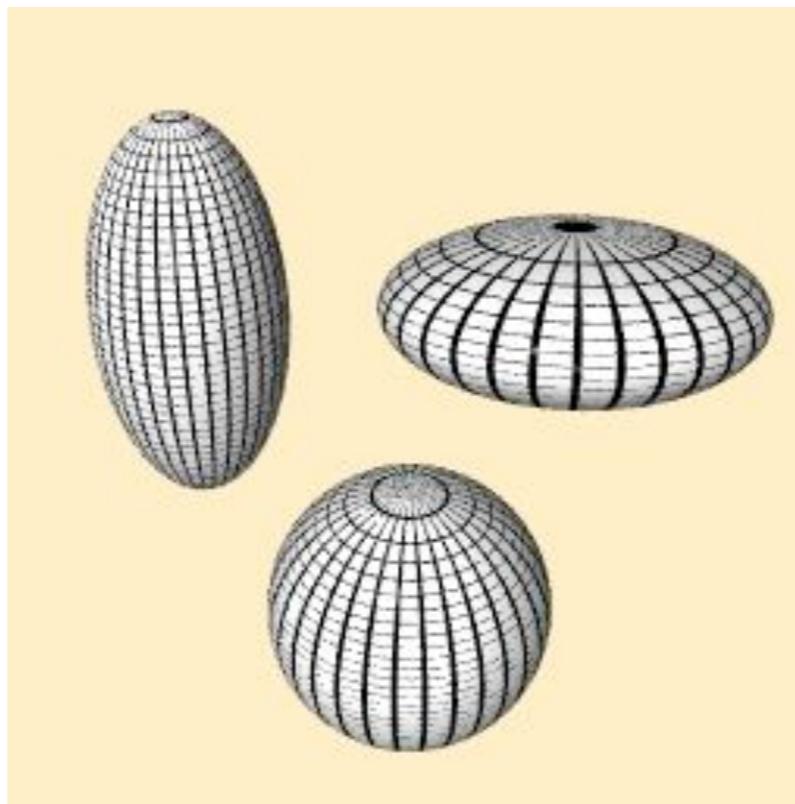
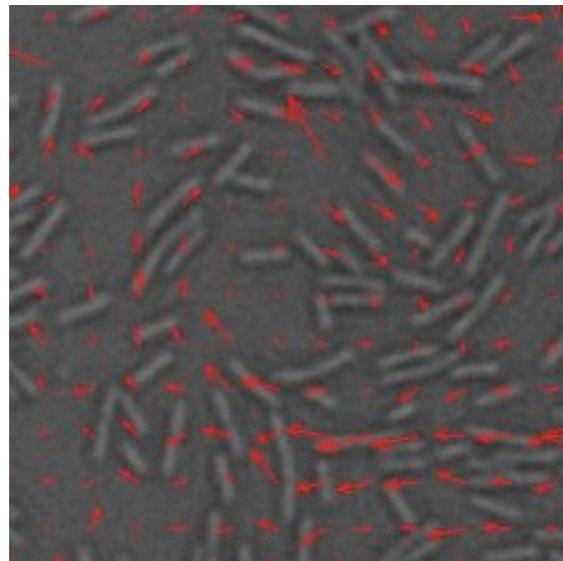


Orientation of non-spherical particles in an axisymmetric random flow

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Swimming micro-organisms
Saintillan & Shelley, *Phys. Rev. Lett.* (2007)

Ice crystals in clouds
Chen & Lamb, *J. Atmos Sci.* (1994)



Turbulent drag reduction
Gyr & Bewersdorff (1995)

Orientation of small particles in flow

Deterministic flows: Analytical results

Jeffery (1922), Bretherton (1962), Bird et al. (1971), Hinch & Leal (1972),
Brenner (1974), Szeri (1993), ...

Turbulent or chaotic flows

Experiments

Krushal & Gallily (1988), Bernstein & Shapiro (1994), Newsom & Bryce (1998), Parsheh et al. (2005), Parsa et al. (2011)

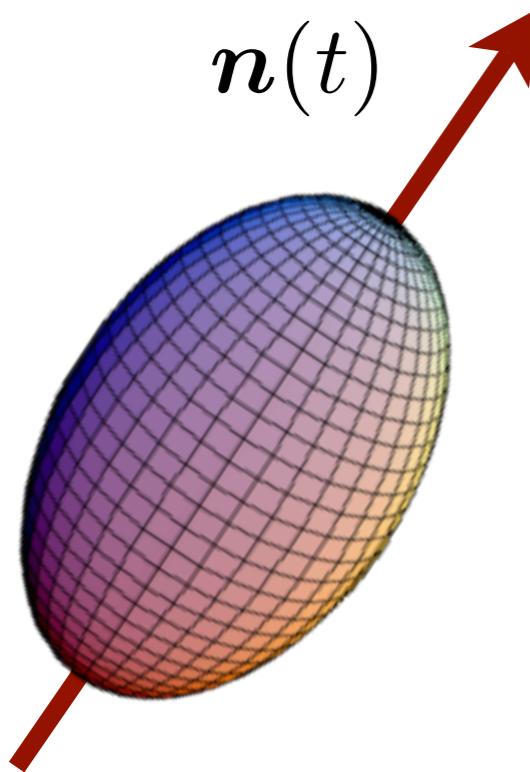
Numerics

Zhang et al. (2001), Mortensen et al. (2008), Wilkinson et al. (2009), Pumir & Wilkinson (2011)

Theory

Turitsyn (2007), Wilkinson & Pumir (2011)

