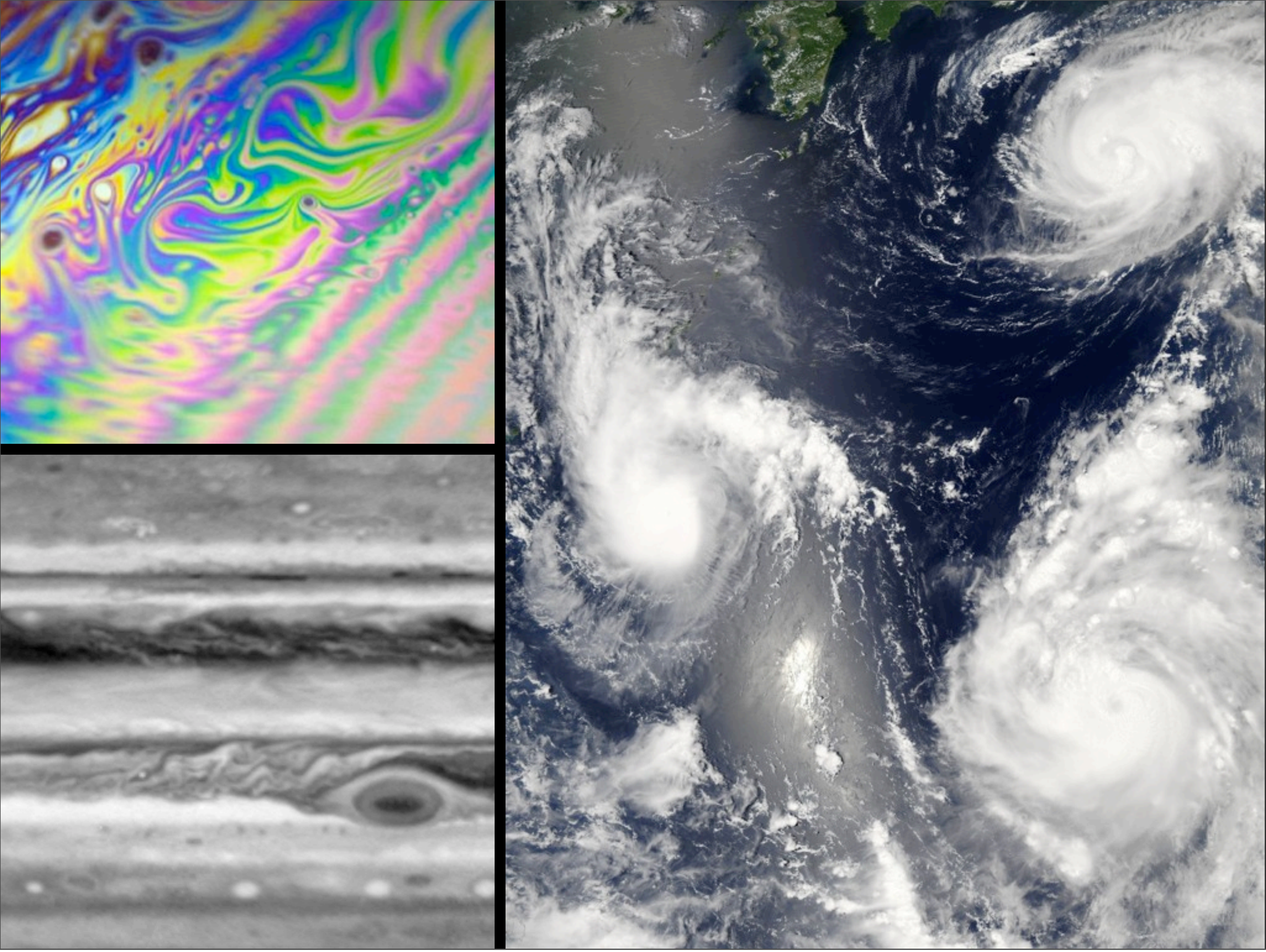
Background shows a condense vortex from our simulation
Color scale is vorticity, red and blue are positive and negative
This talk is about deriving and confirming their properties



Motivation for 2D turbulence

2D Navier-Stokes Equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} P + \nu \nabla^2 \boldsymbol{u} + \mu \boldsymbol{u} + \boldsymbol{f} + \ldots$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

- Without viscosity, Ekman term, forcing, etc

$$\frac{dE}{dt} \equiv \frac{d}{dt}\left[\frac{1}{2}\int d^2 x u^2\right] = 0$$

$$\frac{dZ}{dt} \equiv \frac{d}{dt}\left[\frac{1}{2}\int d^2 x \omega^2\right] = 0$$

where $\omega \equiv \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$

Energy and enstrophy are conserved quantities

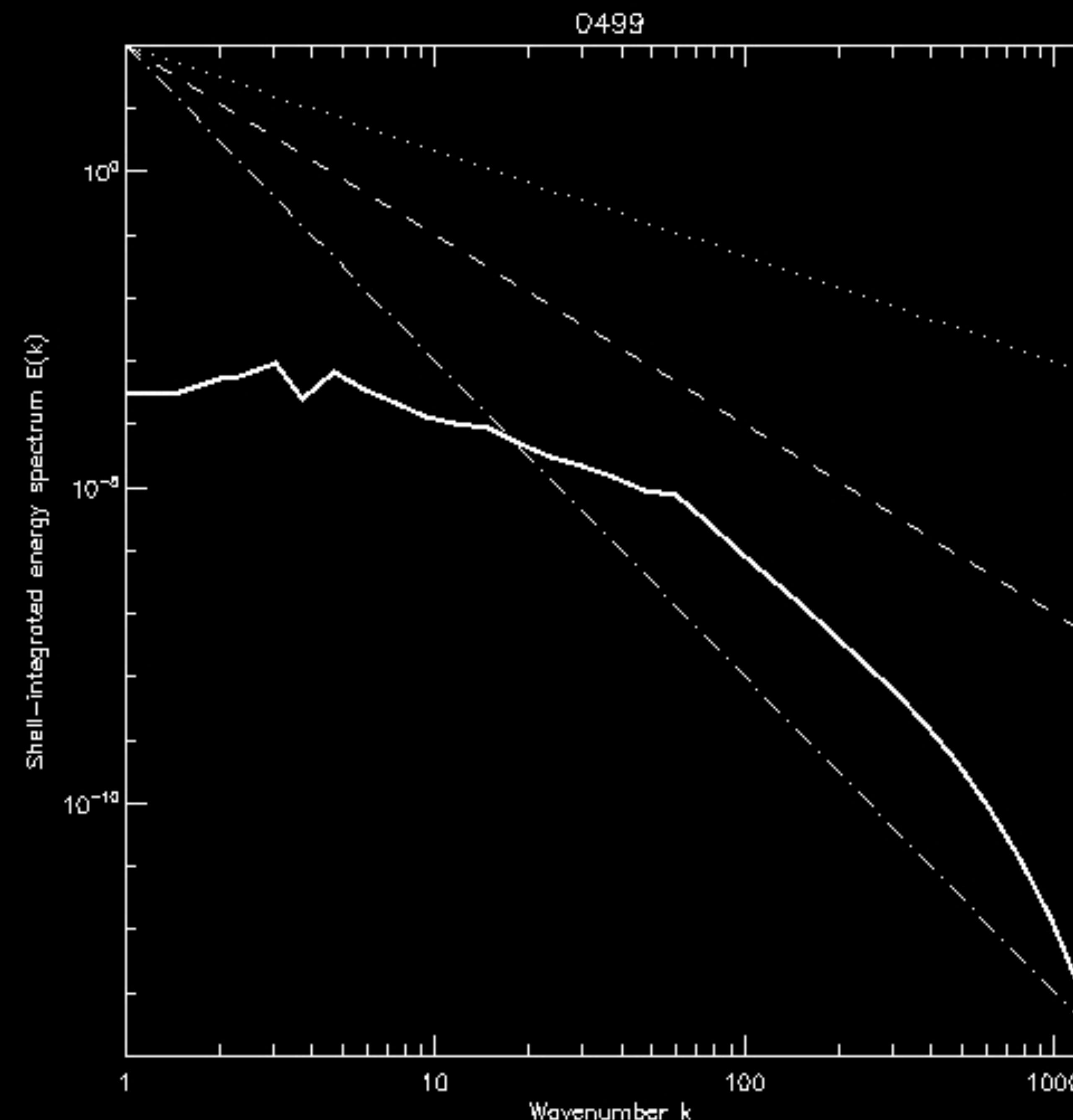
Forward and Inverse Cascades

- With viscosity and Ekman friction
- Constant fluxes ϵ and η
- Two inertial ranges

