NELKIN SCALNG IN "BURGULENCE"

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3D Navier-Stokes turbulence: scaling exponents

Navier-Stokes equation: $$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \nu\nabla^2\boldsymbol{v}, \qquad \nabla\cdot\boldsymbol{v} = 0$$

Longitudinal velocity increments: $$\delta v_{\|}(\boldsymbol{x},\, \boldsymbol{h}) \equiv [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{v}(\boldsymbol{x})]\cdot\frac{\boldsymbol{h}}{h}$$

Structure functions: $$S_p(h) \equiv \left\langle \left(\delta v_{\|}(\boldsymbol{x},\, \boldsymbol{h})\right)^p \right\rangle \propto h^{\zeta_p}$$ (at high Reynolds numbers R and inertial-range separations *h*)

Nelkin scaling: *p*th order "***gradmoment***" $\equiv \langle (\nabla v)^p \rangle \sim R^{\chi_p}$

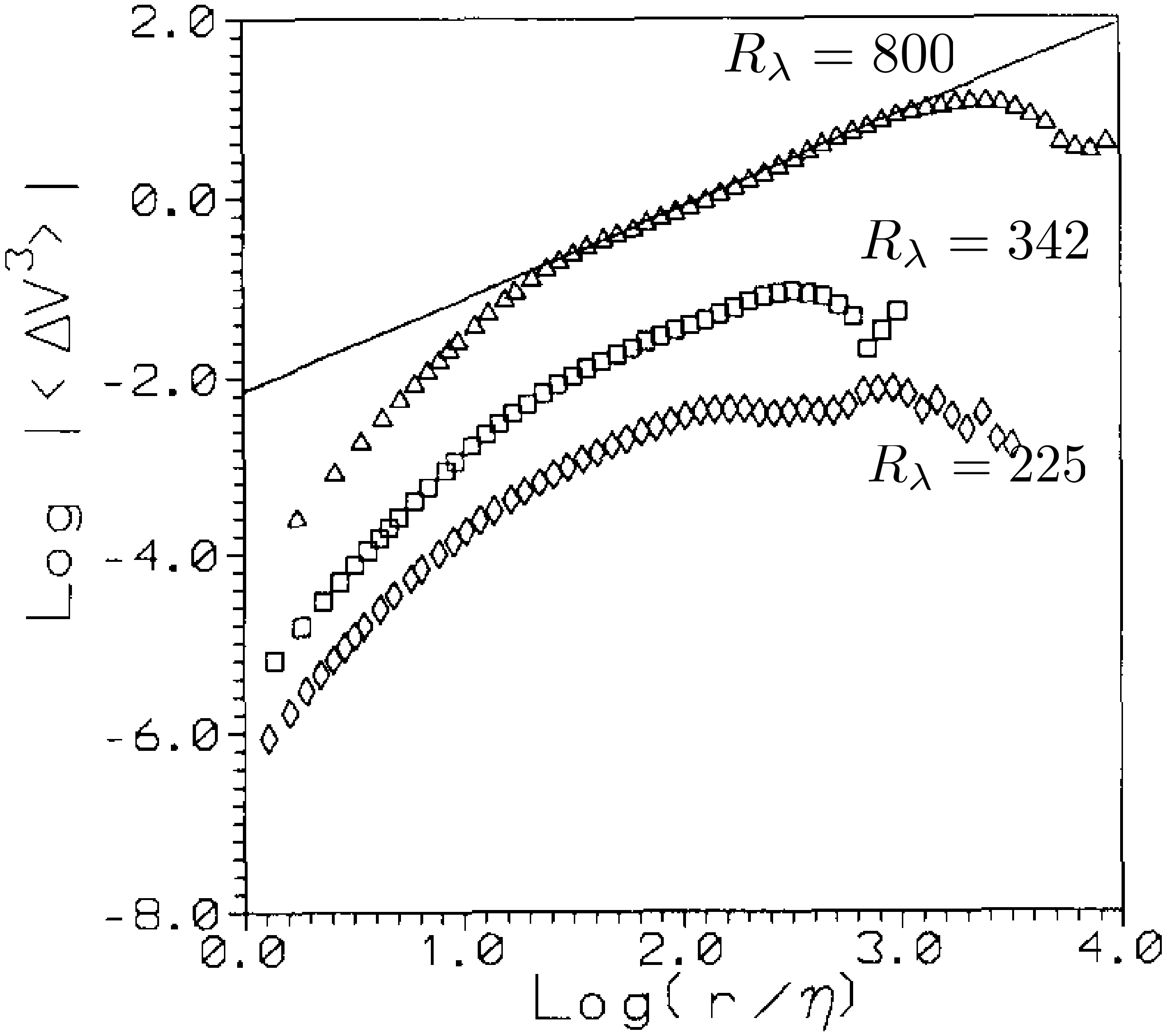
These Nelkin exponents χ_p are expressible in terms of the multifractal structure function exponents ζ_p.