Jeffery's equations



$$\dot{n} = \kappa(t) \cdot n - [\kappa(t) : nn]n + \sqrt{\mathcal{D}} \sigma(t) \cdot \xi(t)$$

$$\sigma = I - nn$$

$\xi(t)$ 3D white noise

$$\kappa = \Omega + \gamma E$$

$$\Omega = \frac{1}{2}[\nabla v - (\nabla v)^T]$$

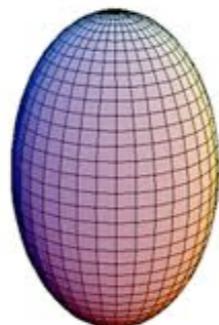
$$E = \frac{1}{2}[\nabla v + (\nabla v)^T]$$

$$|\gamma| \leq 1 :$$

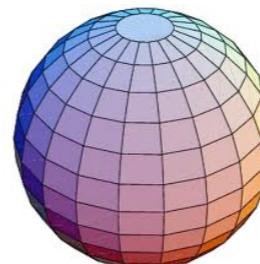
effective aspect ratio

$$r_e = \sqrt{\frac{1+\gamma}{1-\gamma}}$$

$$1 > \gamma > 0$$

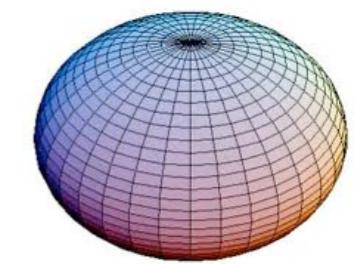


$\gamma = 1$: rod



$$\gamma = 0$$

$\gamma = -1$: disk

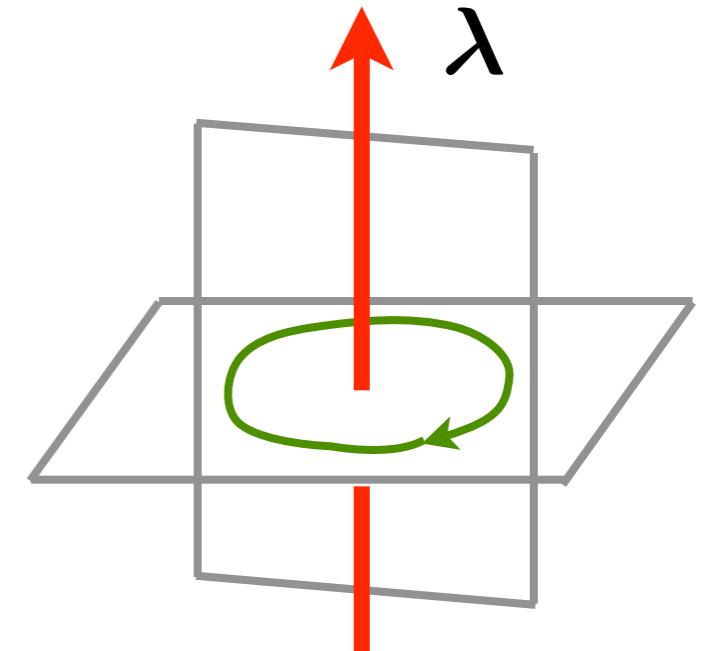


There exists long bodies such that $|\gamma| > 1$ (Bretherton, 1962)

Axisymmetric random flow

Batchelor (1946), Chandrasekhar (1950)

$$\langle \partial_j v_i(\mathbf{x}, t + \tau) \partial_q v_p(\mathbf{x}, t) \rangle = \Gamma_{ijpq} \delta(\tau)$$



$$\begin{aligned} \Gamma_{ijpq} = & (d + 4a)\delta_{jq}\delta_{ip} - a(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{pj}) + (b + c + 5d)\delta_{jq}\lambda_i\lambda_p - b\delta_{ip}\lambda_j\lambda_q \\ & - d[(\delta_{pq}\lambda_i + \delta_{iq}\lambda_p)\lambda_j + (\delta_{pj}\lambda_i + \delta_{ij}\lambda_p)\lambda_q] - c\lambda_i\lambda_p\lambda_j\lambda_q, \end{aligned}$$

If $\boldsymbol{\lambda} \equiv \hat{\mathbf{z}} = (0, 0, 1)$:

$$2a = \Gamma_{1212} - \Gamma_{1111}$$

$$d = \Gamma_{3333} - \Gamma_{1111}$$

$$b = \Gamma_{1212} - \Gamma_{1313}$$

$$2a + b + c + 4d = \Gamma_{3131} - \Gamma_{3333}.$$

$$4a + d > 0$$

$$b < 4a + d$$

$$4a + b + c + 6d > 0$$

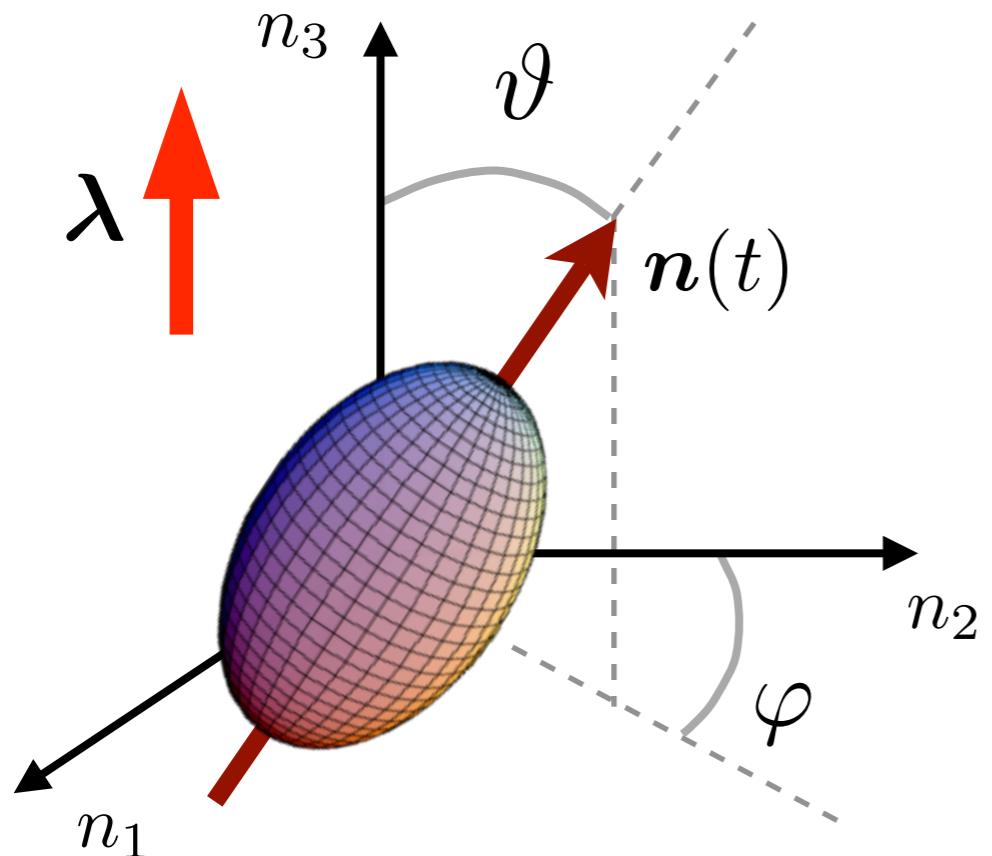
Orientational dynamics

$$\dot{\mathbf{n}} = \boldsymbol{\kappa}(t) \cdot \mathbf{n} - [\boldsymbol{\kappa}(t) : \mathbf{n}\mathbf{n}] \mathbf{n} + \sqrt{\mathcal{D}} \boldsymbol{\sigma} \cdot \boldsymbol{\xi}(t) \quad |\mathbf{n}(t)| = 1$$

PDF of orientations: $p(\mathbf{n}; t)$

Fokker–Planck eq. $\partial_t p = -\partial_{n_i} [\beta_i(\mathbf{n}) p] + \frac{1}{2} \partial_{n_i} \partial_{n_j} [\alpha_{ij}(\mathbf{n}) p]$

$\alpha_{ij}(\mathbf{n})$ and $\beta_i(\mathbf{n})$ depend on γ and on Γ_{ijpq} (and hence on a, b, c, d)



In spherical coordinates: $p(\vartheta, \varphi; t)$

$$\int_0^\pi \int_0^{2\pi} p(\vartheta, \varphi; t) \sin \theta \, d\theta \, d\varphi = 1$$

Fokker–Planck equation

$$\partial_t p = -\partial_\vartheta [\mathcal{B}_\vartheta(\vartheta)p] + \frac{1}{2}\partial_{\vartheta\vartheta}^2 [\mathcal{A}_{\vartheta\vartheta}(\vartheta)p] + \frac{1}{2}\mathcal{A}_{\varphi\varphi}(\vartheta)\partial_{\varphi\varphi}^2 p$$

$$B_\vartheta(\vartheta) = \cos \vartheta (C_1 + C_2 \sin^2 \vartheta + C_3 \sin^4 \vartheta)$$

$$A_{\vartheta\vartheta}(\vartheta) = C_4 \sin \vartheta + C_5 \sin^3 \vartheta + C_6 \sin^5 \vartheta$$

$$A_{\varphi\varphi}(\vartheta) = C_7 \sin \vartheta + C_8 \csc \vartheta$$

The constants C_i depend on γ and on a, b, c, d

Boundary conditions:

$$p(\vartheta + \pi, \varphi; t) = p(\vartheta, \varphi; t) \quad p(\vartheta, \varphi + 2\pi; t) = p(\vartheta, \varphi; t)$$

Evolution of $\vartheta(t)$

$$\dot{\vartheta} = \mathcal{B}_\vartheta(\vartheta) + \sqrt{\mathcal{A}_{\vartheta\vartheta}(\vartheta)} \xi(t)$$

Stationary PDF of orientations (rods: $\gamma = 1$)

$$p_{\text{st}}(\theta) \propto \frac{\chi(\theta)}{[4a + \mathcal{D} - b + d + (2b + 5d) \sin^2 \vartheta + c \sin^4 \vartheta]^{3/4}}$$

$$\chi(\vartheta) = \begin{cases} \exp \left\{ -\frac{3d}{2\sqrt{\Delta}} \arctan \left[\frac{2b+5d+2c \sin^2(\vartheta)}{\sqrt{\Delta}} \right] \right\} & (\Delta > 0) \\ \exp \left\{ \frac{3d}{2[2b+5d+2c \sin^2(\vartheta)]} \right\} & (\Delta = 0) \\ \left| \frac{\sqrt{-\Delta} + 2b + 5d + 2c \sin^2(\vartheta)}{\sqrt{-\Delta} - 2b - 5d - 2c \sin^2(\vartheta)} \right|^{\frac{3d}{4\sqrt{-\Delta}}} & (\Delta < 0) \end{cases}$$