$$E_k = \mathcal{C}\epsilon^{2/3}k^{-5/3}$$

$$E_k = \mathcal{K}\eta^{2/3}k^{-3}$$

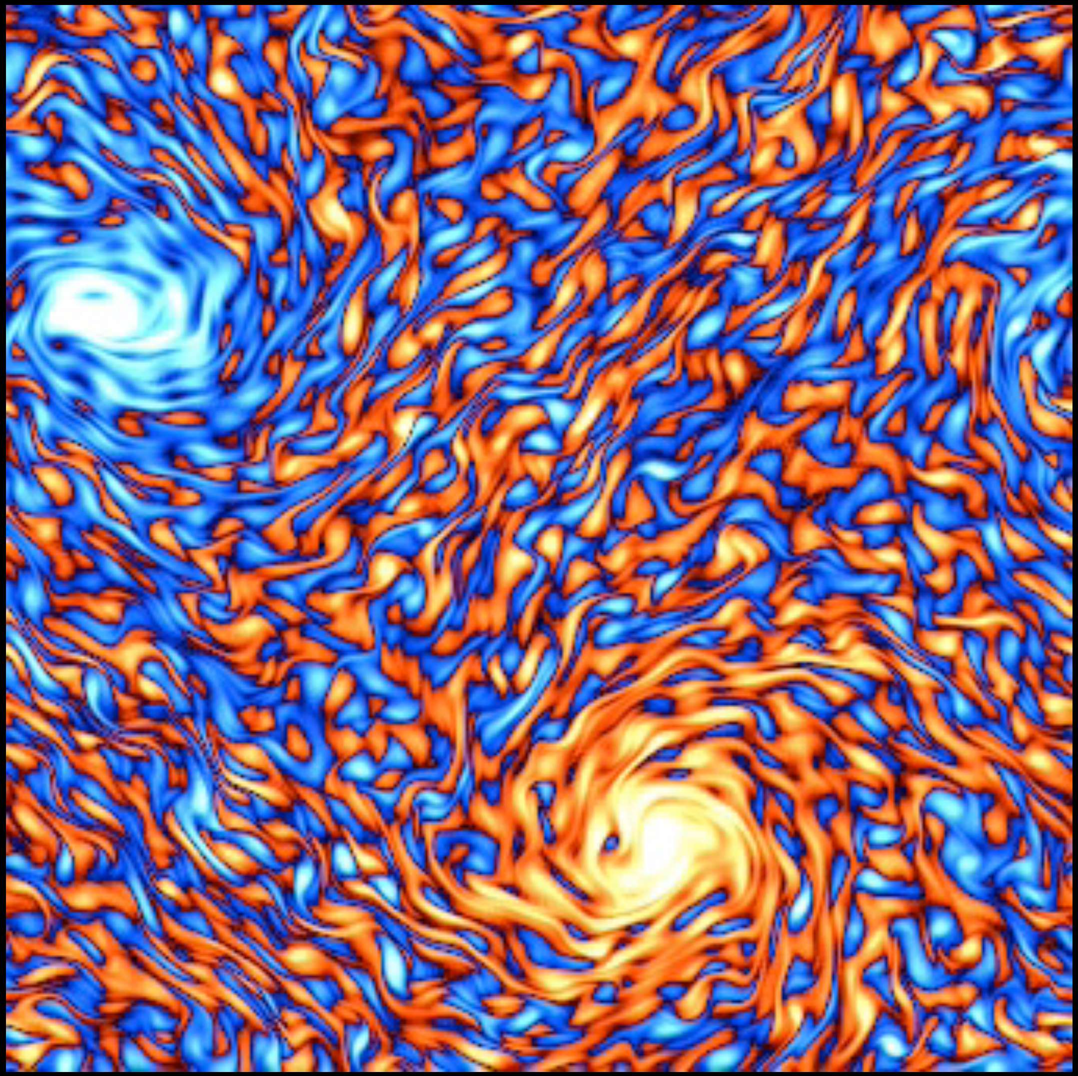
at different wavenumbers



Energy inverse cascades, spectral slope -5/3
Enstrophy forward cascade, spectral slope -3
Movie shows driven turbulence with non-vanishing Ekman friction
Having the Ekman term is too troublesome! Let’s remove it

Energy Condensation

- Vanishing Ekman term
- The rumor: without large scale regulation, the solution will eventually blow up at late time
- Bowman: it saturates; it just takes forever...
- Brandenburg: so it must saturate at viscous time



The rumor says the solution will blow up
Thank the turbulence workshop in KITP

Spectral Galerkin Method on GPUs

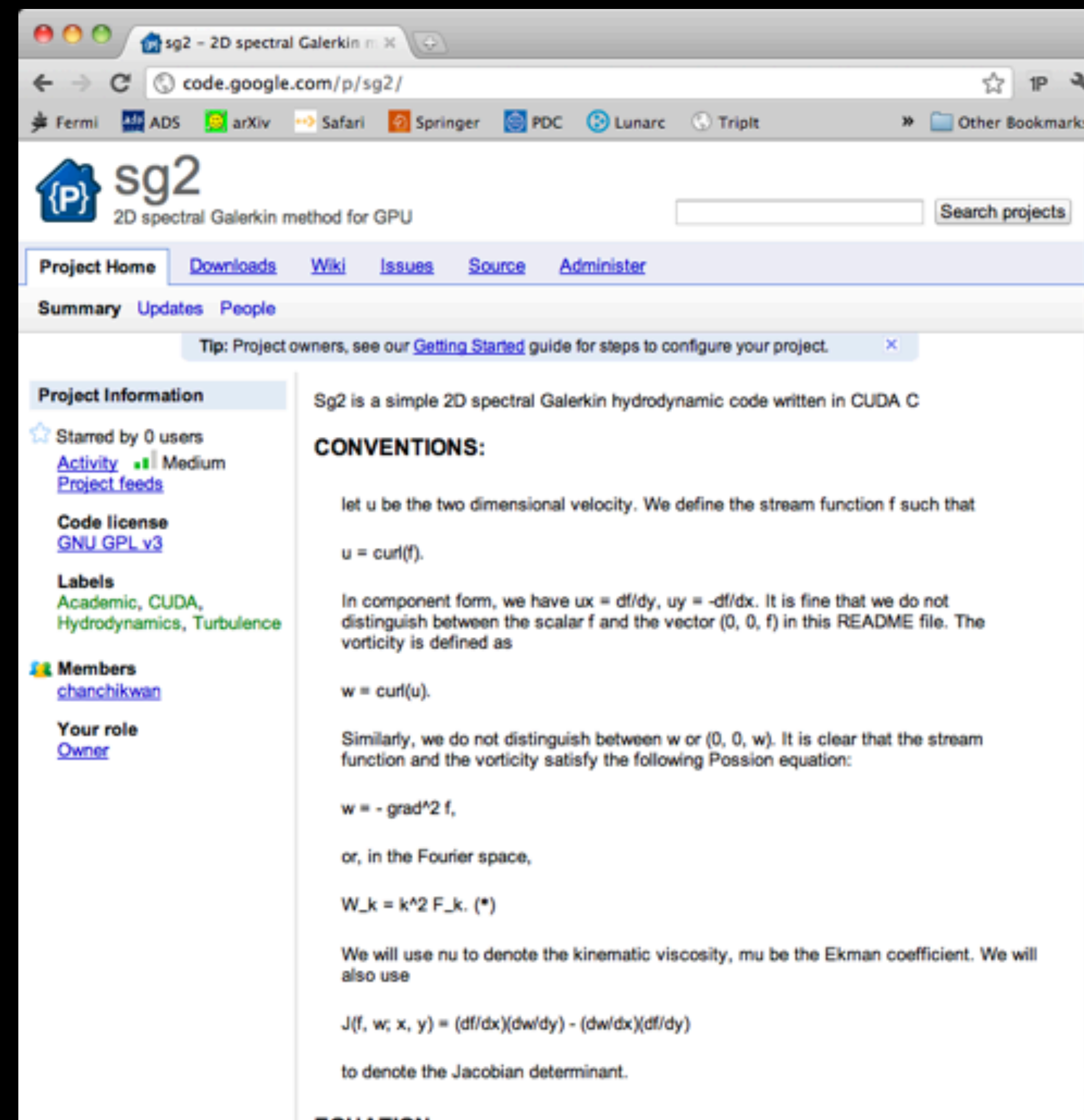
- Simple estimate:

$$E_\infty = f_i^2/2\nu k_1^2$$

$$\tau_E = 1/2\nu k_1^2$$

$$n \sim 10\nu^{-2}$$

- Implemented in CUDA C and runs on GPUs
- Hosted on Google code: `sg2.googlecode.com`



For our nu ~ 10^-4 studies, it takes > 10^9 time steps
GPUs are stream processors that are 10x faster than CPU