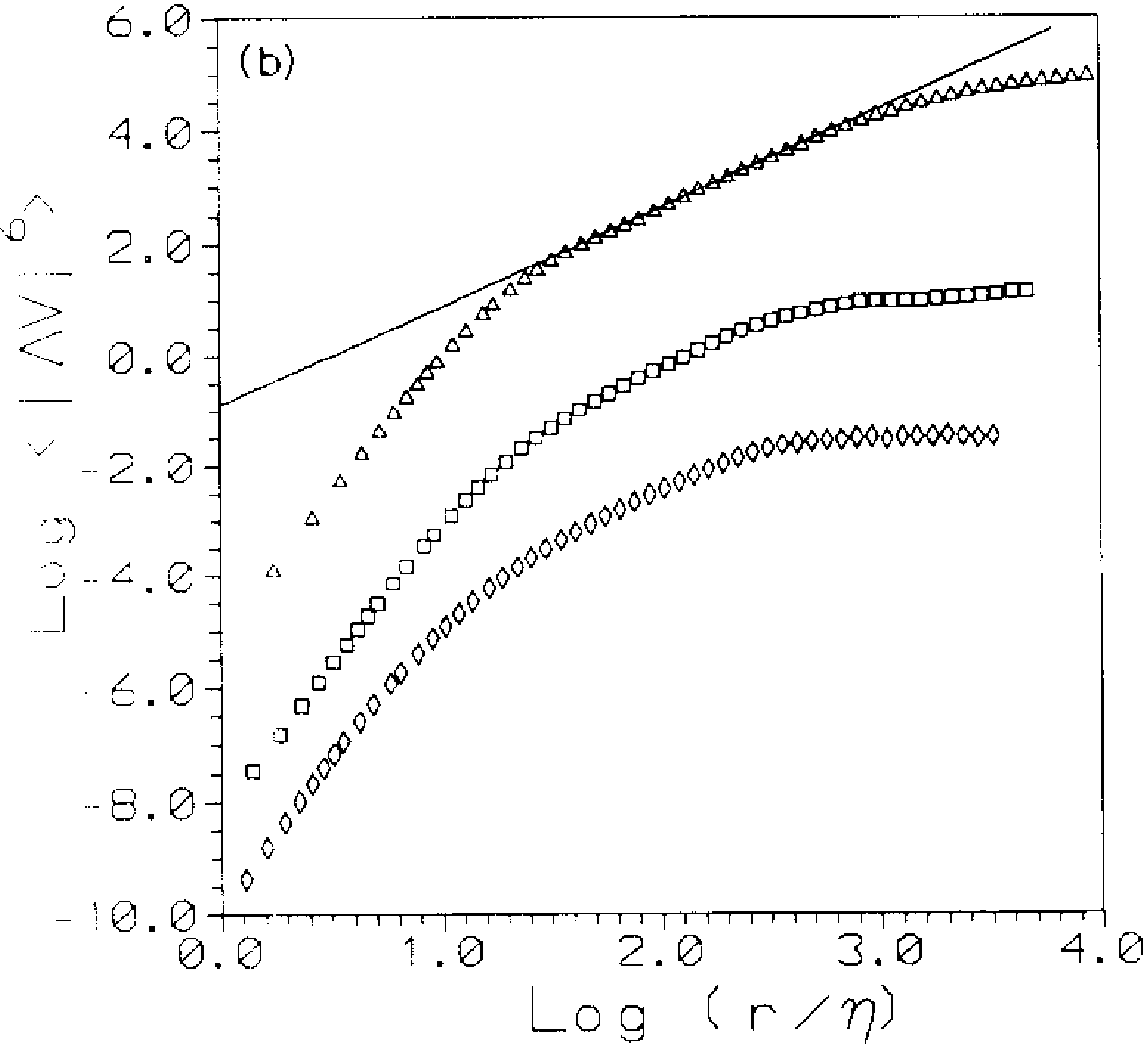
Using very highly resolved 3D direct numerical simulation, it has been checked by Schumacher, Sreenivasan and Yakhot that not only such scaling is present, but is already seen at Reynolds numbers, well below those where structure functions show any inertial-range scaling.

M. Nelkin, Phys. Rev. A **42**, 7226 (1990).

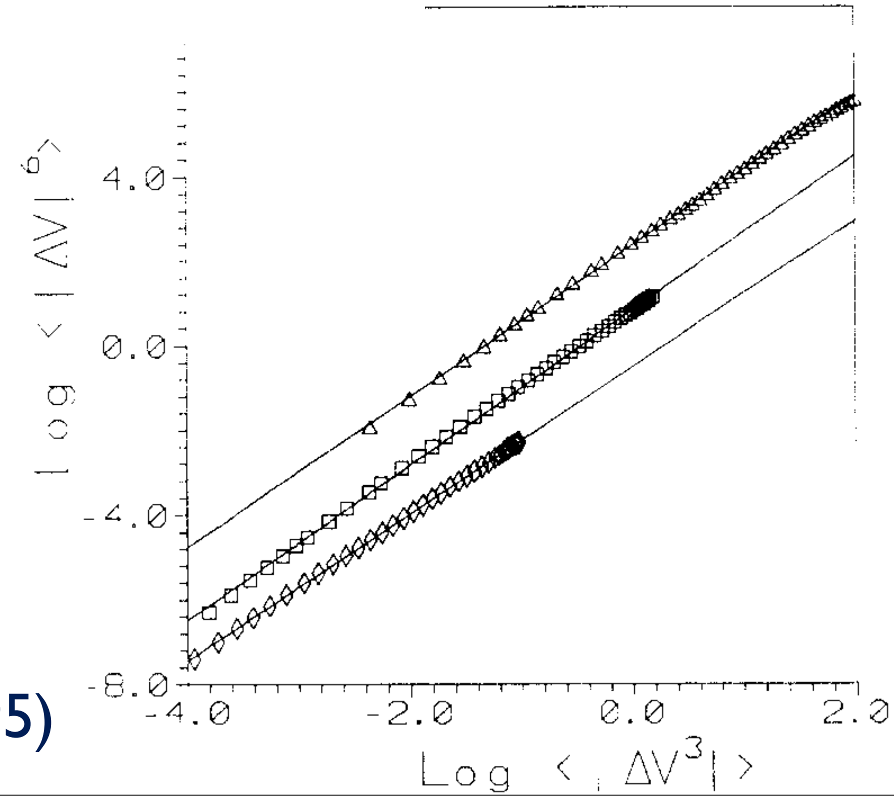
Schumacher et. al., New J. Phys. **9**, 89 (2007).

