INVERSE CASCADE on HYPERBOLIC PLANE

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to Uriel

No road is long
with good company

turkish proverb

- **V.I. Arnold**, Ann. Inst. Fourier (1966): incompressible **Euler equation** describes geodesics on the group of area-preserving diffeomorphisms

- $\Rightarrow$ can be written on any space with **Riemannian metric** (g_{ij}) in the form

$$\partial_t v^i + v^k \underbrace{\nabla_k}_{\substack{\textbf{Levi-Civita} \\ \text{covariant derivative}}} v^i = -\underbrace{g^{ij}}_{\text{inverse meric}} \partial_j p$$

with incompressibility condition $\partial_i(\sqrt{g}\, v^i) = 0$ for $\sqrt{g} \equiv \sqrt{\det(g_{ij})}$

- **Noether symmetries** of the **Euler** equation are given by **currents** (J^0, J^i) that are conserved: $\partial_t(\sqrt{g}\, J^0) + \partial_i(\sqrt{g}\, J^i) = 0$

- In $\boxed{2\mathrm{D}}$ incompressibility implies the existence of **stream function** ψ such that $v^j = \epsilon^{ij} \frac{1}{\sqrt{g}} \partial_i \psi$

- In this case the scalar **vorticity** $\omega = \frac{\epsilon^{ij}}{\sqrt{g}} \nabla_i v_j = g^{ij} \nabla_i \partial_j \psi$ evolves by

$$\partial_t \omega = -v^i \partial_i \omega = \frac{1}{\sqrt{g}} \epsilon^{ij} (\partial_i \omega)(\partial_j \psi)$$

- Upon addition of the viscous dissipation and a source one obtains the forced **Navier-Stokes equation**

$$\partial_t v^i + v^k \nabla_k v^i - \nu(\Delta v)^i = -g^{ij}\partial_j p + f^i$$

Laplace−Beltrami operator

with

$$(\Delta v)^i = g^{jk}\nabla_j \nabla_k v^i + g^{ij}(\nabla_j\nabla_k - \nabla_k\nabla_j)v^k$$

- In 2D we shall assume random force f^i that is a **Gaussian** process with covariance

$$\left\langle f^i(t_1,\boldsymbol{x}_1)\, f^j(t_2,\boldsymbol{x}_2)\right\rangle = \delta(t_1-t_2)\,\frac{\epsilon^{ki}\epsilon^{lj}}{\sqrt{g}(\boldsymbol{x}_1)\,\sqrt{g}(\boldsymbol{x}_2)}\,\partial_{x_1^k}\partial_{x_2^l}\,\mathcal{C}\Big(\frac{\rho(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\ell_f}\Big)$$

for $\mathcal{C}(\,\cdot\,)$ a fast decreasing function, $\rho(\boldsymbol{x}_1,\boldsymbol{x}_2)$ the **geodesic distance**, and ℓ_f the **forcing scale**

- **Arnold** considered general geometries to study topological properties of flows: **Arnold-Khesin**, *Topological Methods in Hydrodynamics*, Springer 1998

- **Our motivation**: search of **conformal symmetry** in **2D inverse cascasde** signaled by numerical discovery of **SLE** statistics of 0-vorticity lines in **Bernard-Boffetta-Celani-Falkovich**, Nature Physics **2** (2006)

- **Main idea**: conformal symmetry may be easier to find in different 2D geometries supporting inverse cascade

- This appeared to be the case for the **NS** flows on the **hyperbolic plane** but it did not yet throw light on the (conjectured) **SLE** statistics

Why **hyperbolic plane** ?