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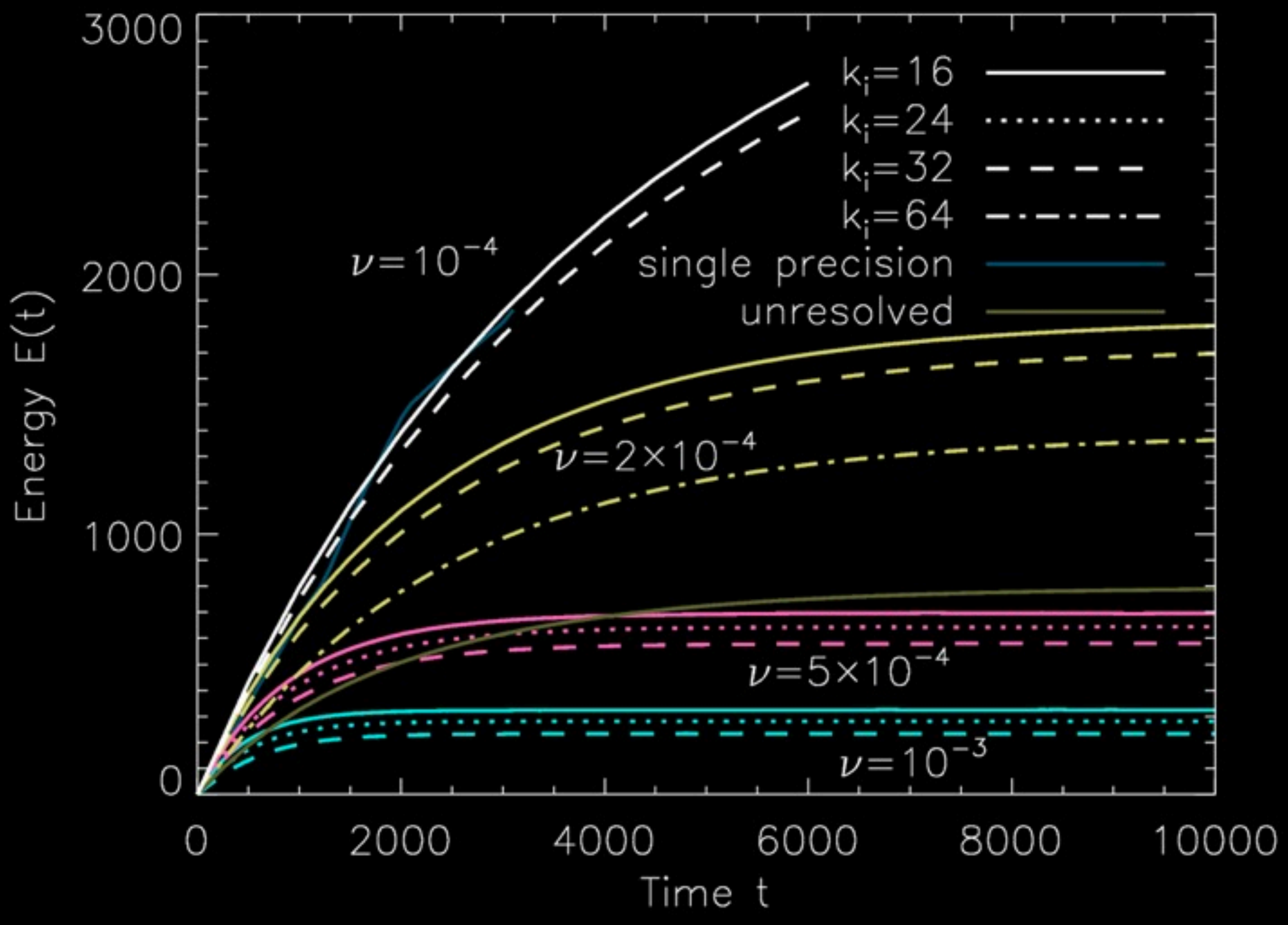
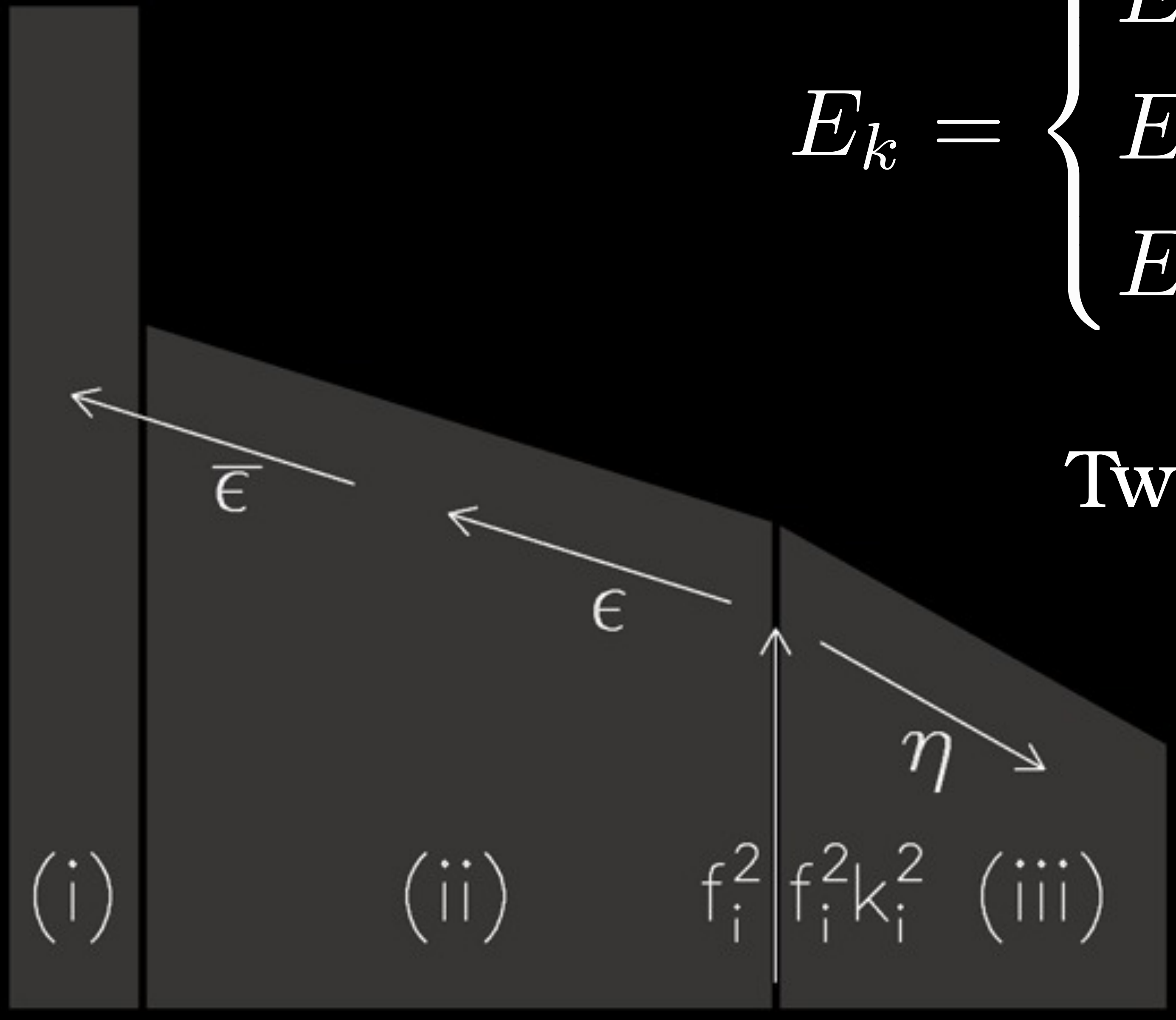
Saturation of Condensate



Summary of the simulations
We used the GPU code and reach saturation
The viscosity dependence makes sense
But what’s going on with the forcing scale?

A Three-Scale Model



$$E_k = \begin{cases} E_1 & k = k_1 \\ E_\mathrm{i}(k/k_\mathrm{i})^{-5/3} & k_1 < k \leq k_\mathrm{i} \\ E_\mathrm{i}(k/k_\mathrm{i})^{-3} & k_\mathrm{i} < k \leq k_\mathrm{d} \end{cases}$$

Two time-dependent values:
$E_1(t)$ and $E_\mathrm{i}(t)$

A simple 3-scale model
f_i^2 and f_i^2 k_i^2 are energy and enstrophy inputs
Only 2 time-dependent values E_1(t) and E_i(t)

Energy and Enstrophy Balances

$$\partial_t \overline{E} = -2\nu k_1^2 \overline{E} + \bar{\epsilon}$$

$$\partial_t E' = -2\nu Z' + f_{\mathrm{i}}^2 - \bar{\epsilon}$$

$$\partial_t Z' = -2\nu P' + f_{\mathrm{i}}^2 k_{\mathrm{i}}^2 - k_1^2 \bar{\epsilon}$$

$$P' = \gamma k_{\mathrm{d}}^2 Z'$$

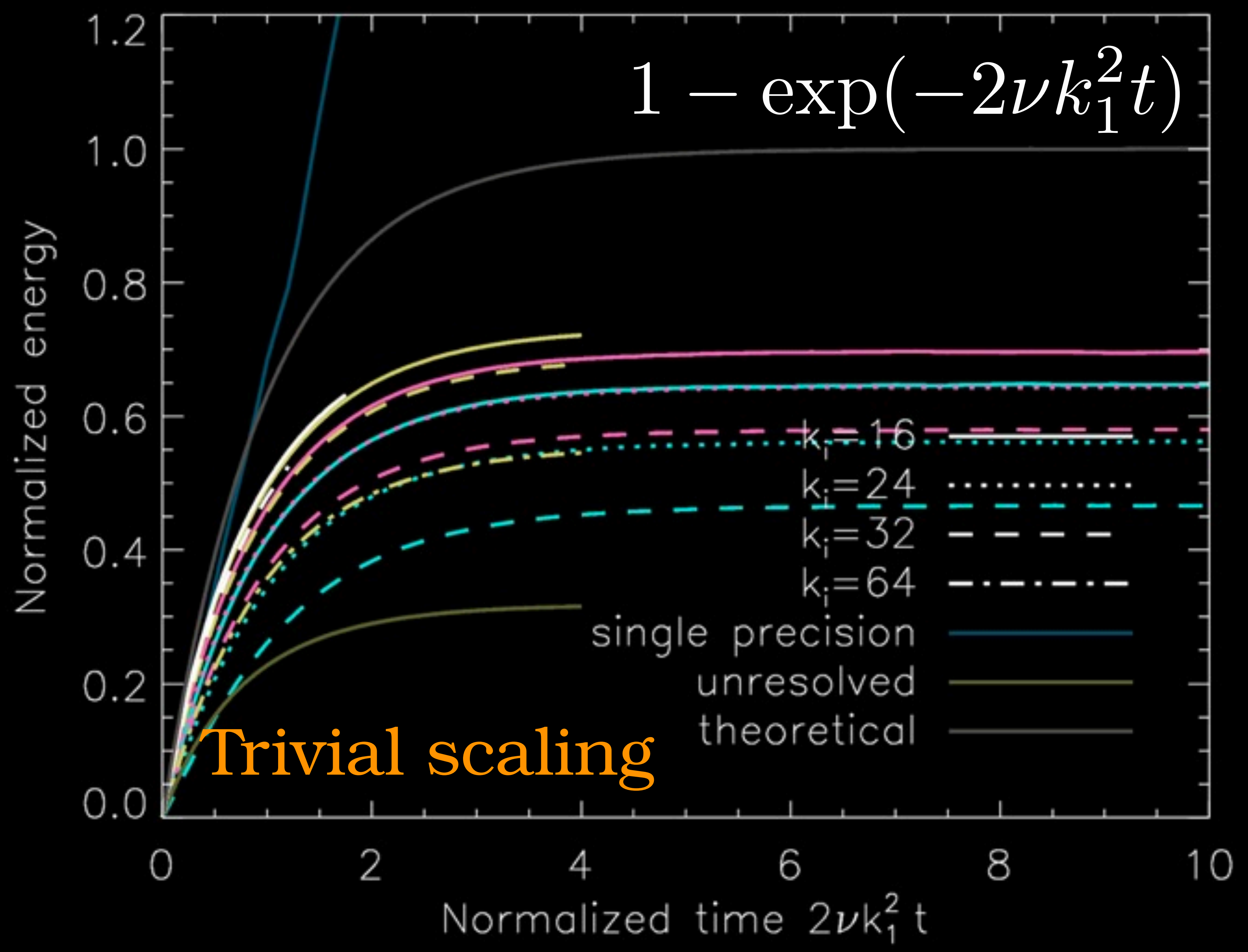
$$Z' = \Gamma k_{\mathrm{i}}^2 E'$$

$$Z'(t) = \frac{f_{\mathrm{i}}^2}{2\nu} \frac{k_{\mathrm{i}}^2 - k_1^2}{\gamma k_{\mathrm{d}}^2 - k_1^2} \Big[1 - \exp\left(-2\nu\gamma k_{\mathrm{d}}^2 t\right) \Big]$$