$$\Delta = 4(4a + \mathcal{D} - b + d)c - (2b + 5d)^2$$

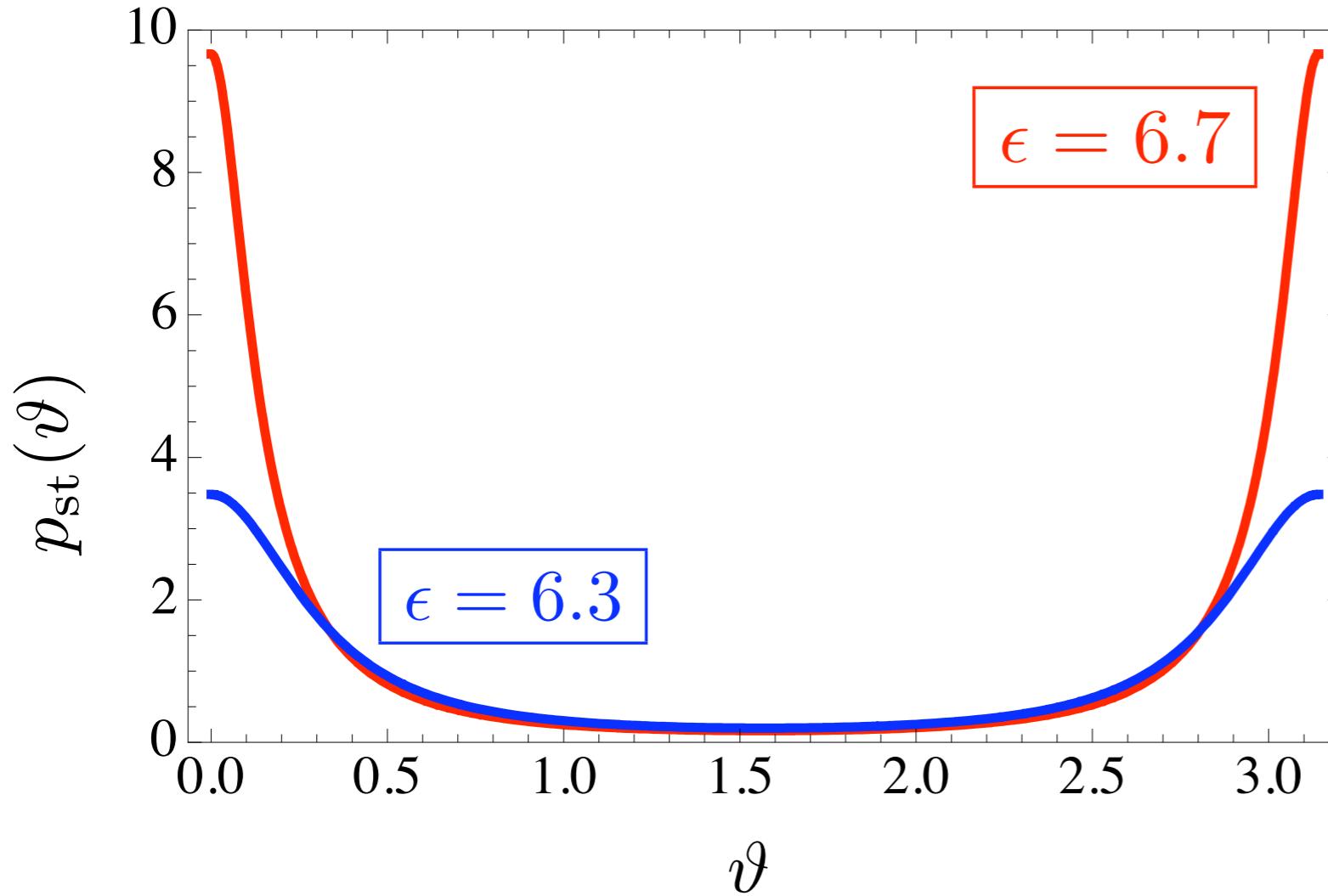
$$2a = \overline{(\partial_2 v_1)^2} - \overline{(\partial_1 v_1)^2}$$

$$d = \overline{(\partial_3 v_3)^2} - \overline{(\partial_1 v_1)^2}$$

$$b = \overline{(\partial_2 v_1)^2} - \overline{(\partial_3 v_1)^2}$$

$$2a + b + c + 4d = \overline{(\partial_1 v_3)^2} - \overline{(\partial_3 v_3)^2}$$

Tumbling motion



$$\epsilon = \frac{3b + c}{4a}$$

$$d = \overline{(\partial_3 v_3)^2} - \overline{(\partial_1 v_1)^2} = 0$$

$$p_{\text{st}}(\vartheta) \propto [4a + \mathcal{D} - b + 2b \sin^2 \vartheta + c \sin^4 \vartheta]^{-3/4}$$