STANDARD PLOTTING VS ESS PLOTTING





UV ←

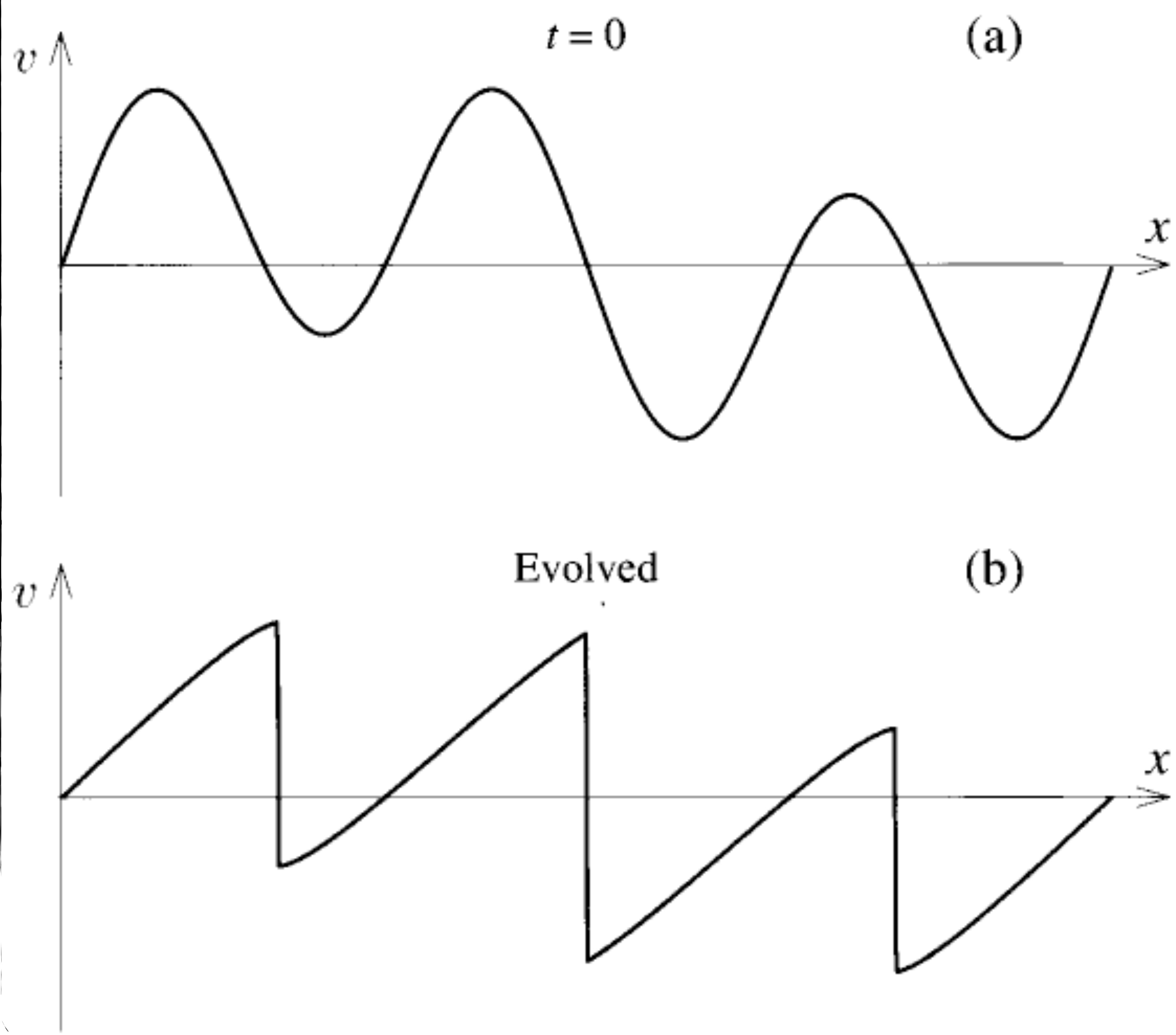


→ IR

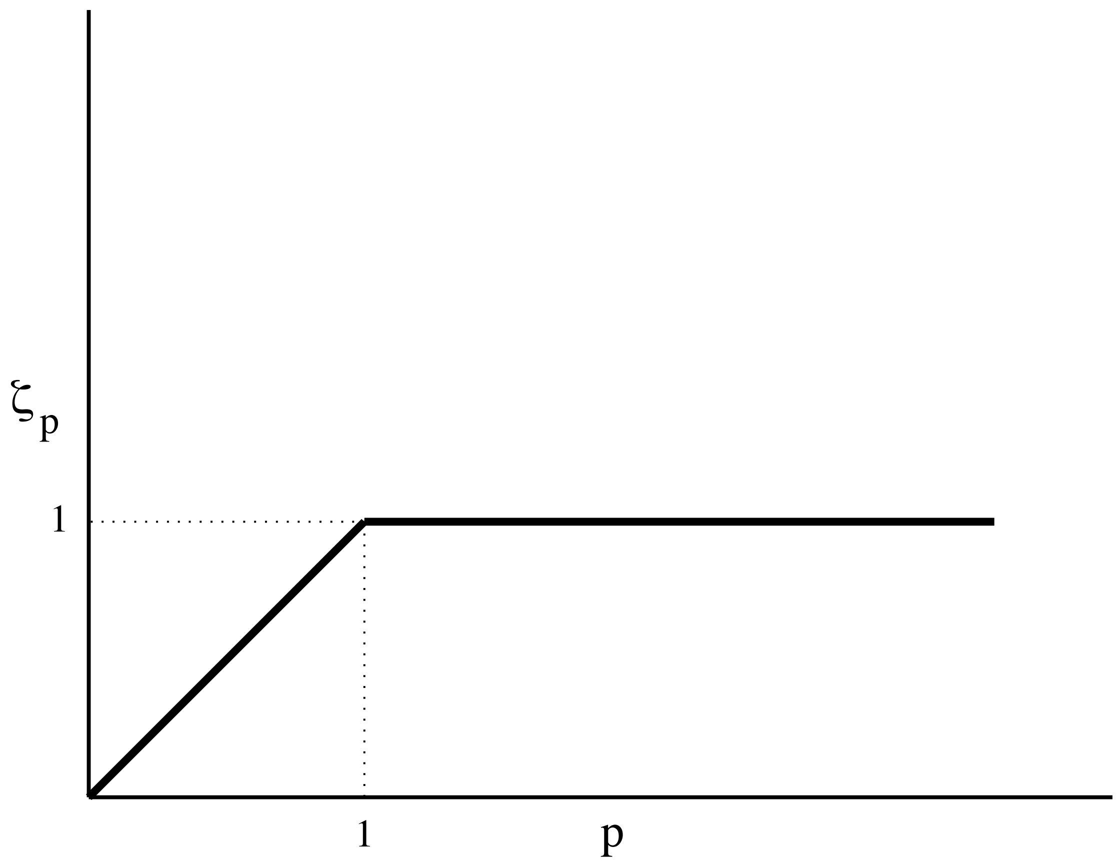
Benzi et. al., Physica D **80**, 385 (1995)