$$\overline{E}(t) = \frac{f_{\mathrm{i}}^2}{2\nu k_1^2} \frac{\gamma k_{\mathrm{d}}^2 - k_{\mathrm{i}}^2}{\gamma k_{\mathrm{d}}^2 - k_1^2} \Big[1 - \exp\left(-2\nu k_1^2 t\right) \Big]$$

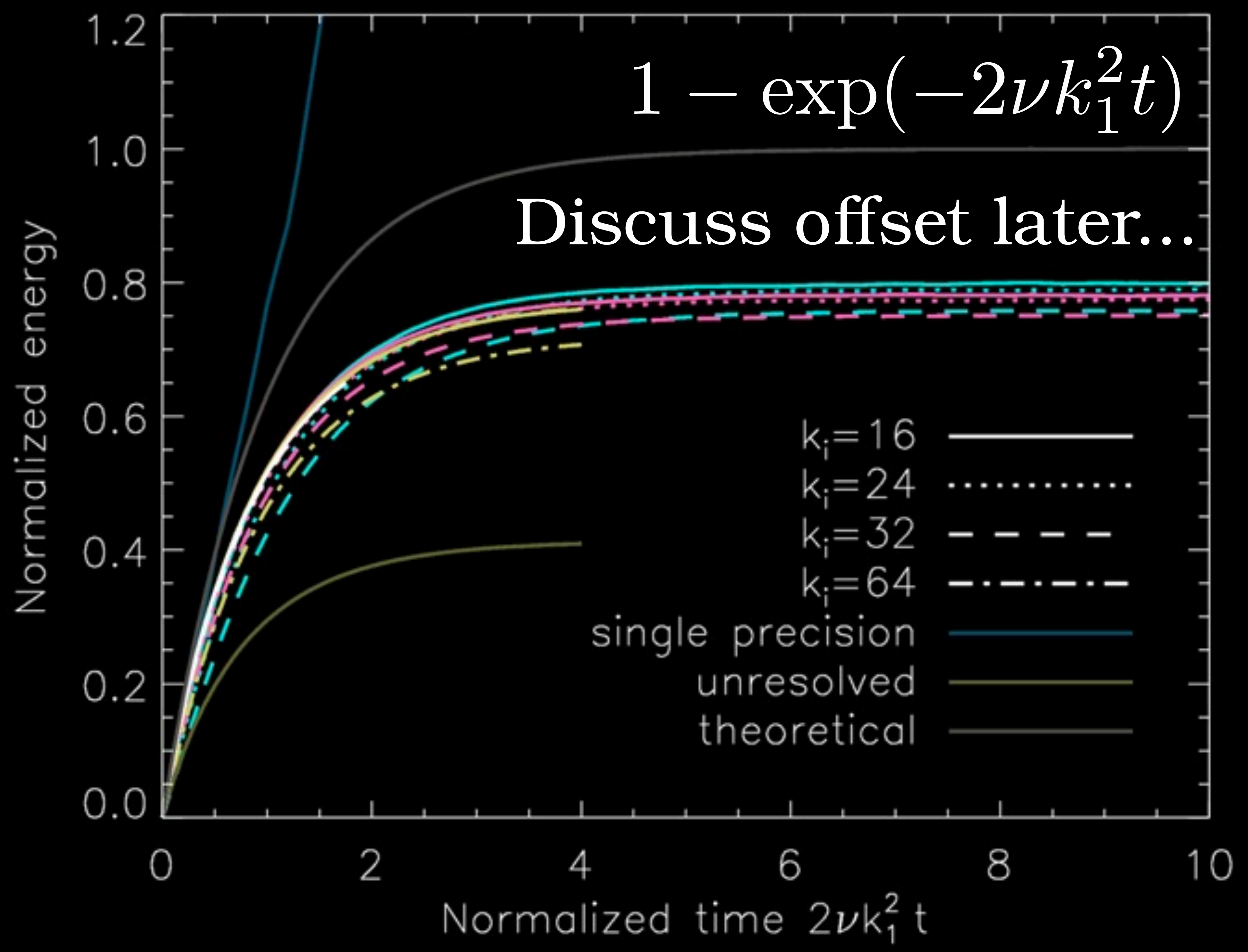
Split the conservation laws into condensate and fluctuation
Use 3-scale model to reduce number of variables
The solutions are similar to our guess, with correction factors

Non-Trivial Agreement!



Normalize solution, curves are on top of each other
3-scale model is better than the simple scaling argument
There is an overall offset... will talk about it later

Non-Trivial Agreement!