- it is a 2D space with 3-dimensional symmetry group (as the flat plane) and a **constant negative curvature** $-2R^{-2}$

- the 2D sphere with constant positive curvature $2R^{-2}$ also has a 3-dimensional symmetry group but no space to develope **inverse cascade**

- the hyperbolic plane has more room at large scales than the flat space: the circumference of the circle of radius ρ is equal here to $2\pi R \sinh \frac{\rho}{R}$

- no geometric mechanism that would block the development of **inverse cascade**

Hyperbolic (upper half-)plane

$\|$

Lobachevsky-Bolyai plane

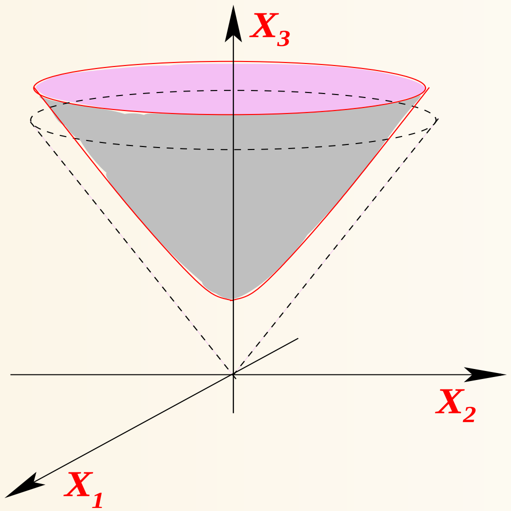
$\|$

Poincaré disc

$\|$

Upper hyperboloid $\boldsymbol{H}_R$ in 3D **Minkowski space** $\boldsymbol{M}^3$ with signature $(+,+,-)$

$$\boldsymbol{H}_R \;=\; \left\{\,(X_1, X_2, X_3)\;\middle|\; X_1^2 + X_2^3 - X_3^2 = -R^2, \quad X_3 > 0\,\right\}$$



- **Isometry group** of $\boldsymbol{H}_R \;=\;$ 3D **Lorentz group** $\;=\; SL(2,\boldsymbol{R})/\{\pm 1\}$

- Convenient parametrization of $\boldsymbol{H}_R$:

$$X_1 = r\cos\varphi\,, \quad X_2 = r\sin\varphi\,, \quad X_3 = \sqrt{R^2 + r^2}$$

- In terms of **stream function** ψ such that

$$v^r = -\frac{\sqrt{R^2+r^2}}{R\,r}\,\partial_\varphi\psi\,, \qquad v^\varphi = \frac{\sqrt{R^2+r^2}}{R\,r}\,\partial_r\psi$$

and **vorticity**

$$\omega \;=\; \left(\frac{\sqrt{R^2+r^2}}{R\,r}\,\partial_r\,\frac{r\sqrt{R^2+r^2}}{R}\,\partial_r + \frac{1}{r^2}\,\partial_\varphi^2\right)\psi$$

the **Euler** equation on $\boldsymbol{H}_R$ becomes

$$\partial_t\omega \;=\; \frac{\sqrt{R^2+r^2}}{R\,r}\Big((\partial_r\omega)(\partial_\varphi\psi) - (\partial_\varphi\omega)(\partial_r\psi)\Big)$$

- **Noether** **symmetries** of **Euler** equation on $\boldsymbol{H}_R$ correspond to conserved currents $(J^0,\ J^r,\ J^\varphi)$

  - time translation invariance gives

$$J^0_E = \frac{1}{2}\left(\frac{R^2}{R^2+r^2}(v^r)^2 + r^2(v^\varphi)^2\right), \qquad J^{r,\varphi}_E = (J^0_E + p)v^{r,\varphi}$$

energy density

  - 3D **Lorentz** group invariance gives

$$J^0_X = -(v_r X^r + v_\varphi X^\varphi), \qquad J^{r,\varphi}_X = J^0_X v^{r,\varphi} - pX^{r,\varphi}$$

momentum density

for

$$X^r = \begin{cases} \sqrt{R^2+r^2}\,\sin\varphi \\ -\sqrt{(R^2+r^2)}\,\cos\varphi \\ 0 \end{cases}, \qquad X^\varphi = \begin{cases} -1+r^{-1}\sqrt{R^2+r^2}\,\cos\varphi \\ r^{-1}\sqrt{R^2+r^2}\,\sin\varphi \\ -2 \end{cases}$$

  - no analogue of **Galilean** invariance**!**

- In the **Navier-Stokes** equation with the viscous dissipation and random **Gaussian** forcing (f^r, f^φ) as given before one has the **energy balance**

$$\partial_t \langle J_E^0 \rangle = - \underbrace{\nu \langle \omega^2 \rangle}_{\substack{\textbf{dissipation}\\ \textbf{rate}\ \varepsilon}} + \underbrace{\left(- \ell_f{}^{-2} \mathcal{C}''(0) \right)}_{\substack{\textbf{injection}\\ \textbf{rate}\ \iota}}$$