If $b/c \geq 0$: two maxima in 0 and π and one minimum in $\pi/2$

Tumbling motion

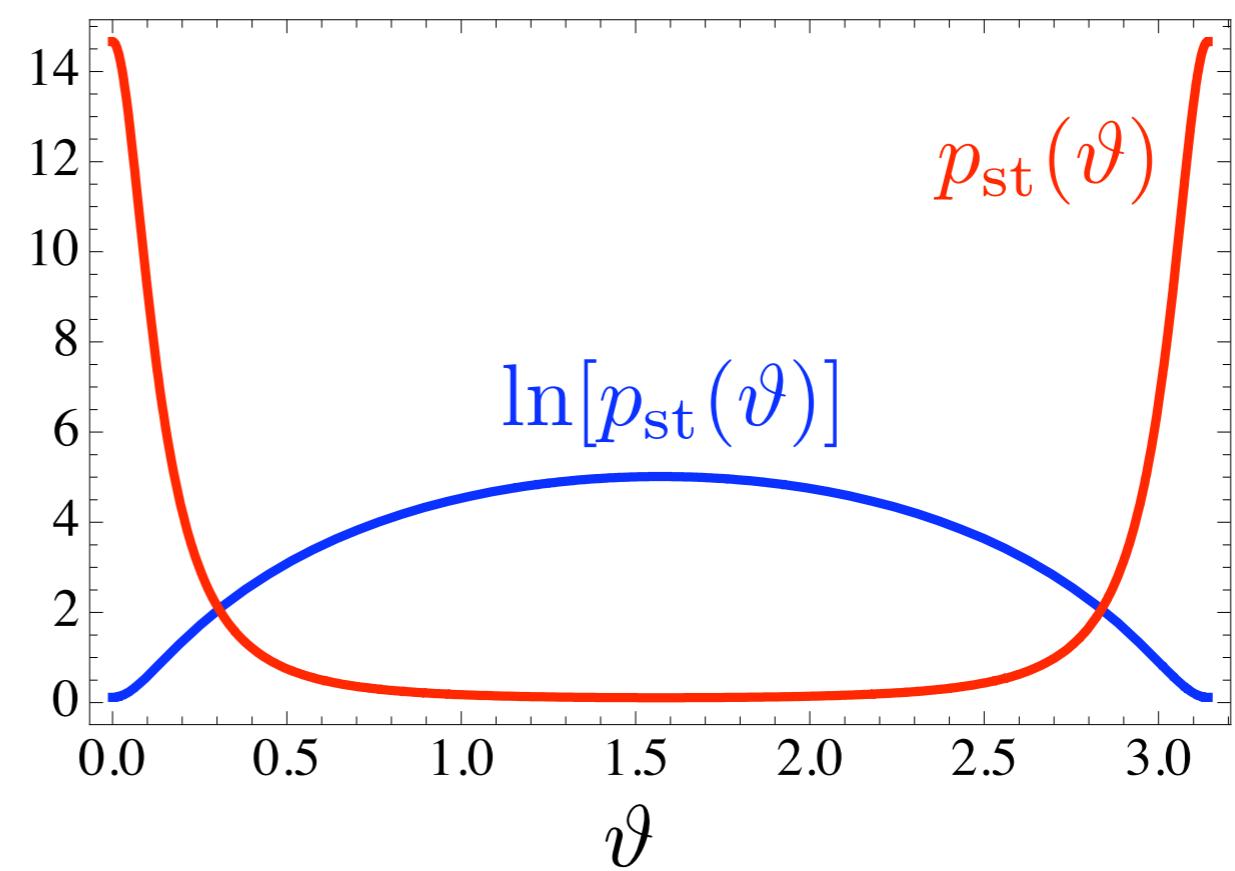
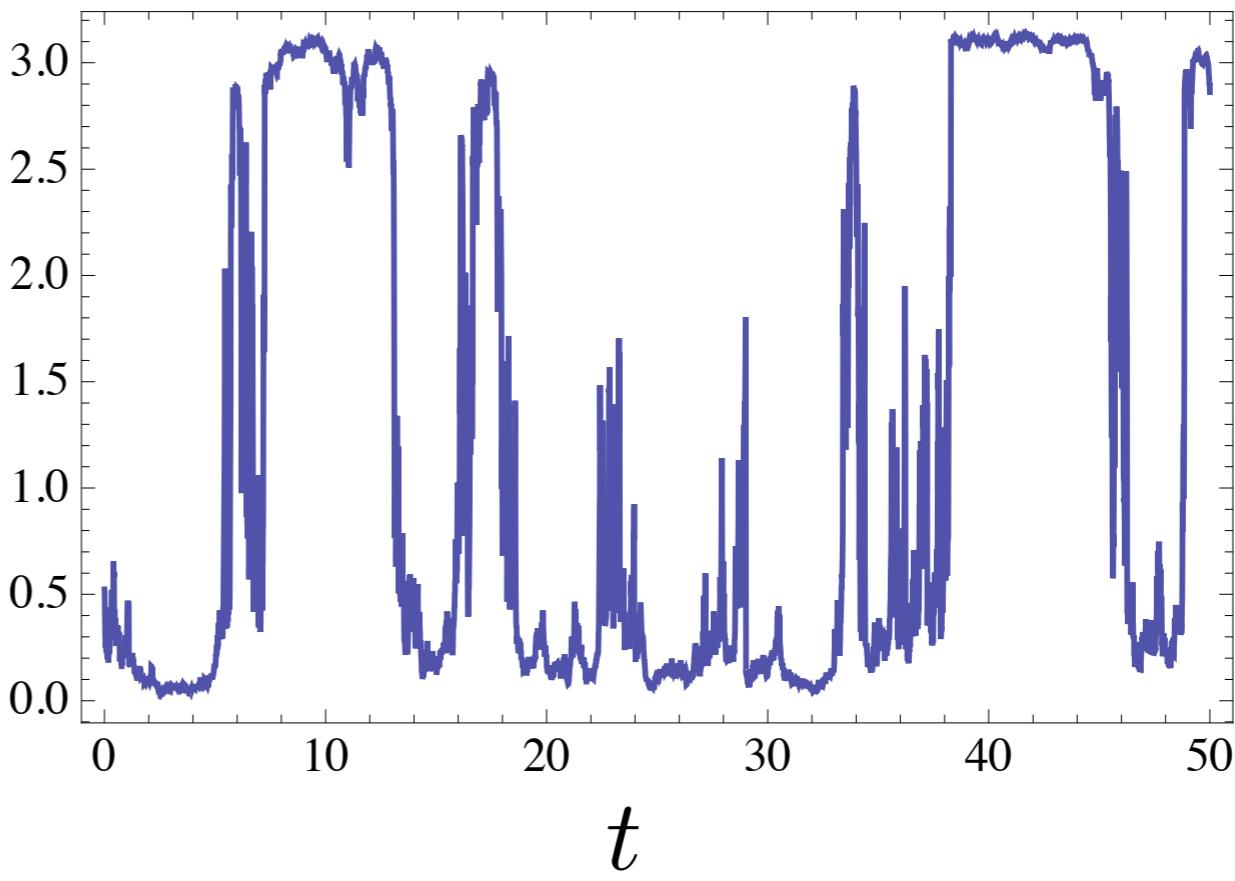
$$a = 0.01$$