UNDERSTANDING ESS VIA THE BURGERS MODEL

$$\partial_t u + u\partial_x u = \nu \partial_x^2 u$$

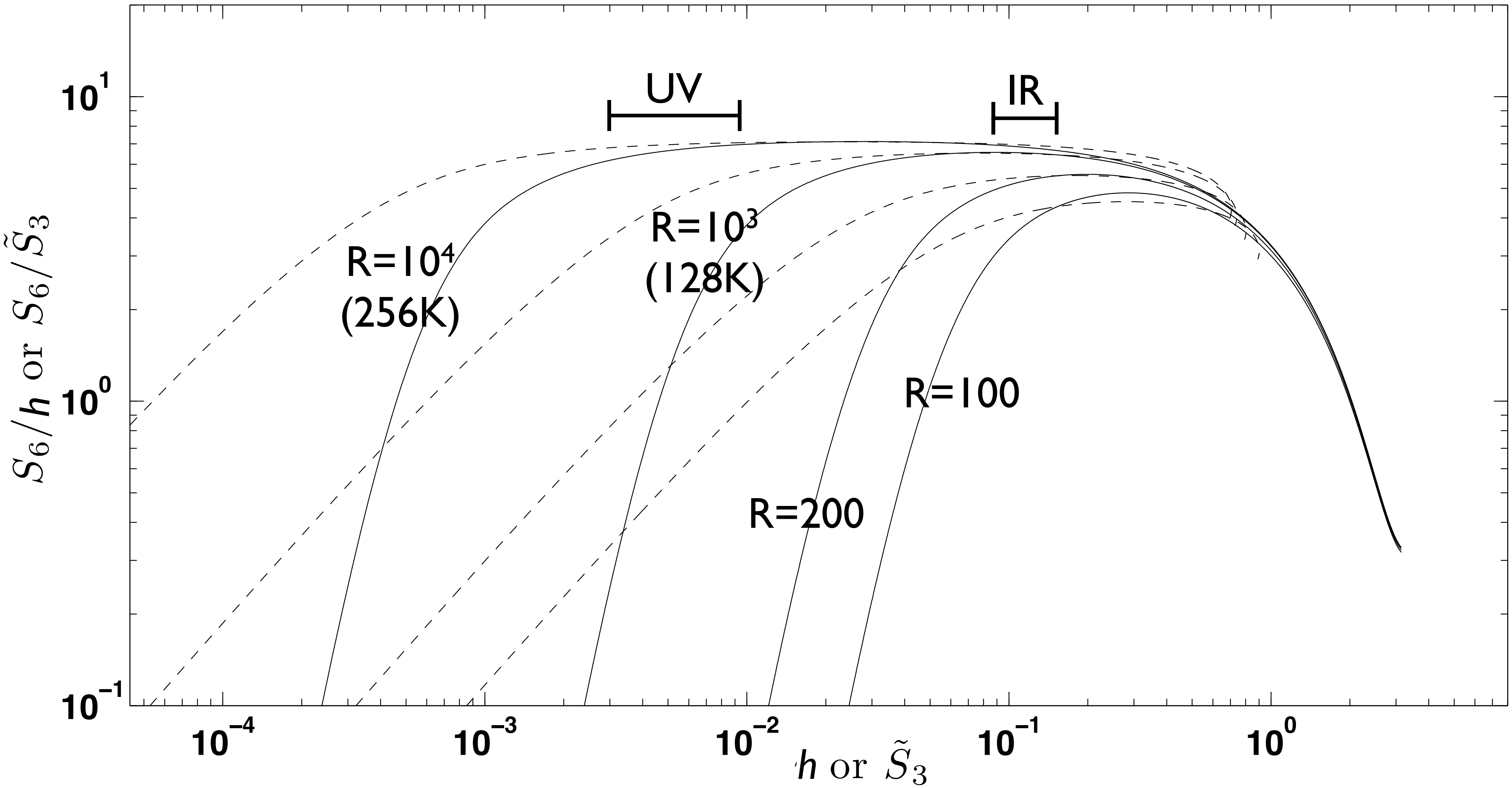


$$S_p \equiv (1/2\pi)\int_0^{2\pi} dx\,[u(x+h)-u(x)]^p \sim h^{\zeta_p}$$



“It has been shown that ESS does not hold for the Burgers equation...” (Benzi et. al. 1995)

ESS WORKS FOR BURGERS (HIGH RESOLUTION)



Compensated sixth-order structure function in standard and ESS plotting

For Burgers: $S_3(h) = -12\varepsilon h + \mathrm{hot};$ Here ESS uses: $\tilde{S}_3(h) \equiv \dfrac{S_3(h)}{-12\varepsilon}$

Chakraborty, Frisch and Ray, J. Fluid Mech. **649**, 275-285 (2010).

GRADMOMENTS: MOMENTS OF VELOCITY GRADIENTS

Can moderate Reynolds number scaling (with the Reynolds number) for moments of velocity gradients, reported by Schumacher, Yakhot and Sreenivasan, be explained along similar lines?
Yes, and it can be combined with an ESS-type plotting.

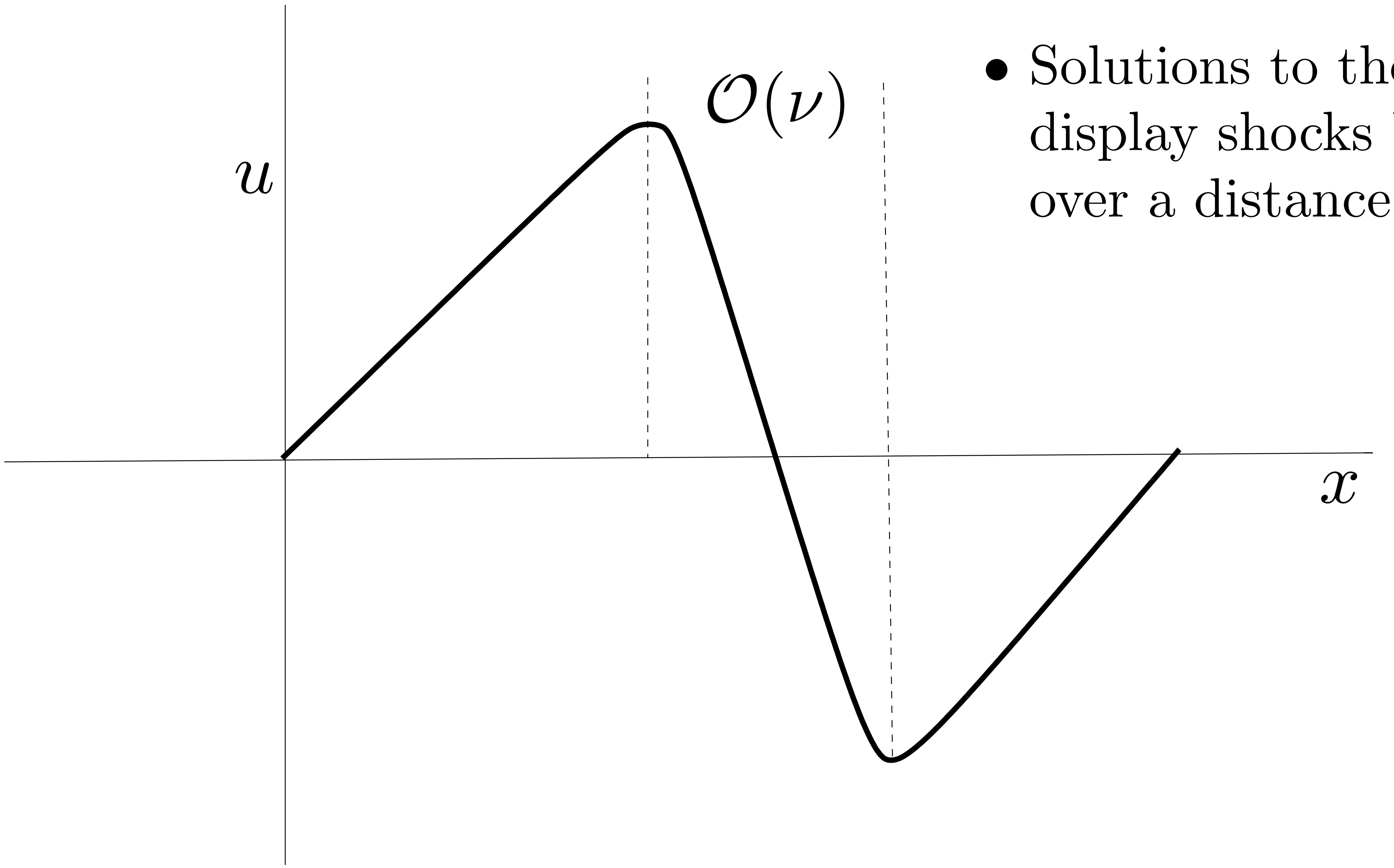
The gradmoments of integer order p, as a function of the Reynolds number $R \equiv 1/\nu$, are defined as the spatial average over the period 2π:

$$M_p(R) \equiv \frac{1}{2\pi} \int_0^{2\pi} dx \left[\frac{\partial u(x,t)}{\partial x} \right]^p; \quad \varepsilon = \nu M_2 = (1/R) M_2$$

$$M_p(R) = A_p R^{\chi_p} + B_p R^{\chi_P^{(1)}} + C_p R^{\chi_P^{(2)}} + \dots$$

Chakraborty, Frisch, Pauls and Ray, (2011) *(to be submitted)*.

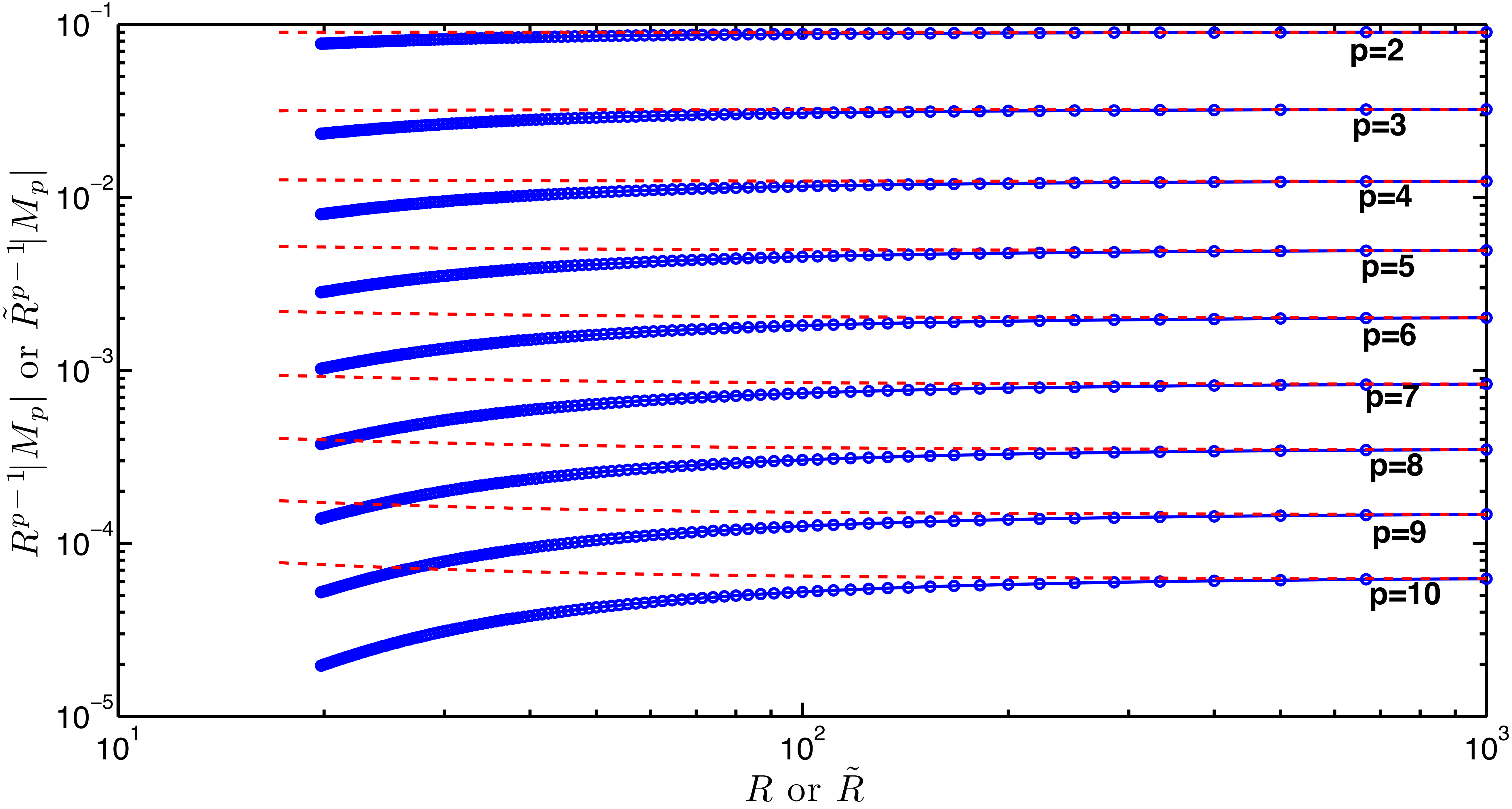
GRADMOMENTS: LEADING ORDER EXPONENT



- Solutions to the Burgers equation display shocks broadened by viscosity over a distance $\mathcal{O}(\nu) = \mathcal{O}\left(R^{-1}\right)$.