- Flat space **inverse cascade** scenario of **Kraichnan** (1967) and **Batchelor** (1969) - well substantiated by theory, simulations, and experiments

  - **Scales**: $\underbrace{\ell_\nu}_{\substack{\textbf{dissipative}\\ \textbf{scale}}} \ll \underbrace{\ell_f}_{\substack{\textbf{forcing}\\ \textbf{scale}}} \ll \underbrace{\rho}_{\substack{\textbf{running}\\ \textbf{scale}}}$

  - At scales ρ and long times energy flows into a **condensate mode**

  - In terms of stream functions:

$$\langle \psi(t, \boldsymbol{x}_1)\, \psi(t, \boldsymbol{x}_2) \rangle \approx -\frac{1}{2}\, \iota\, \rho^2\, t + const.\, \iota^{2/3}\, \rho^{8/3} + \ \ldots$$

with $\rho = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$ and $\ldots$ not contributing to velocity 2-pt function

Inverse cascade - condensation scenario on $\boldsymbol{H}_R$

- We postulate that

$$\boxed{\langle\psi(t,\boldsymbol{x}_1)\,\psi(t,\boldsymbol{x}_2)\rangle\ \approx\ \Psi_0(x)\,t\,+\,\Psi_{st}(x)\ +\ \ldots}\qquad \text{for}\qquad x\equiv\cosh(\tfrac{\rho}{R})$$

with ρ the hyperbolic distance, $\cosh(\frac{\rho}{R})=\frac{\sqrt{R^2+r_1^2}\sqrt{R^2+r_2^2}-r_1r_2\cos(\varphi_1-\varphi_2)}{R^2}$

- One more **length-scale** present: $R\gg\ell_f$ **!**

- For $\ell_f\ll\rho\ll R$ this expression should agree with the flat space one, in particular, we should have

$$-R^{-2}\Psi_0'(1)\ =\ \iota$$

energy injection rate

- We shall try to find the form of modes $\Psi_0,\ \Psi_{st}$ for $R\ll\rho$ using scaling arguments

- The scenario seems self-consistent but, at the end, its credibility should be tested numerically**!**

Scaling theory

- For equal-time velocity correlation n-pt functions

$$F^{n,m}_{\boldsymbol{H}_R,\nu,\mathcal{C},\ell_f}(t;r_1,\varphi_1,\dots,r_n,\varphi_n) \;=\; \Big\langle \prod_{j=1}^{m} v^r(t;r_j,\varphi_j) \prod_{j=m+1}^{n} v^\varphi(t;r_j,\varphi_j) \Big\rangle$$

  one has a tautological **scaling relation**

$$\lambda^{\frac{2}{3}n-m}\, F^{n,m}_{\boldsymbol{H}_R,\nu,\mathcal{C},\ell_f}(\lambda^{\frac{2}{3}}t;\lambda r_1,\varphi_1;\dots;\lambda r_n,\varphi_n)$$

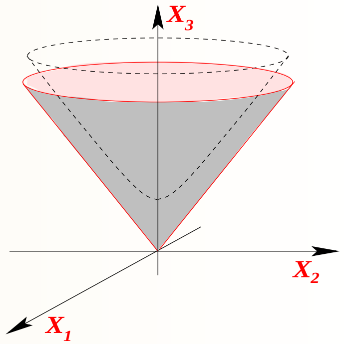
$$= \;\; F^{n,m}_{\boldsymbol{H}_{R/\lambda},\lambda^{-4/3}\nu,\lambda^{-2}\mathcal{C},\ell_f/\lambda}(t;r_1,\varphi_1;\dots;r_n,\varphi_n)$$

- The forcing on both sides corresponds to the same **energy injection rate** ι

- **Scaling limit** $\lambda\to\infty$ of RHS should describe stochastic **Euler** equation on the (light-)**cone** $\boldsymbol{H}_0$ in 3D **Minkowski** space yielding the long time - large distance asymptotics of the inverse cascade on $\boldsymbol{H}_R$

- A geometric effect: far away $\boldsymbol{H}_R$ looks like $\boldsymbol{H}_0$!