$$b = 0.49$$

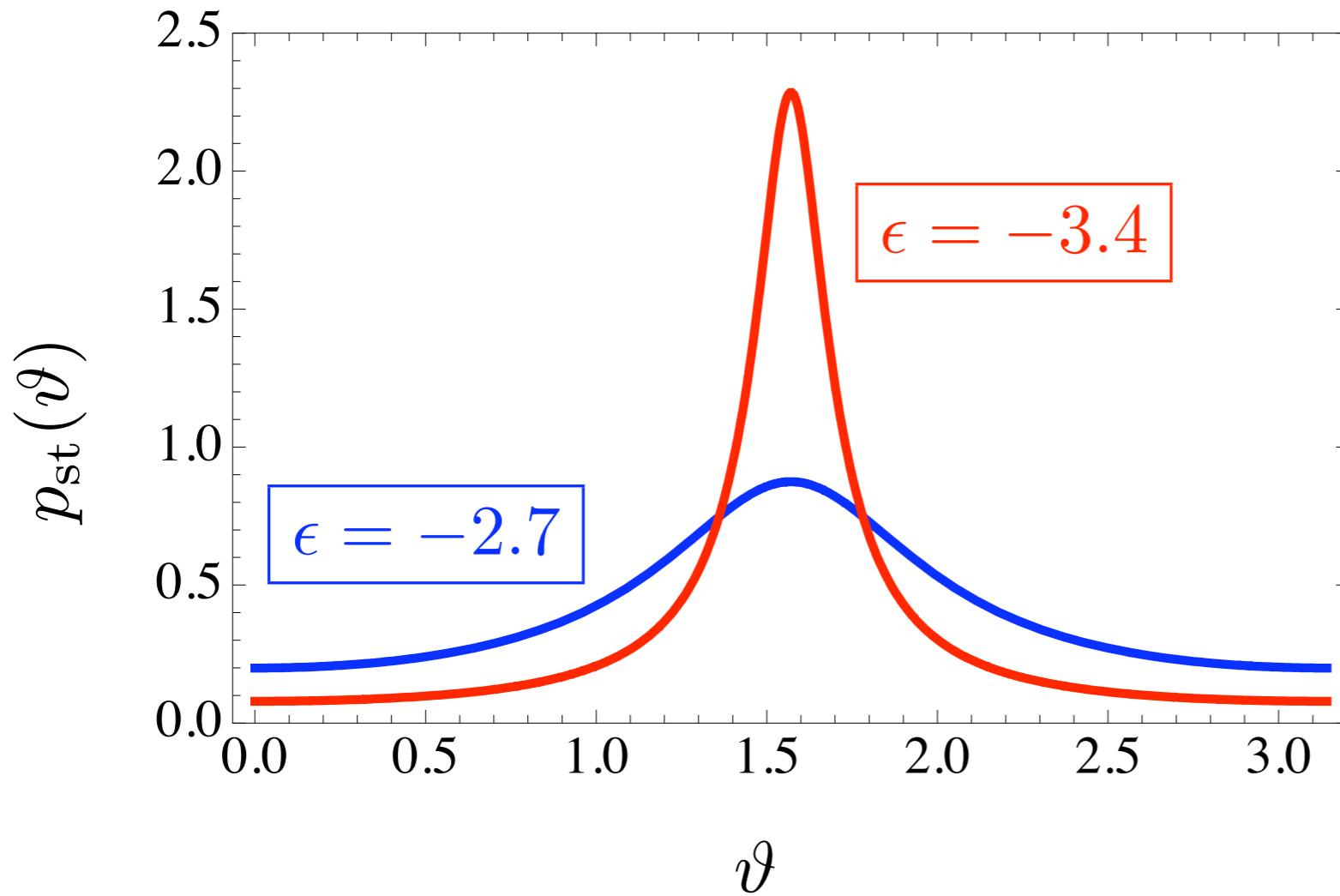
$$c = 20$$

$$d = 0$$

$$\vartheta(t)$$



Rotation in the plane perpendicular to λ



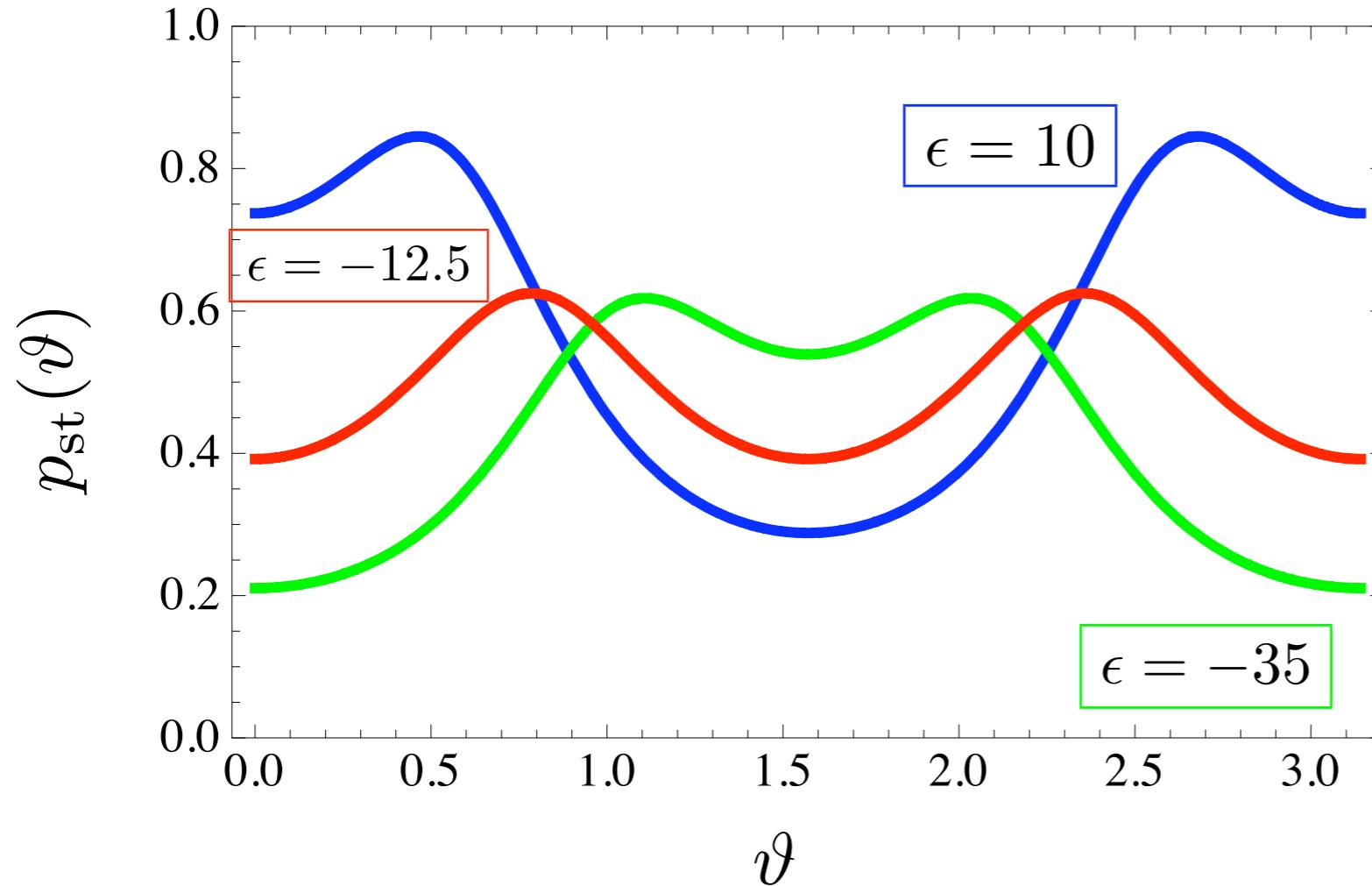
$$\epsilon = \frac{3b + c}{4a}$$

$$d = \overline{(\partial_3 v_3)^2} - \overline{(\partial_1 v_1)^2} = 0$$

$$p_{\text{st}}(\vartheta) \propto [4a + \mathcal{D} - b + 2b \sin^2 \vartheta + c \sin^4 \vartheta]^{-3/4}$$

If $b/c \leq -1$: two minima in 0 and π and one maximum in $\pi/2$

Intermediate preferential orientation



$$\epsilon = \frac{3b + c}{4a}$$

$$d = \overline{(\partial_3 v_3)^2} - \overline{(\partial_1 v_1)^2} = 0$$

$$p_{\text{st}}(\vartheta) \propto [4a + \mathcal{D} - b + 2b \sin^2 \vartheta + c \sin^4 \vartheta]^{-3/4}$$

If $-1 < b/c < 0$: two additional extrema in $\cos \vartheta_\star = \pm \sqrt{1 + b/c}$

General axisymmetric flow

$$p'(\vartheta) = \omega(\vartheta)[2b + c + 6d - c \cos(2\vartheta)] \sin(2\vartheta)$$