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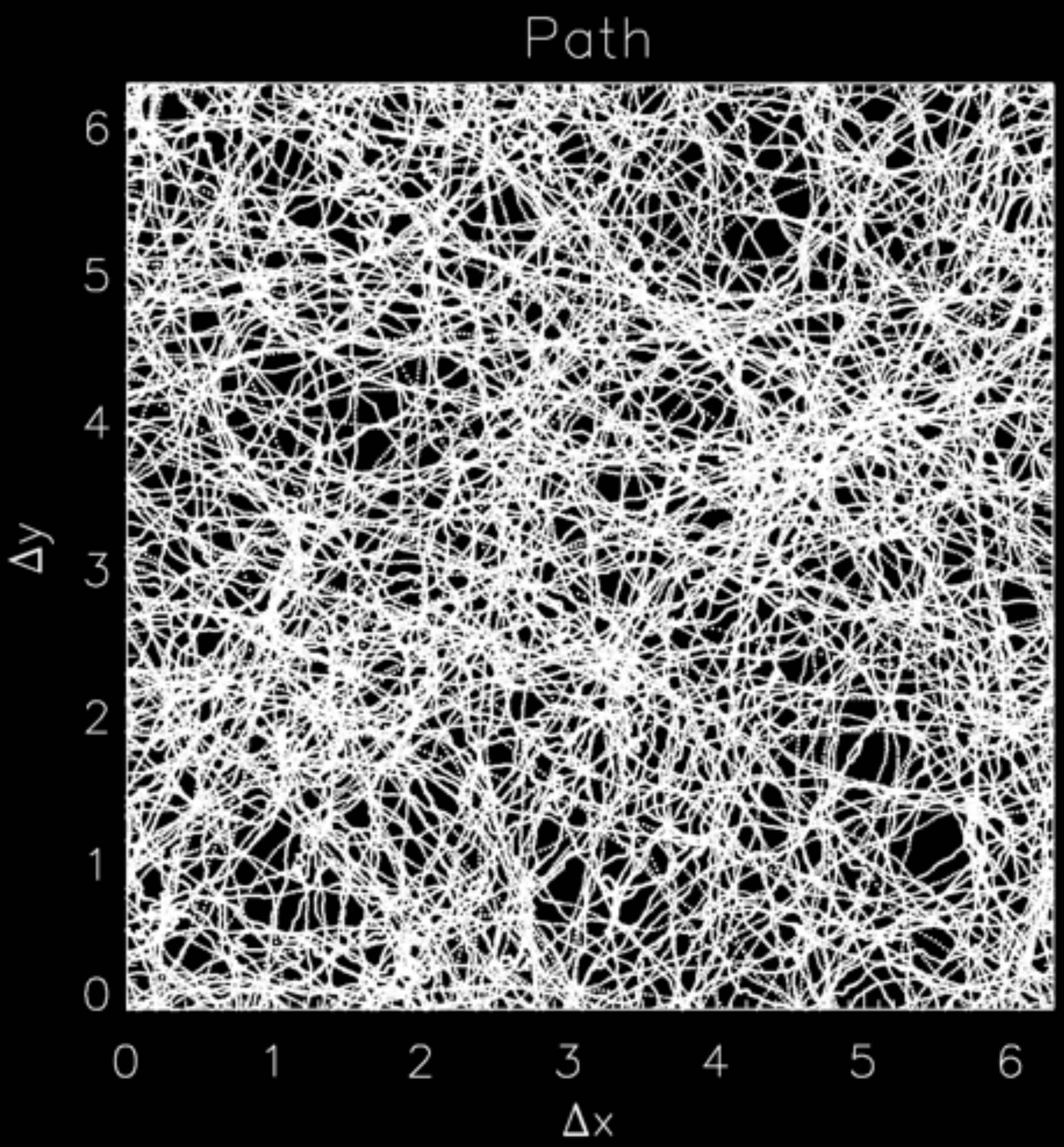
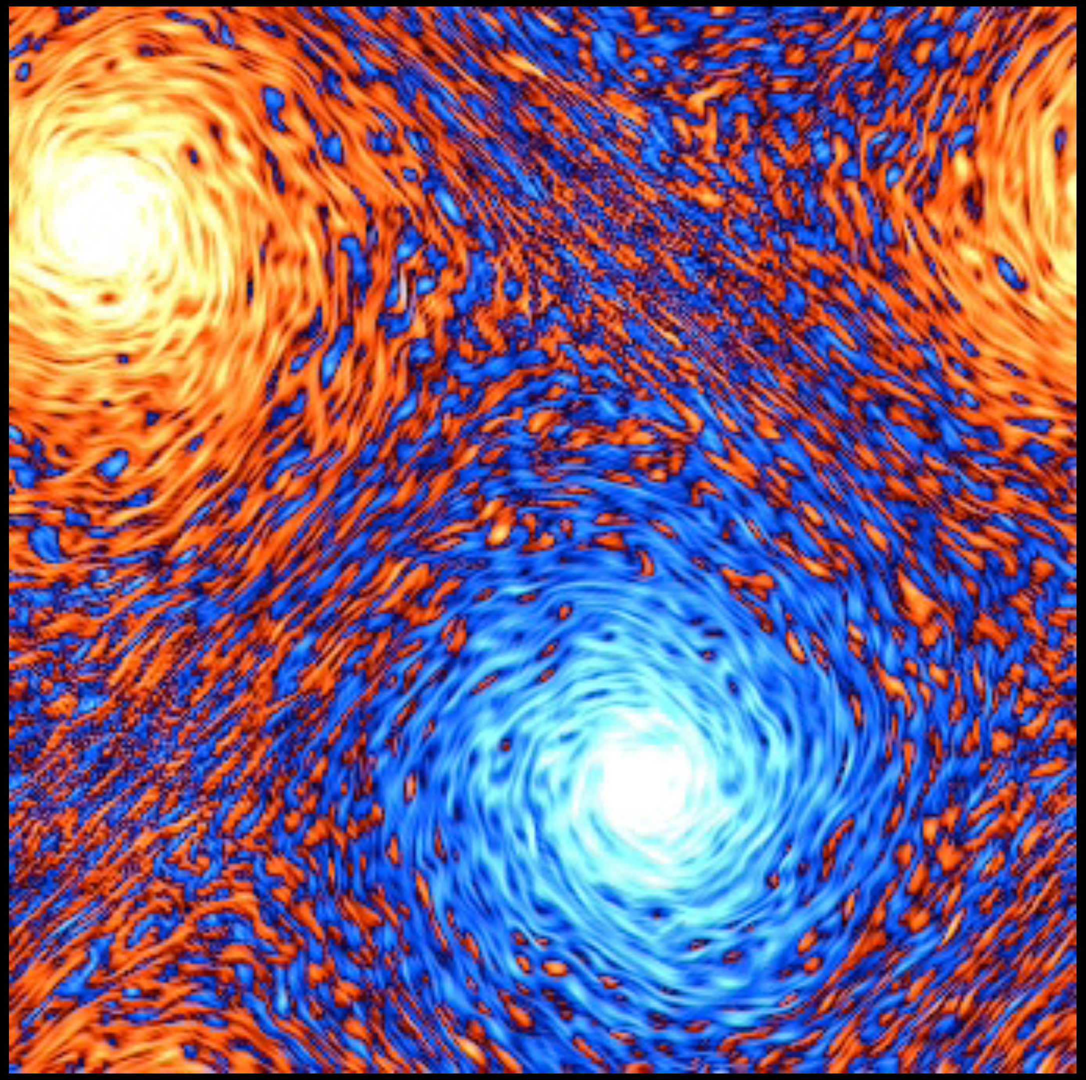
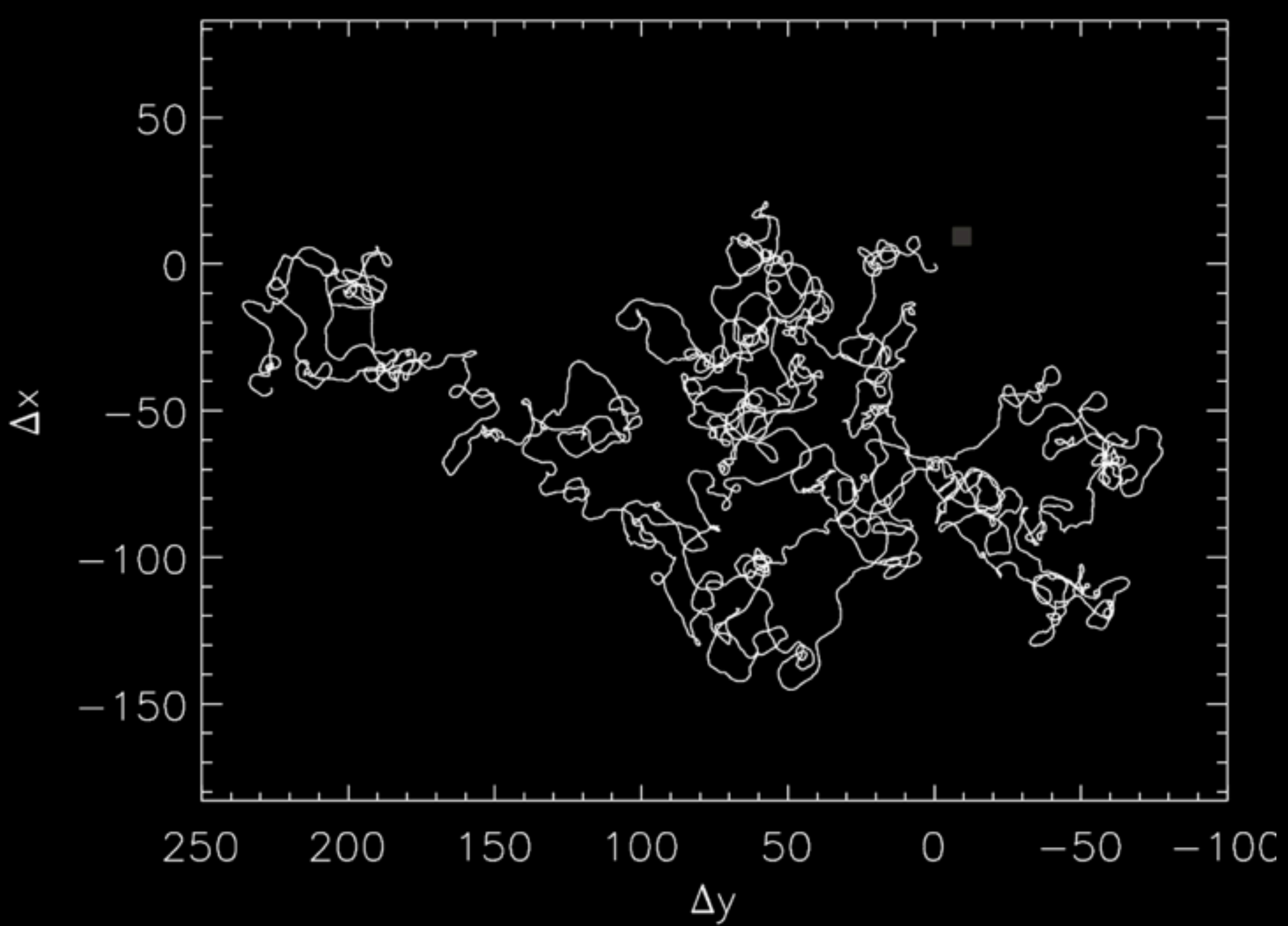
Condensation Vortices Movement





Saturation level is not the full story, need position
Look at trajectory

Unfolded Path



Unfolded path
Small box indication the domain size
It looks like random walk

Turbulent "Velocity"

$$\text{Mean vorticity} \quad \overline{\omega}(\boldsymbol{x}) = \sum_{|\boldsymbol{k}|=k_1} \omega_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\partial_t \overline{\omega} + \boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}}\,\overline{\omega}) = -\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\mathcal{F}}} + \nu\nabla^2\overline{\omega}$$

$$\overline{\mathcal{F}}_i \equiv \overline{u_i\omega} - \overline{u}_j\overline{\omega} = \alpha_i\overline{\omega} + \beta_{ij}\partial_j\overline{\omega}$$

$$(\partial_t + \alpha_i\partial_i)\overline{\omega} = (\nu\delta_{ij} - \beta_{ij})\partial_i\partial_j\overline{\omega}$$

$$\overline{\boldsymbol{\mathcal{F}}} \sim \boldsymbol{u}_{\sqrt{2}k_1}\overline{\omega} \qquad \langle\alpha^2\rangle \sim k_1 E_{\sqrt{2}k_1} = \frac{f_{\mathrm{i}}^{4/3}}{2k_1^{2/3}}$$

Construct mean field theory
Transport coefficients alpha and beta
alpha is velocity
Estimate alpha by condensate-fluctuation interaction

Simplest Stochastic Model

- Ornstein-Uhlenbeck process

$$\partial_t \boldsymbol{\alpha} = -\boldsymbol{\alpha}/\tau_\alpha + \boldsymbol{\phi}$$