Euler equation on $\boldsymbol{H}_0 = \{(X_1, X_2, X_3) \mid X_1^2 + X_2^2 = X_3^2\}$



- Parametrization of $\boldsymbol{H}_0$:

$$X_1 = r\cos\varphi\,, \quad X_2 = r\sin\varphi\,, \quad X_3 = r$$

- $\boldsymbol{H}_0$ inherits from 3D **Minkowski** space a degenerate metric

- $\boxed{\textbf{Isometry group of } \boldsymbol{H}_0 = \mathit{Diff}(S^1) = 1\text{D } \textbf{conformal group}}$

  (= half of 2D **conformal group**) $\supset$ 3D **Lorentz group**

- Conformal symmetry arises similarly as in the **AdS-CFT** correspondence!

- In terms of **stream function** ψ such that $v^r = -\partial_\varphi\psi, \quad v^\varphi = \partial_r\psi$ and the "**vorticity**" $\omega = \partial_r(r^2 v^\varphi) = \partial_r(r^2\partial_r\psi)$ the **Euler** equation takes the standard looking form

$$\partial_t\omega = -(v^r\partial_r + v^\varphi\partial_\varphi)\omega = (\partial_r\omega)(\partial_\varphi\psi) - (\partial_\varphi\omega)(\partial_r\psi)$$

- **Noether** **symmetries** of **Euler** equation on $\boldsymbol{H}_0$ correspond to conserved currents

$$J^0_E = \frac{1}{2}\, r^2 (v^\varphi)^2\,, \qquad J^{r,\varphi}_E = (J^0_E + p)\, v^{r,\varphi}$$
$$J^0_\zeta = -r^2 v^\varphi \zeta\,, \qquad J^r_\zeta = J^0_\zeta v^r + rp\zeta'\,, \qquad J^\varphi_\zeta = J^0_\zeta v^\varphi - p\zeta$$

for any periodic function $\zeta(\varphi)$ (integral of $r^2 v^\varphi$ along each light-ray in $\boldsymbol{H}_0$ is separately conserved)

- $Diff(S^1)$ symmetry is spontaneously broken to the **Lorentz** one in the $\lambda \to \infty$ **scaling limit** of the stochastic **NS** equation on $\boldsymbol{H}_R$ **!** The precise nature of this breaking remains to be understood

- The scaling limit has a tautological **scale invariance**:

$$\lambda^{\frac{2}{3}n-m}\, F^{n,m}_{\boldsymbol{H}_0}(\lambda^{\frac{2}{3}} t; \lambda r_1, \varphi_1; \ldots; \lambda r_n, \varphi_n)$$
$$= \; F^{n,m}_{\boldsymbol{H}_0}(t; r_1, \varphi_1; \ldots; r_n, \varphi_n)$$

- For the velocity 2-pt functions on $\boldsymbol{H}_0$ there are 2 scale invariant solutions that in terms stream functions have the form

$$\big\langle\psi(t,\boldsymbol{x}_1)\,\psi(t,\boldsymbol{x}_2)\big\rangle \;=\; \big(A(\ln x)^2+B\ln x\big)t-6A(\ln x)\,t\ln t\,+\,\dots$$

$$\big\langle\psi(t,\boldsymbol{x}_1)\,\psi(t,\boldsymbol{x}_2)\big\rangle \;=\; Cx^{1/3}\,+\,\dots$$

for $x=\frac{r_1r_2\big(1-cos(\varphi_1-\varphi_2)\big)}{L^2}$ with an arbitrary length scale L

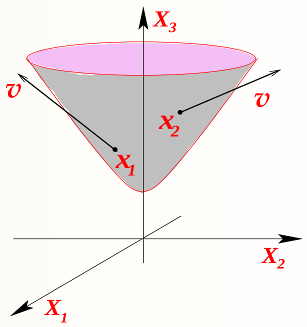
- In the **inverse cascade-condensate** scenario for stochastic **NS** equation on $\boldsymbol{H}_R$ they imply for the condensate and stationary modes the behavior:

$$\Psi_0(x)\quad\underset{x\gg1}{\approx}\quad\iota\,R^2\big((\ln x)^2-\ln x\big)\,+\,\dots$$

$$\Psi_{st}(x)\quad\underset{x\gg1}{\approx}\quad const.\,\iota^{2/3}R^{8/3}\,x^{1/3}\,+\,\dots$$