- Within a shock, the pth power of the velocity gradient is $\mathcal{O}\left(R^{p}\right)$.
- Shocks cover a fraction $\mathcal{O}\left(R^{-1}\right)$ of the one-dimensional spatial domain.
- Hence, for the Burgers equation we expect $\chi_p = p - 1$.

NELKIN SCALING IN BURGERS



Dashed lines for ESS-type plots; $\tilde{R} \equiv M_2/\varepsilon$

THEORY: IT IS ALL ABOUT SUBDOMINANT TERMS

$$u(x,t) = -2\nu\partial_x \ln\theta(x,t)$$

where,

$$\theta(x,t) = \int_0^{2\pi} \mathrm{e}^{\cos(x-x')/(2\nu)}\, G(x',t)\, dx'$$

and

$$G(x',t) = \sum_{k=-\infty}^{k=\infty} \mathrm{e}^{ikx'-\nu k^2 t} \quad .$$

Using the method of steepest descent, one can show that, for large R and any integer $p \geq 2$: $M_p(R) = A_p R^{p-1} + B_p R^{p-2} + C_p R^{p-3} + \dots$

- $\tilde{R} \equiv \dfrac{M_2}{A_2} = R^1 + \dfrac{B_2}{A_2} R^0 + \mathcal{O}(R^{-1}); \quad [M_2 = A_2 R + B_2 + \ldots],$
- $M_p = A_p \tilde{R}^{p-1} + \tilde{B}_p \tilde{R}^{p-2} + \mathcal{O}(\tilde{R}^{p-3}); \quad \tilde{B}_p = B_p - \dfrac{(p-1)A_p}{A_2} B_2$

DATA PROCESSING USING ASYMPTOTIC EXTRAPOLATION

order(p)	χ_p	A_p	$\chi_p^{(1)}$	B_p	$\chi_p^{(2)}$	C_p
2	0.9999987	+0.09032605	− 0.002	− 0.2290236	− 1.04	+0.201
3	1.999998	− 0.03245271	1.00001	+0.1736854	0.03	− 0.132
4	2.999996	+0.01249279	2.00001	− 0.090466	1.0001	+0.08417
5	3.999995	− 0.00498725	3.00001	+0.045622	1.99988	− 0.08209
6	4.999994	+0.00203621	4.00001	− 0.022523	2.99993	+0.06103
7	5.999993	− 0.00084414	5.000008	+0.010955	4.0002	− 0.0398
8	6.999992	+0.0003539	5.999993	− 0.00526	5.002	+0.024
9	7.999994	− 0.0001495	6.99991	+0.0025	6.009	− 0.01
10	9.00001	+0.000063	7.9995	− 0.0012	7.03	+0.03

Table I: Dominant scaling exponents χ_p and the first two subdominant exponents $\chi_p^{(1)}$ and $\chi_p^{(2)}$ together with the corresponding coefficients A_p, B_p, and C_p for the large-R behavior of gradmoments of order p, obtained by asymptotic extrapolation processing of a 400-digit precision determination of gradmoments from the Hopf-Cole solution. The theoretical values are $\chi_p = p-1$, $\chi_p^{(1)} = p-2$, and $\chi_p^{(2)} = p-3$.

J. van der Hoeven, J. Symb. Comput. **44**, 1000 (2009),
W. Pauls and U. Frisch, J. Stat. Phys. **127**, 1095 (2007).

ESS-TYPE EXTENSION OF GRADMOMENTS' SCALING

order(p)	$R_p^\star = \|B_p/A_p\|$	$\tilde{R}_p^\star = \|\tilde{B}_p/A_p\|$
2	2.5344	0.0
3	5.3520	0.2827
4	7.2414	0.3622
5	9.1477	0.9906
6	11.0613	1.6116
7	12.9785	2.2290
8	14.8980	2.8440
9	16.8222	3.4544
10	19.0604	3.7507

TABLE II. Estimates of Reynolds numbers beyond which subdominant corrections become small in the Reynolds number representation (middle column) and the ESS-type representation (last column).

- $M_p = A_p R^{p-1} + B_p R^{p-2} + \mathcal{O}(R^{p-3})$
- $\tilde{R} \equiv \dfrac{M_2}{A_2} = R^1 + \dfrac{B_2}{A_2} R^0 + \mathcal{O}(R^{-1}); \quad [M_2 = A_2 R + B_2 + \ldots],$
- $M_p = A_p \tilde{R}^{p-1} + \tilde{B}_p \tilde{R}^{p-2} + \mathcal{O}(\tilde{R}^{p-3}); \quad \tilde{B}_p = B_p - \dfrac{(p-1)A_p}{A_2} B_2$