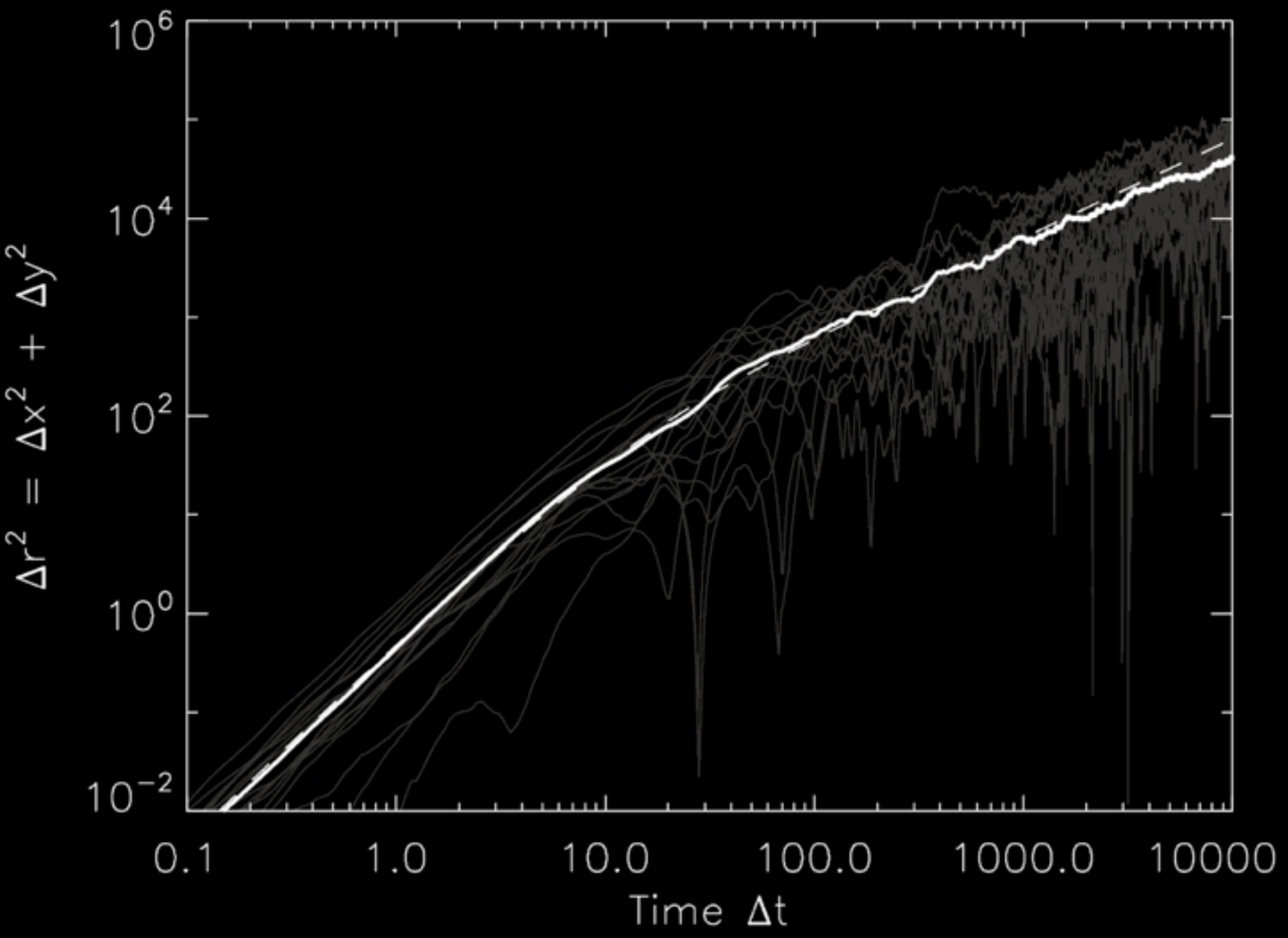
- Fit a time scale

$$\begin{aligned}\langle \boldsymbol{\alpha}(s) \cdot \boldsymbol{\alpha}(t) \rangle &= \alpha(0)^2 e^{-(t+s)/\tau_\alpha} \\ &\quad + \xi f_{\mathrm{i}}^2 \frac{\gamma k_{\mathrm{d}}^2 - k_{\mathrm{i}}^2}{\gamma k_{\mathrm{d}}^2 - k_1^2} \tau_\alpha \left[e^{-(t-s)/\tau_\alpha} - e^{-(t+s)/\tau_\alpha} \right]\end{aligned}$$

- “Inertial” Brownian motion

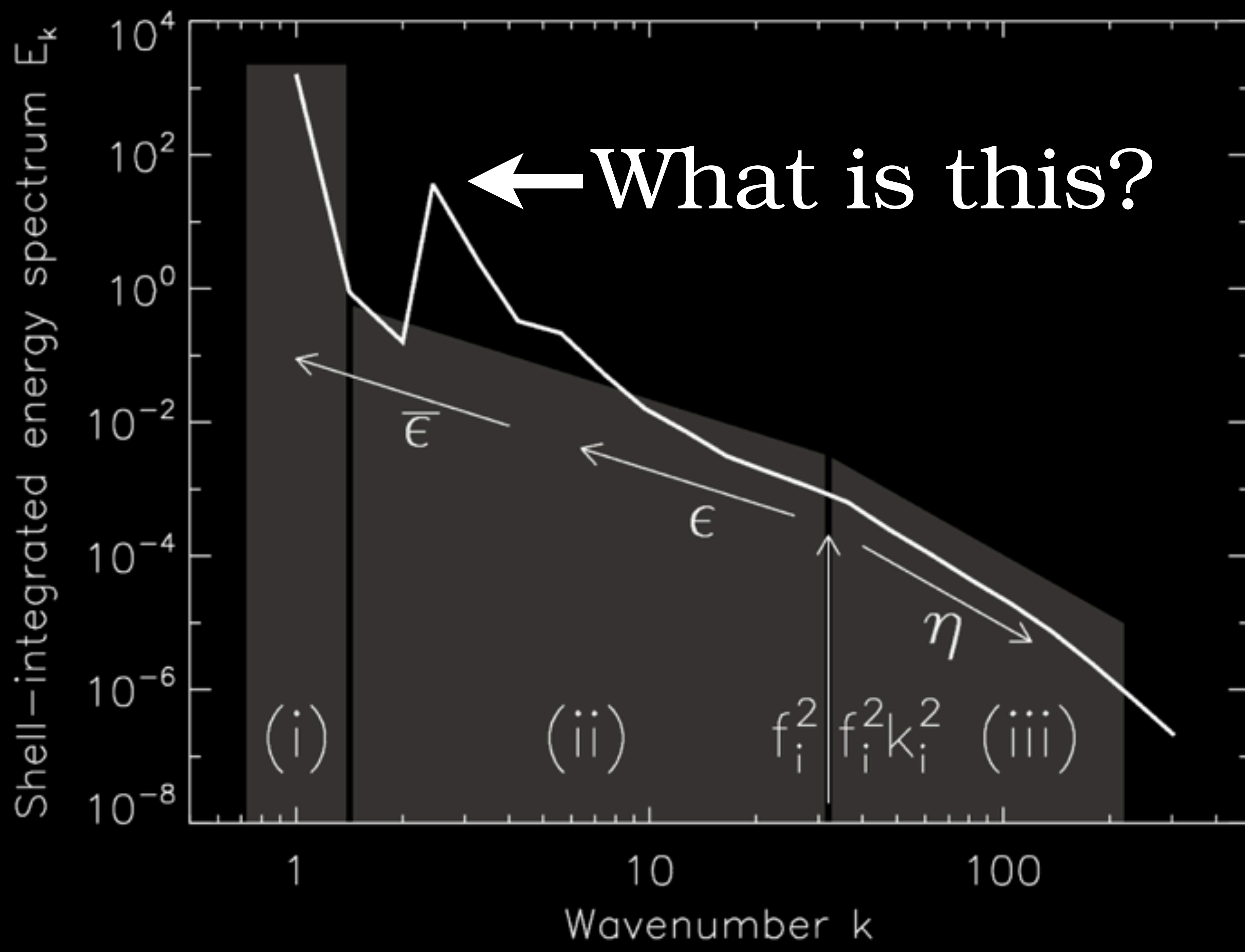
The simplest model we can come up with
phi is effective forcing
There is one fit parameter xi
“Inertial” Brownian motion