($t\ln t$ term absorbes a logarithmic divergence of the rescaled 2-pt function $\lambda^{-2/3}F^{2,2}_{\boldsymbol{H}_R,\nu,\mathcal{C},\ell_f}(\lambda^{2/3}t;\lambda r_1,\varphi_1;\lambda r_2,\varphi_2)$)

- For the **Lorentz-invariant** 2-pt function using **Minkowskian** scalar product $\boldsymbol{v}(t,\boldsymbol{x}_1)\cdot\boldsymbol{v}(t,\boldsymbol{x}_2)$ of vectors tangent to $\boldsymbol{H}_R$



$$\big\langle v(t,\boldsymbol{x}_1)\cdot v(t,\boldsymbol{x}_2)\big\rangle \;=\; -R^{-2}\,\partial_x(x^2-1)\,\partial_x\left(\Psi_0(x)\,t+\,\Psi_{st}(x)\right)$$

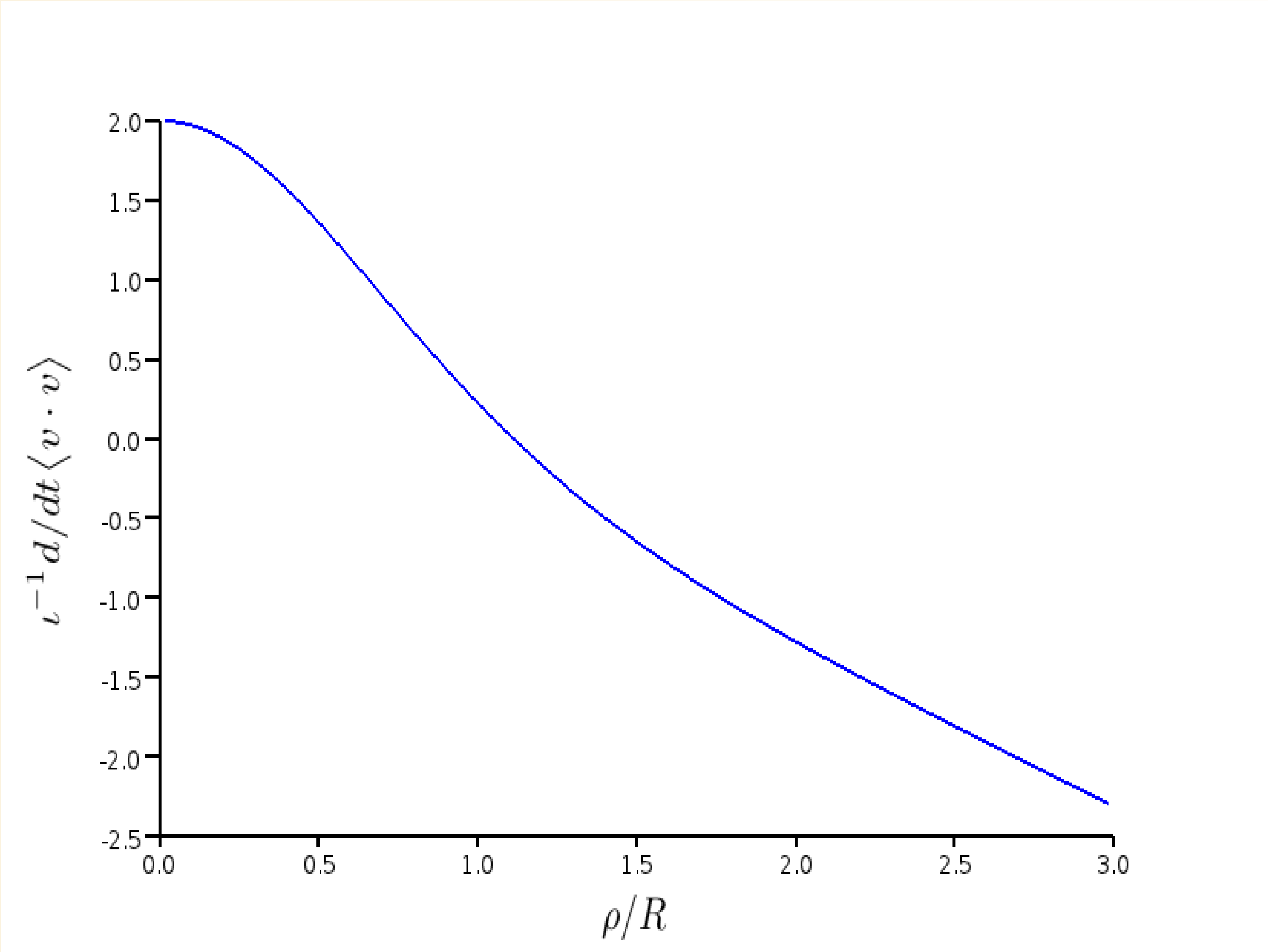
where $x=\cosh\left(\frac{\rho}{R}\right)$ this implies that