RESOLUTION VS PRECISION

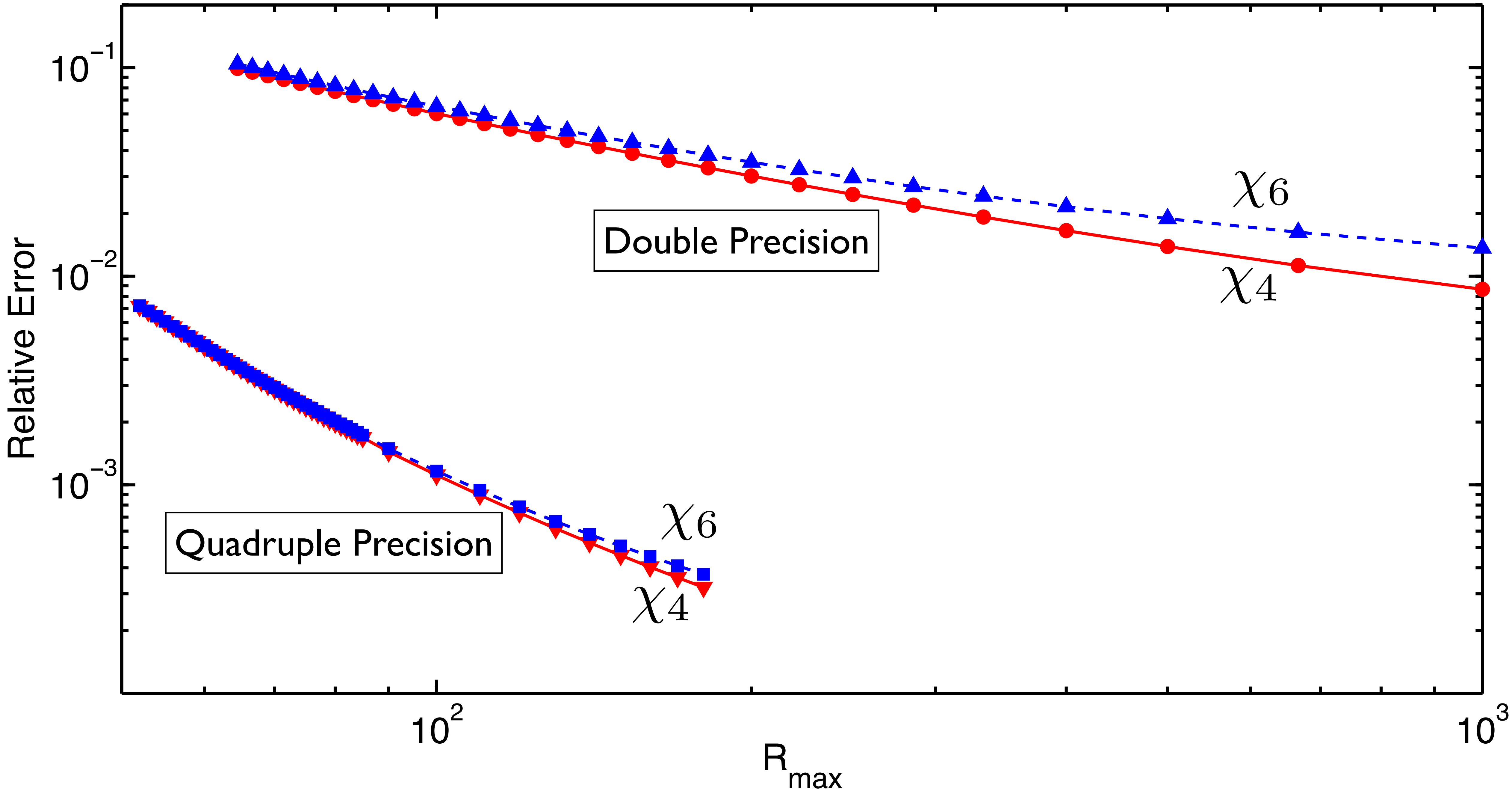


Fig: Relative error of Nelkin exponents χ_4 and χ_6 obtained by asymptotic extrapolation from pseudo-spectral calculations up to a maximum Reynolds number $R_{\max}$. Upper set of curves: double precision calculations (χ_4: red filled circles, χ_6: blue filled triangles); lower set of curves: quadruple precision (χ_4: red inverted triangles, χ_6: blue filled squares).

CONCLUSIONS

- ESS works for the Burgers equation. ESS improves scaling because it reduces the intensity of subdominant corrections.
- Nelkin scaling for moments of velocity gradients (gradmoments) can be explained *along similar lines*; and it can be combined with an ESS-type plotting in the Burgers equation.
- It will be interesting to check whether ESS-type plotting works for Nelkin scaling in 3D NS turbulence.
- It seems that increasing the precision, rather than the Reynolds number may be a good strategy for determining scaling exponents in Burgers case.
- Also, it remains to be seen if this result precision vs. resolution carries over to a much broader class of equations including multi-dimensional problems.

THANK YOU

THEORY: IT IS ALL ABOUT SUBDOMINANT TERMS

- Understanding why ESS works and by how much the scaling is improved in the IR and UV directions can be done for the Burgers equation.

- Simplest setting: deterministic periodic initial conditions, no forcing. Extension to random initial conditions + forcing can be done too.

- $$S_p(h) \equiv (1/2\pi) \int_0^{2\pi} dx\, [u(x+h,t) - u(x,t)]^p \, .$$

- There are two small parameters: $\frac{h}{L}$ at IR end and $\frac{\eta}{h}$ at UV end.

DOMINANT PLUS SUBDOMINANT IR TERM

Let there be a shock at x_s with left and right velocities u_- and u_+.
Left and right limits of the velocity are denoted u_- and u_+.
Left and right limits of the velocity gradient are denoted s_- and s_+.

Shock amplitude $\Delta = u_- - u_+$ For $0 < h \ll L = 2\pi$,

$$2\pi S_3 = -\Delta^3 h + \frac{3}{2}\Delta^2(s_- + s_+)h^2 + \text{hot} \qquad \text{(standard)}$$

$$2\pi S_6 = +\Delta^6 h - 3\Delta^5(s_- + s_+)h^2 + \text{hot}$$

$$\tilde{S}_3 = -\frac{2\pi S_3}{\Delta^3} = h - \frac{3}{2\Delta}(s_- + s_+)h^2 + \text{hot}$$

$$2\pi S_6 = \Delta^6 \tilde{S}_3 - \frac{3}{2}\Delta^5(s_- + s_+)(\tilde{S}_3)^2 + \text{hot} \qquad \text{(ESS)}$$

The coefficient of the first subdominant correction with ESS is half of what it is in the standard representation. Hence scaling extends further by a factor two. For *random* initial conditions and large time *t*, scaling is extended by *at least* a factor two [From $s_- \approx s_+ \approx 1/t$ and $\langle\Delta^6\rangle/\langle\Delta^5\rangle \geq \langle\Delta^3\rangle/\langle\Delta^2\rangle$, which follows from the convexity of $q \mapsto \ln\langle\Delta^q\rangle$.

DOMINANT PLUS SUBDOMINANT UV TERM

- Due to viscosity, the shock has a tanh structure which causes UV corrections.
- Shift spatial origin to shock location, rescale distances by $\eta = 4\nu t/\Delta$ and velocities by Δ.
- To leading order, the shock becomes $u(x) = -\tanh x$ and

$$2\pi S_p = \int_{-\infty}^{\infty} dx\,[\tanh(x) - \tanh(x+h)]^p.$$

Thus, for large h

$$2\pi S_3 = -8h + 12 + \mathrm{tst}$$

$$2\pi S_6 = 64h - 2192/15 + \mathrm{tst} \qquad \text{(standard)}$$

$$\tilde{S}_3 \equiv -\frac{2\pi S_3}{8} = h - 3/2 + \mathrm{tst}$$

$$2\pi S_6 = 64\tilde{S}_3 - 752/15 + \mathrm{tst} \qquad \text{(ESS)}$$

The coefficient of the first subdominant correction with ESS is smaller by a factor $2192/752 \approx 2.91$ to what it is in the standard representation. Hence scaling extends further by a factor 2.91. For statistically homogeneous forces and (random) initial conditions; and if they have rapidly decreasing spatial correlations, ergodicity helps us to conclude that ESS extends the scaling into UV regime by *at least* 2.91.