“Inertial” Brownian Motion



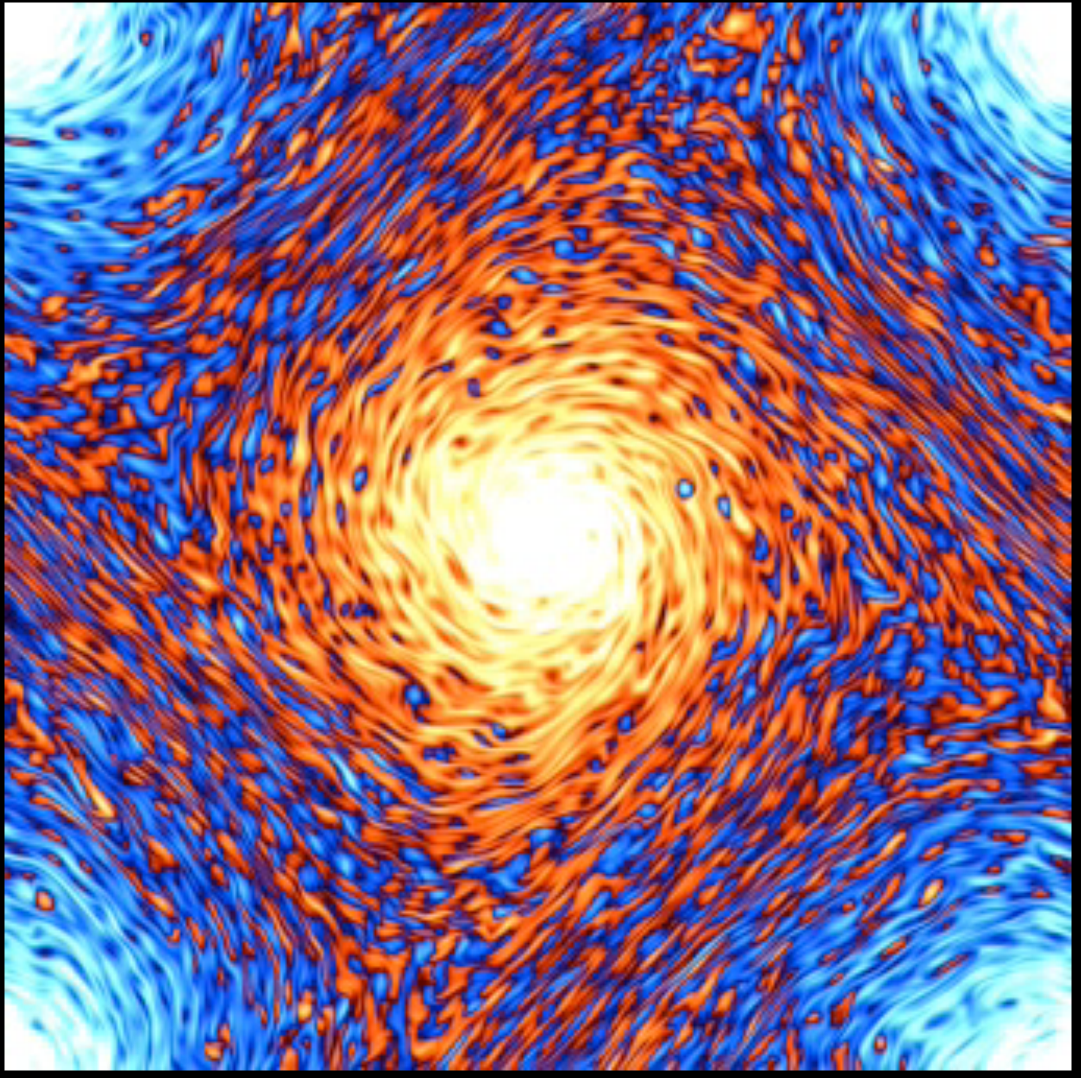
Dark grey are 16 different runs
Solid white is average
The initial normalization given by the 3-scale model
The late time diffusion is just fitting, but it is still nice

Modeled vs. Numerical Spectra



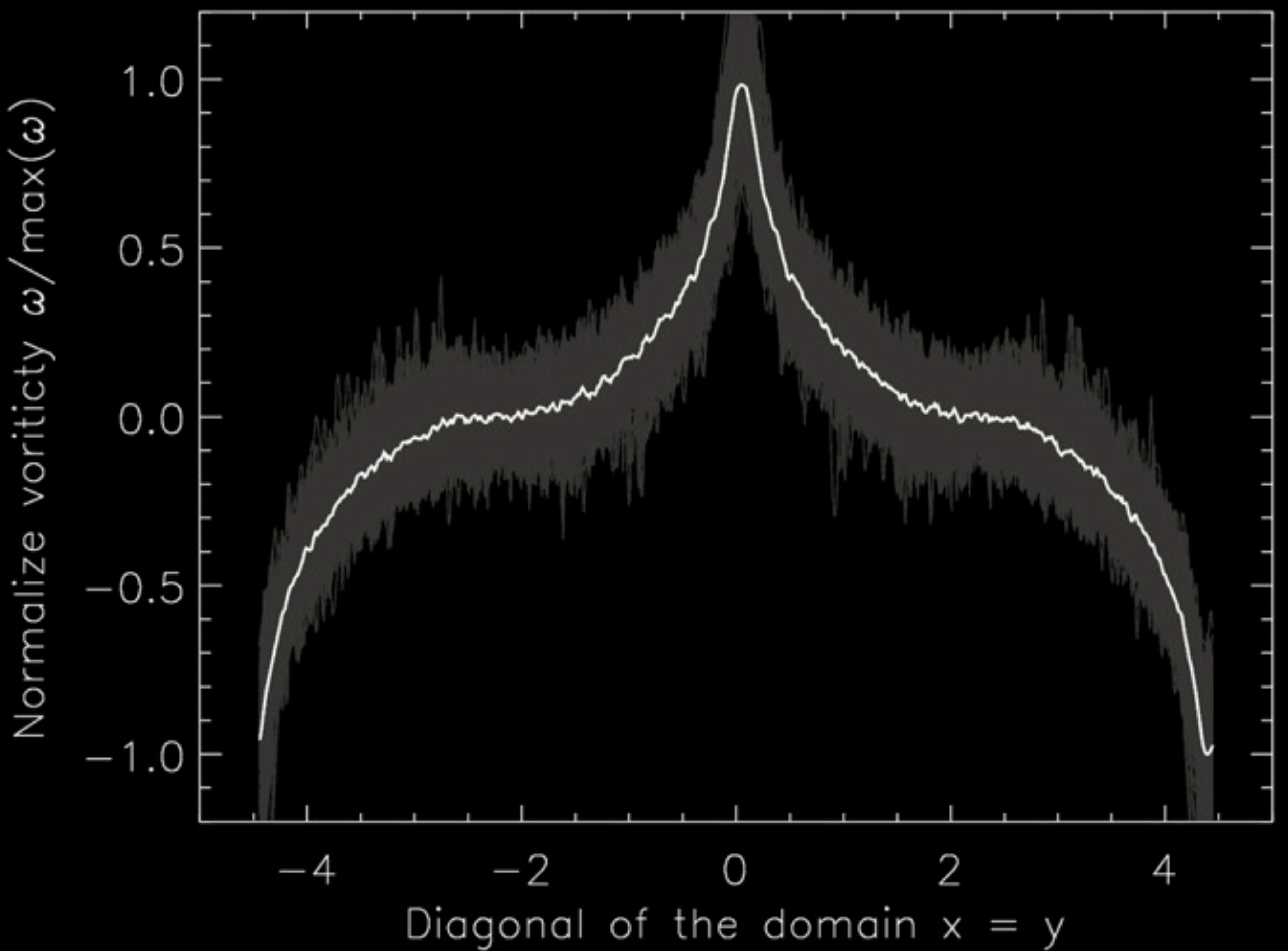
Remember the offset from the prediction?
Compare the simulation with model
On top of the broken power law there is a peak

Shape of Condensate Vortices



If we can compute the trajectory, we can do the opposite
Fix the vortices, compute the average
The shape is non-trivial

Shape of Condensate Vortices



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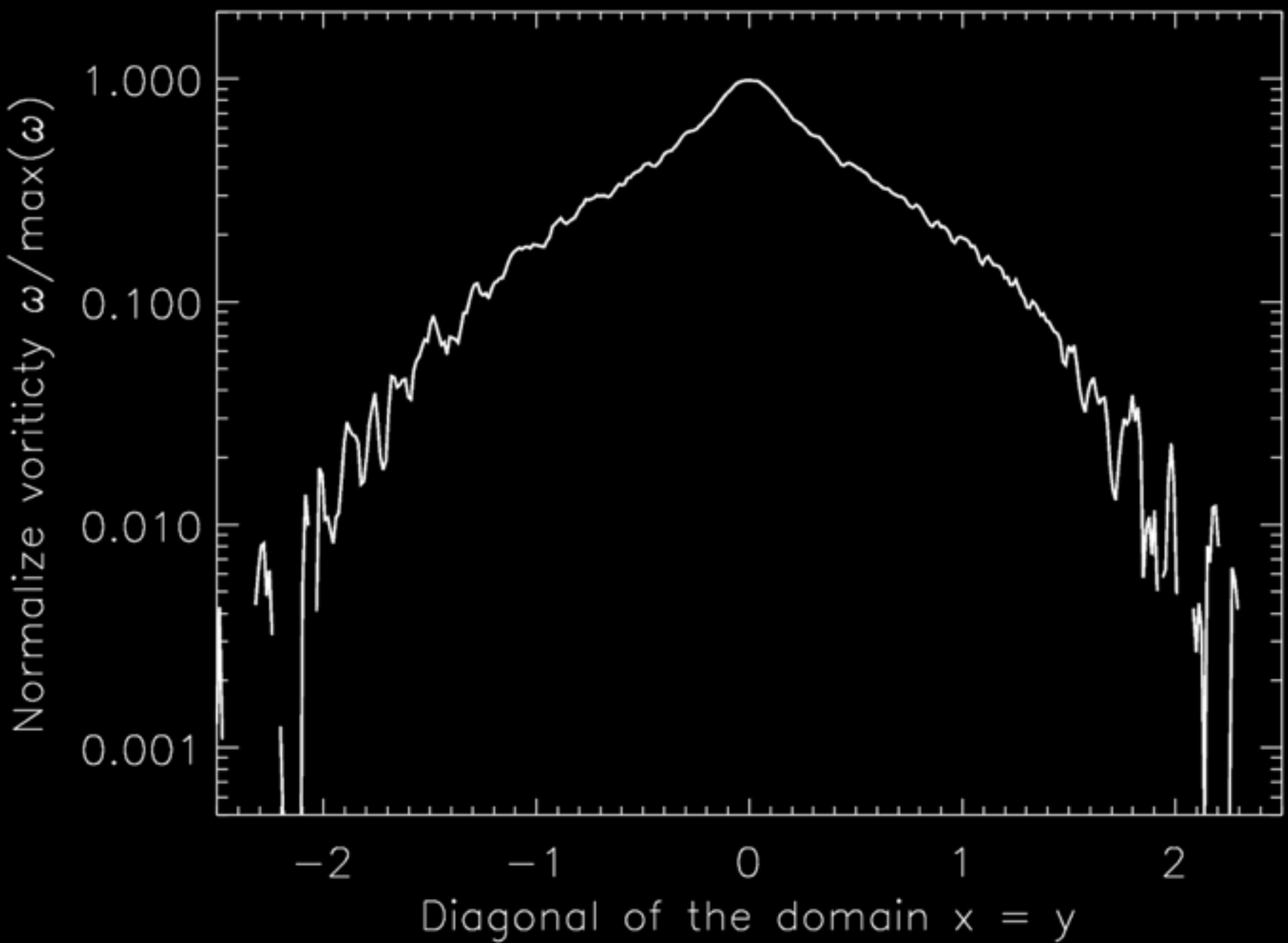