- the condensate contribution $\propto t$ and equal to $2\,\iota\,t$ for $\rho\ll R$ decreases **linearly** in hyperbolic distance ρ for $\rho\gg R$

- the stationary contribution $\propto -\iota^{2/3}\rho^{2/3}$ for $\rho\ll R$ decreases **exponentially** $\propto -R^{2/3}\iota^{2/3}\,\mathrm{e}^{\rho/(3R)}$ for $\rho\gg R$

- no obvious contradiction but what's the **physics** behind**?**

- **Spectral** interpretation via $SL(\boldsymbol{R},2)$-related **Fourier** analysis on $\boldsymbol{H}_R$ is possible



Condensate contribution to the invariant velocity 2-pt function

Flux relation

- In the flat space the inverse cascade scenario implies the **flux relation**

$$\Theta \equiv \Big\langle \Big(\tfrac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|} \cdot \boldsymbol{v}(t, x_1) \Big) \Big(\boldsymbol{v}(t, \boldsymbol{x}_1) \cdot \boldsymbol{v}(t, \boldsymbol{x}_2) \Big) \Big\rangle \; = \; \frac{1}{2}\, \iota\, \rho$$

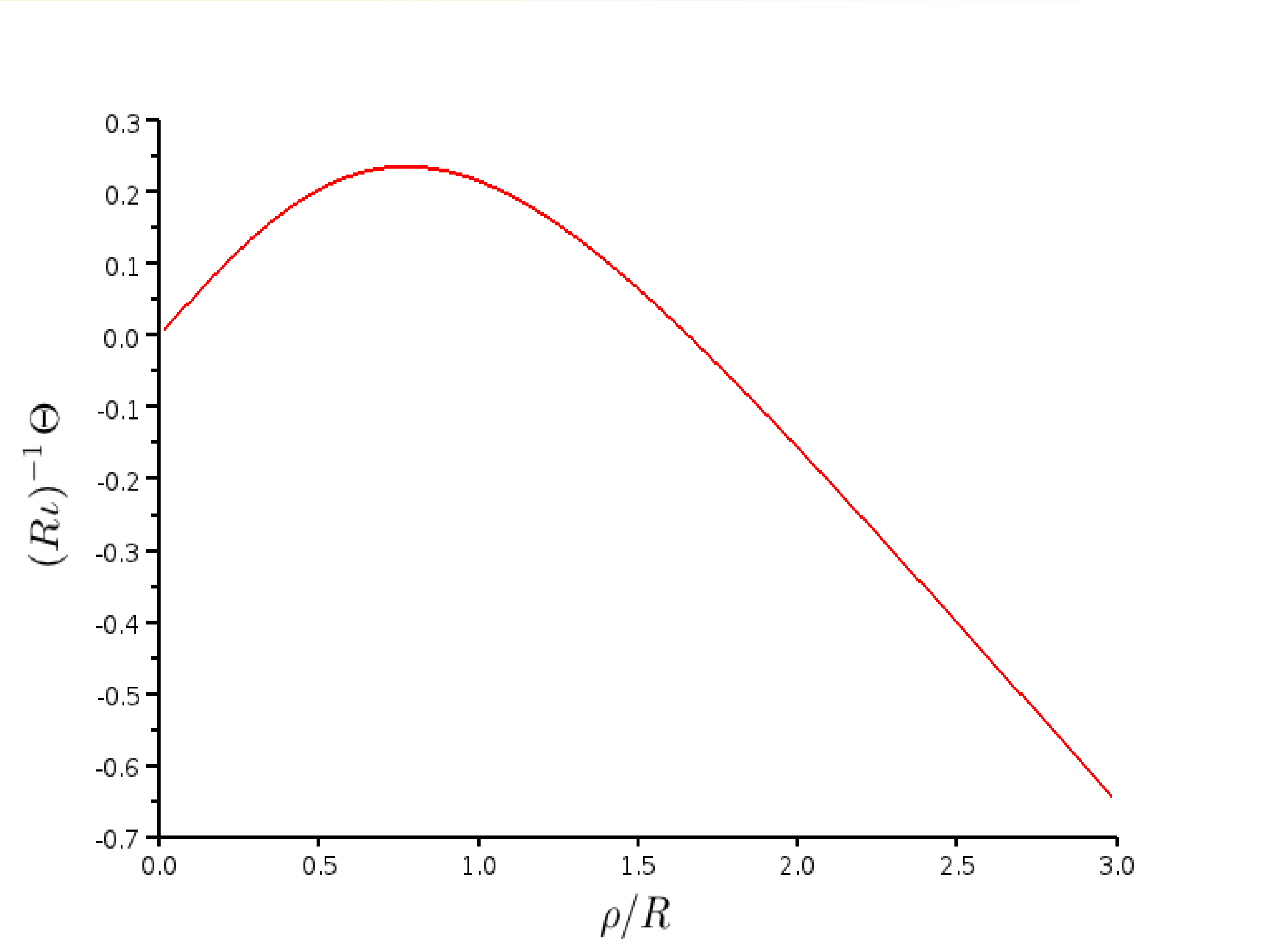
- On $\boldsymbol{H}_R$ the **flux relation** takes the form

$$\Theta \equiv \Big\langle \Big(\boldsymbol{e}(\boldsymbol{x}_1, \boldsymbol{x}_2) \cdot \boldsymbol{v}(t, \boldsymbol{x}_1) \Big) \Big(\boldsymbol{v}(t, \boldsymbol{x}_1) \cdot \boldsymbol{v}(t, \boldsymbol{x}_2) \Big) \Big\rangle$$

$$= - \tfrac{1}{2R} \sinh \big(\tfrac{\rho}{R}\big)\, \Psi_0' \Big(\cosh \big(\tfrac{\rho}{R}\big) \Big)$$