ASYMPTOTIC EXTRAPOLATION

- Given a function $G(r)$ with assumed leading-order expansion $(r \to \infty)$

$$G(r) \simeq C r^{-\alpha} e^{-\delta r}$$

on a regular 1D grid $r_0, 2r_0, ..., Nr_0$

$$G_n = G(nr_0), \qquad n = 1, 2, ..., N$$

can we determine C, α and δ numerically with high accuracy? What about subleading terms?

- Naive method: least square fit

- Improvement: take second ratio (Shelley, Caflisch, Pauls–Matsumoto–Frisch–Bec)

$$R_n \simeq \frac{G_n G_{n-2}}{G_{n-1}^2} = \left(1 - \frac{1}{(n-1)^2}\right)^{-\alpha}$$

Ignore subleading corrections

$$\alpha = -\frac{\ln R_n}{\ln(1 - 1/(n-1)^2)}$$

- Is there a more systematic approach?

ASYMPTOTIC EXTRAPOLATION

- Interpolate the sequence G_n in the "most asymptotic" region $n = L, ..., N$
- Transformations:

  I Inverse: $G_n \longrightarrow \frac{1}{G_n}$

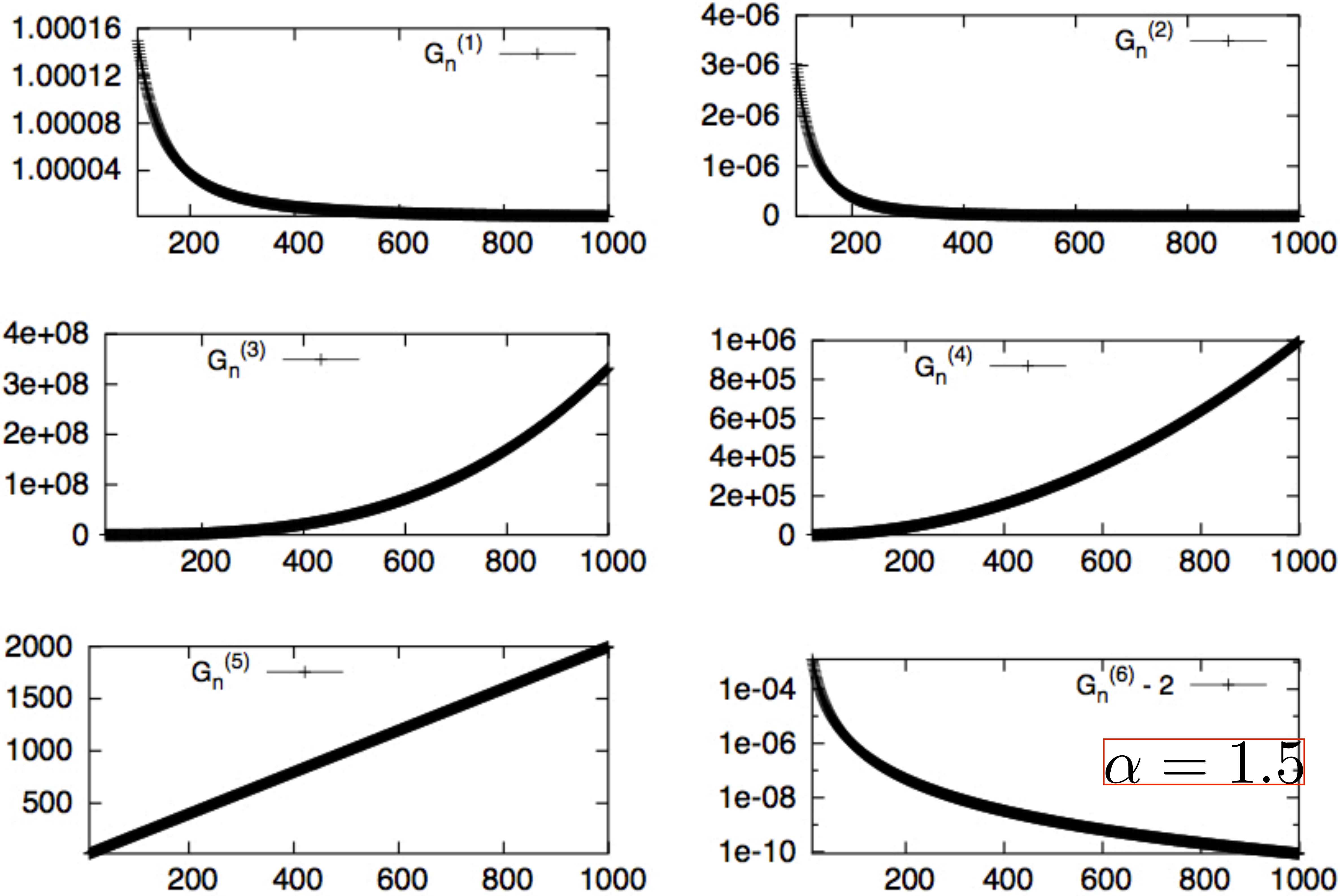
  R Ratio: $G_n \longrightarrow \frac{G_n}{G_{n-1}}$

  SR Second ratio: $G_n \longrightarrow \frac{G_n G_{n-2}}{G_{n-1}^2}$

  D Difference: $G_n \longrightarrow G_n - G_{n-1}$

- Going down (assuming $G_n > 0$):
  - Test 1: if $G_n < 1$ apply **I**
  - Test 2: Does G_n grow faster than $n^{5/2}$?
    * Yes: if the growth is exponential apply **SR**, otherwise **R**
    * No: apply **D**
  - Continue untill obtaining data which are easy to interpolate and clean enough
- Go back by inverting transformations **I**, **R**, **SR** and **D**

ASYMPTOTIC EXTRAPOLATION



$$Ck^{-\alpha}e^{-\delta k} \xrightarrow{\mathbf{SR}} 1+\frac{\alpha}{k^2} \xrightarrow{-\mathbf{D}} \frac{2\alpha}{k^3} \xrightarrow{\mathbf{I}} \frac{k^3}{2\alpha} \xrightarrow{\mathbf{D}} \frac{3k^2}{2\alpha} \xrightarrow{\mathbf{D}} \frac{3k}{\alpha} \xrightarrow{\mathbf{D}} \frac{3}{\alpha}$$