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Symmetry of the Vortices

- Assuming $\tilde{\omega} \approx f(\overline{\omega})$, the "shape function" $f(x)$ is odd
- Taylor expanding,

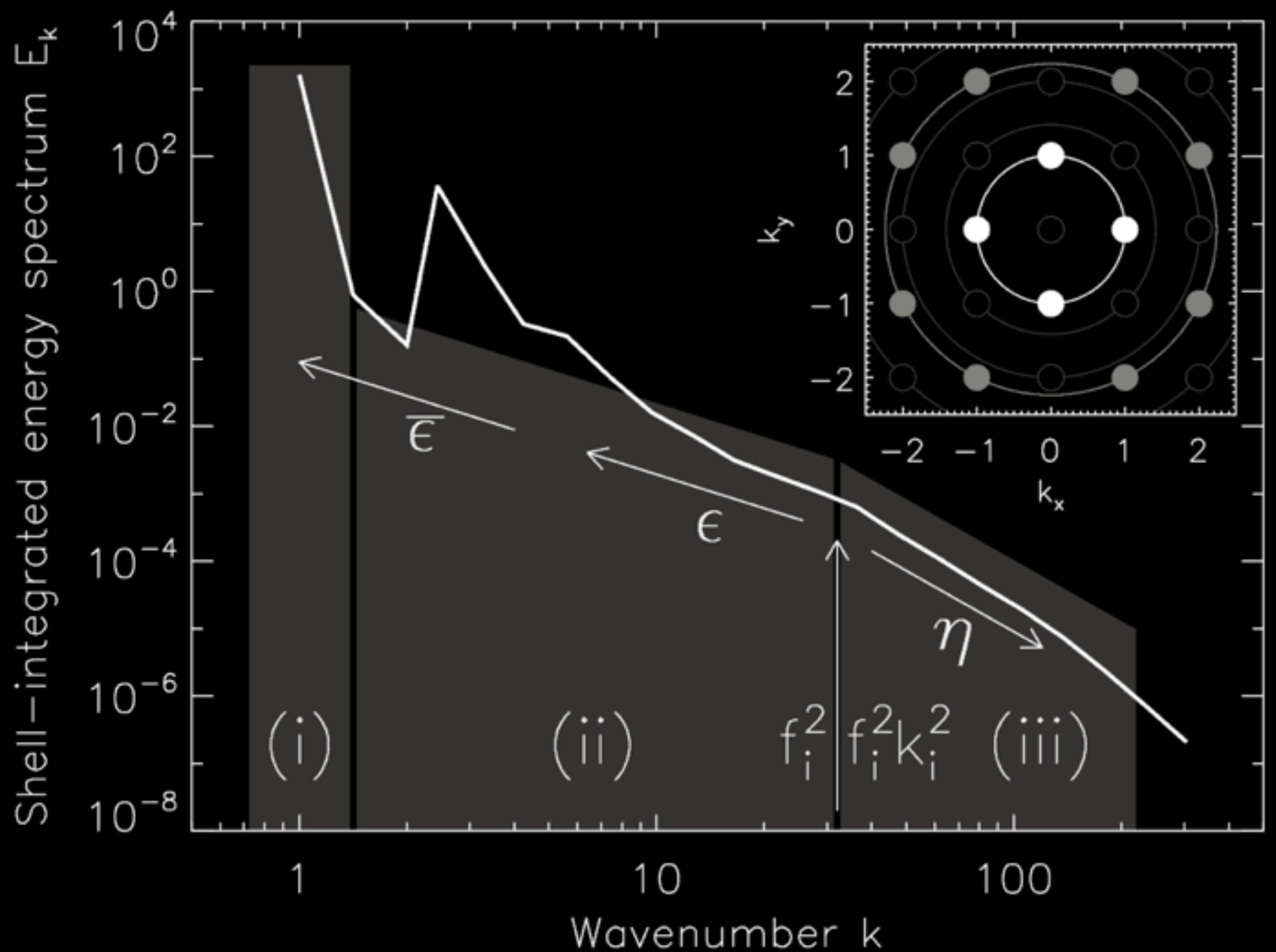
$$f(x) \approx f_0' x + \frac{1}{3!} f_0^{(3)} x^3 + \frac{1}{5!} f_0^{(5)} x^5 + \dots$$

- Hence,

$$\begin{aligned} f(\overline{\omega}) \approx & \dots e^{ix} + \dots e^{iy} \\ & + \dots e^{3ix} + \dots e^{2ix+iy} + \dots e^{ix+2iy} + \dots e^{3iy} \\ & + \dots \end{aligned}$$

Use a symmetry argument
The large scale vortex occupy modes with odd k_x + k_y

Higher Harmonics



In inset, white circles are k_1 modes, grey are first harmonics
The harmonics increase the enstrophy Z‘
So they decrease the condensate energy E

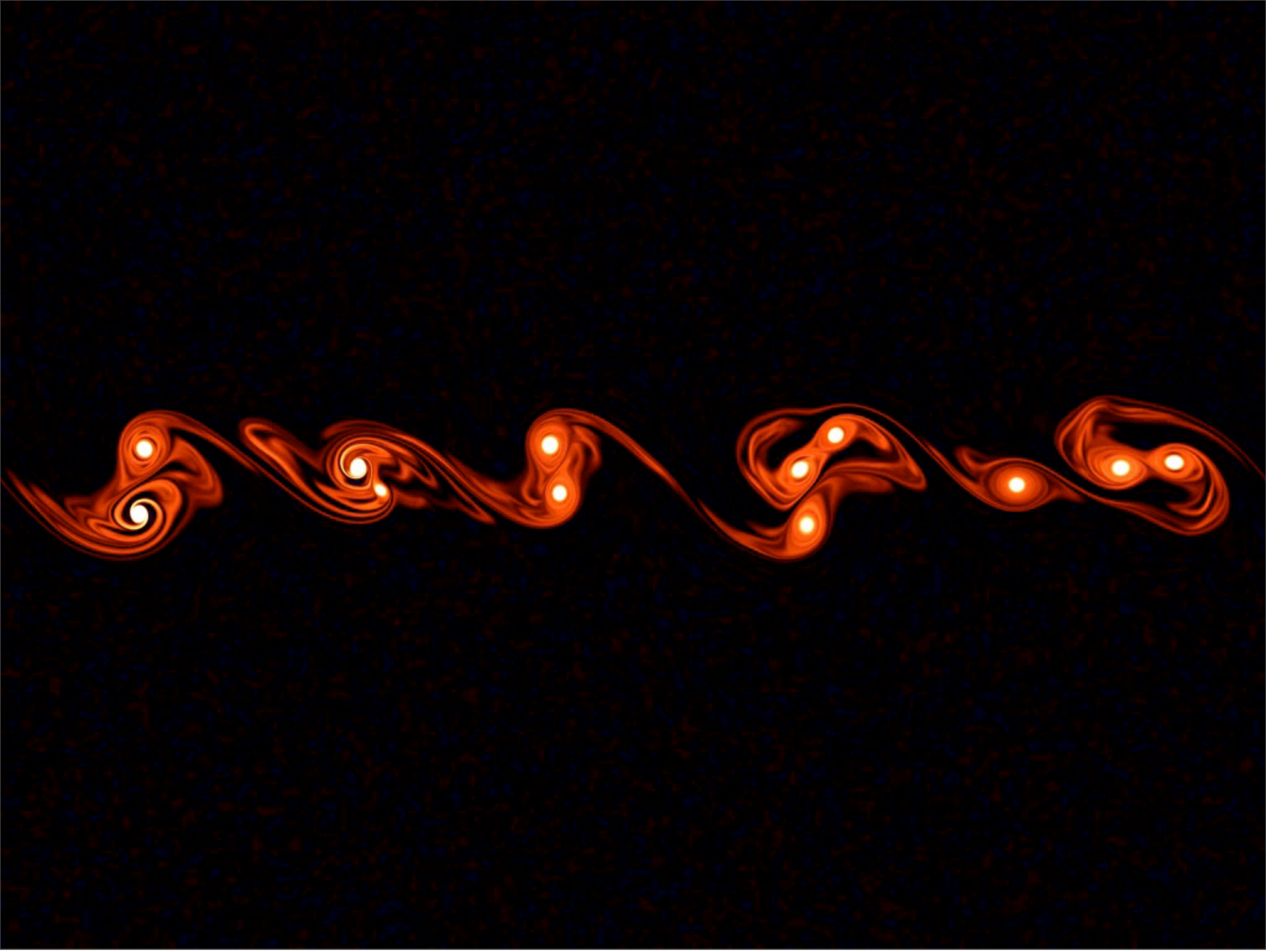
Summary

- The rumor is not wrong, it is just not useful!

$$\lim_{t\to\infty}\left(\lim_{\nu\to 0}\overline{E}\right) = f_{\mathrm{i}}^2 t \quad \text{vs.} \quad \lim_{\nu\to 0}\left(\lim_{t\to\infty}\overline{E}\right) = \frac{f_{\mathrm{i}}^2}{2\nu k_1^2}$$

- Condensate saturates at viscous time scale
- Three-scale model predicts/explains saturation level
- Condensate movement is “inertial” Brownian
- Higher harmonics probably offsets the saturation
- GPUs rock! `http://sg2.googlecode.com`

Uriel: order of taking limit is important!



Show off the GPU code, 4096^2 simulation of KH instability, done in couple hours