  where $\boldsymbol{e}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the unit vector tangent at $\boldsymbol{x}_1$ to the geodesic curve joining $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$

- This agrees with the flat space ezpression for $\rho \ll R$ since $-R^{-2}\Psi_0'(1) = \iota$ but behaves like $-\frac{1}{2}\,\iota\,\rho$ for $\rho \gg R$



What is the physics of the inversion of sign of $\partial_\rho \Theta$ around $\rho = R$?

Conclusions

- Our analysis confirms the **inverse cascade - condensation** scenario in stochastic **NS** equation on the **hyperbolic plane** but questions about physical interpretation of the result remain
- Asymptotic behavior of the condensate and stationary modes at distances $\rho \gg R$ were determined by a **scaling limit** that lives on the **cone** $\boldsymbol{H}_0$
- Precise way in which this limit breaks the $Diff(S^1)$ symmetry of the **Euler** dynamics on $\boldsymbol{H}_0$ remains to be understood
- It may be a clue to an eventual link between this asymptotic symmetry and the **SLE** statistics of 0-vorticity lines
- Numerical simulations of forced **NS** flows on $\boldsymbol{H}_R$ would be welcome
- Experimental realizations of such flows are difficult since $\boldsymbol{H}_R$ cannot be embedded isometrically into the 3D **Euclidian** space**!**
- In particular, **soap films** would not do but they may provide indication on the effect of negative curvature on the inverse cascade

Croyez que tout mortel a besoin d'indulgence

Fénelon