with $\omega(\vartheta) > 0$

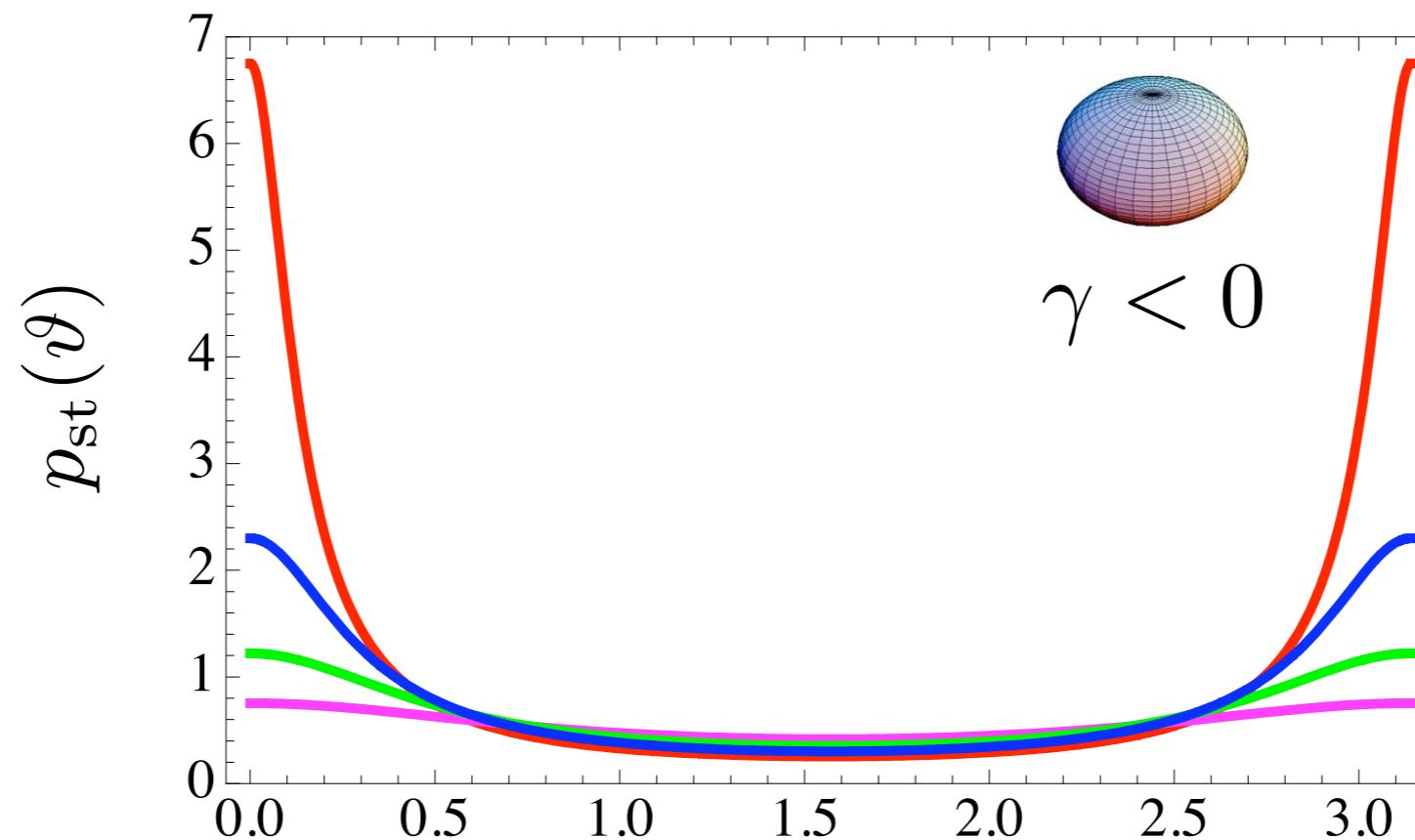
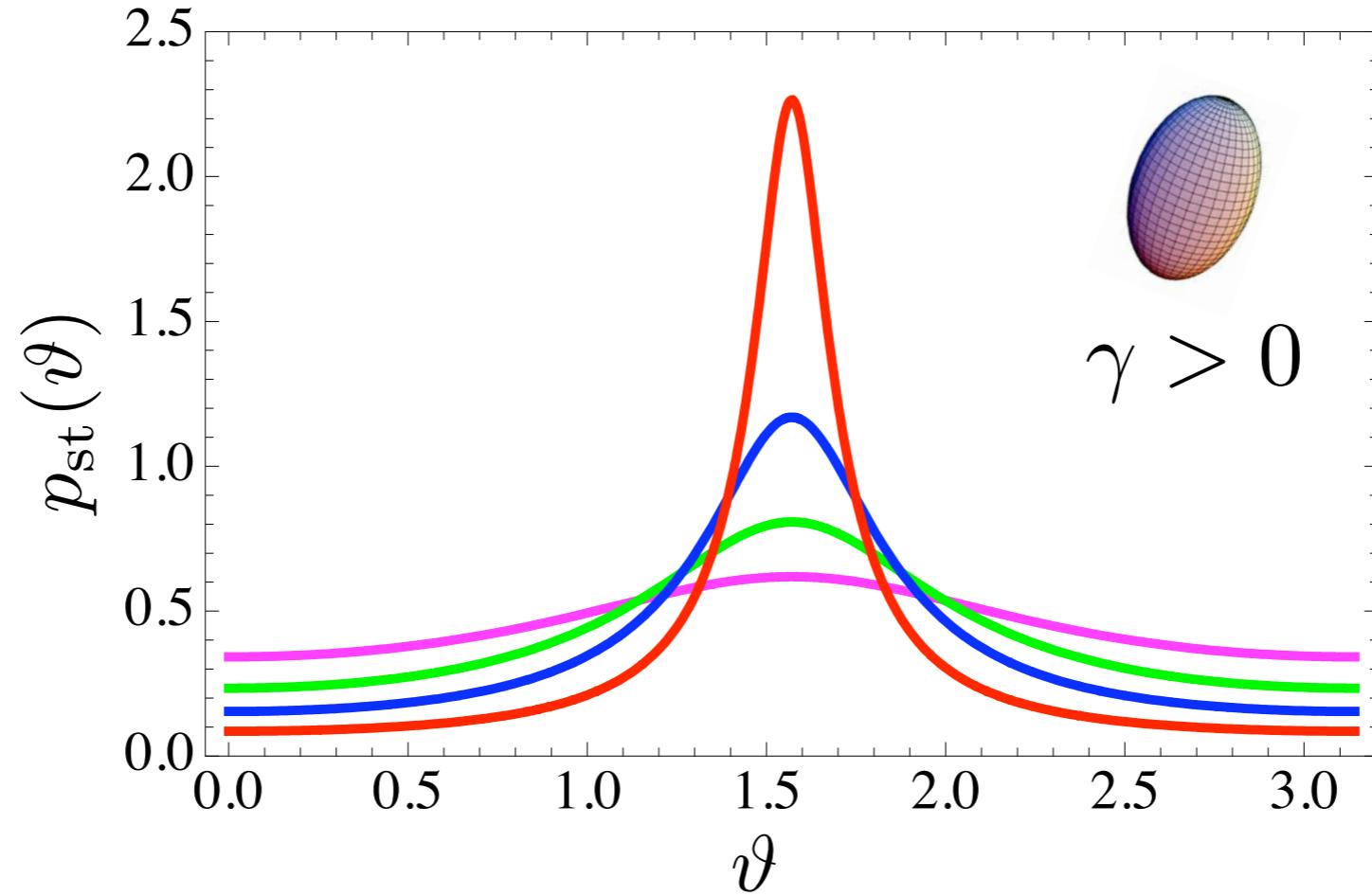
Extrema:

$$\begin{cases} \vartheta_* = 0, \frac{\pi}{2}, \pi \\ \cos \vartheta_* = \pm \sqrt{1 + \frac{b + 3d}{c}} \end{cases}$$

Three or five extrema in $(0, \pi)$

Bodies of revolution of a general shape

$|\gamma| = 1$
 $|\gamma| = 3/4$
 $|\gamma| = 1/2$
 $|\gamma| = 1/4$



$a = 0.1$
 $b = -3.9$
 $c = 0$
 $d = 0$