STABILITY AND BIFURCATIONS OF HETEROCLINIC CYCLES OF TYPE Z

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(submitted to "Nonlinearity")

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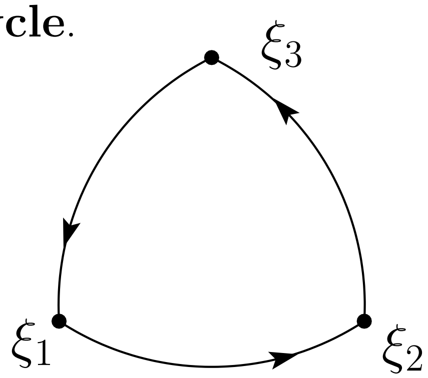
We consider a smooth dynanical system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f : \mathbb{R}^n \to \mathbb{R}^n.$$

Suppose it possesses:
equilibria $\xi_1, \ldots, \xi_m \in \mathbb{R}^n$;
connecting trajectories $\kappa_j : \xi_j \to \xi_{j+1}$ for all $1 \leq j \leq m$ (where $\xi_{m+1} = \xi_1$).

The union of the **equilibria** and the **connecting trajectories** is called a **heteroclinic cycle**.



Example of a heteroclinic cycle with three equilibria.

In a generic system heteroclinic cycles are of codimension at least one.

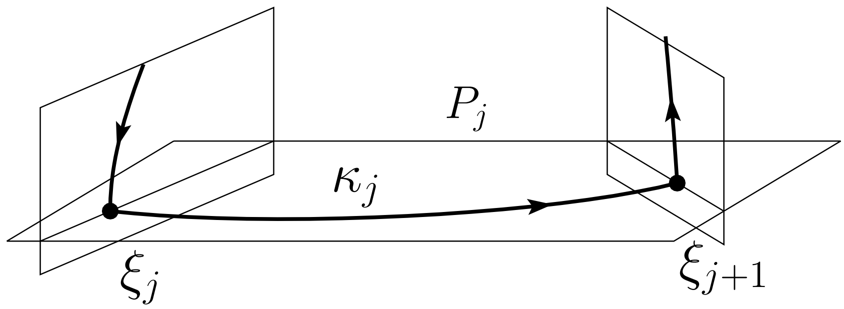
In a Γ-equivariant system, i.e.

$$\gamma f(\mathbf{x}) = f(\gamma \mathbf{x}) \text{ for any } \gamma \in \Gamma,$$

where $\Gamma \subset O(n)$, heteroclinic cycles can be of codimension zero.

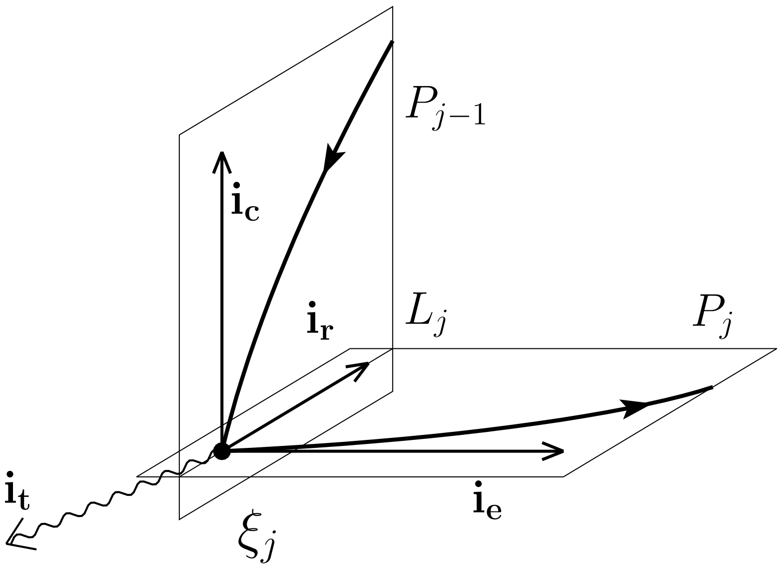
A heteroclinic cycle is **structurally stable (or robust)**, if for any j there exists an invariant subspace P_j, such that

- ξ_{j+1} is a sink in P_j;
- $\kappa_j \subset P_j$.



Eigenvalues of $df(\xi_j)$ can be divided into four classes:

- **radial** – eigenvalues with associated eigenvectors in $L_j = P_j \cap P_{j-1}$
- **contracting** – eigenvalues with associated eigenvectors in $P_{j-1} \ominus L_j$
- **expanding** – eigenvalues with associated eigenvectors in $P_j \ominus L_j$
- **transverse** – other eigenvalues



A structurally stable heteroclinic cycle $X \in \mathbb{R}^n \setminus \{0\}$ is **simple**, if for any j

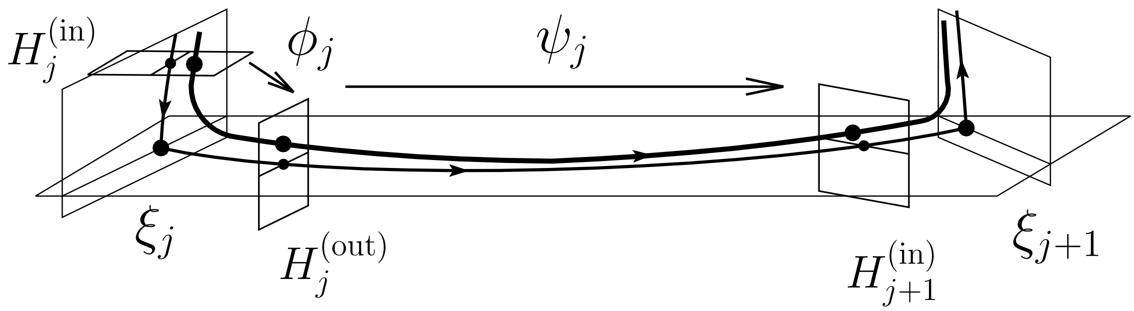
- all eigenvalues of $df(\xi_j)$ are distinct;
- $\dim(P_j \ominus L_j) = 1$.

Poincaré map is a superposition of **local** and **global** maps.

Eigenvectors of $df(\xi_j)$ constitute a basis, used in a neighbourhood of ξ_j.
The respective local coordinates of a point are $(\mathbf{u}, v, w, \mathbf{z})$,
split into the radial, contracting, expanding and transverse components.
$H_j^{(\mathrm{in})}$ intersects κ_{j-1} at $(\mathbf{u}_0, v_0, 0, 0)$, $|(\mathbf{u}_0, v_0)| = \tilde{\delta}$;
$H_j^{(\mathrm{out})}$ is the plane $w = \tilde{\delta}$.

Local map $\phi_j : H_j^{(\mathrm{in})} \to H_j^{(\mathrm{out})}$

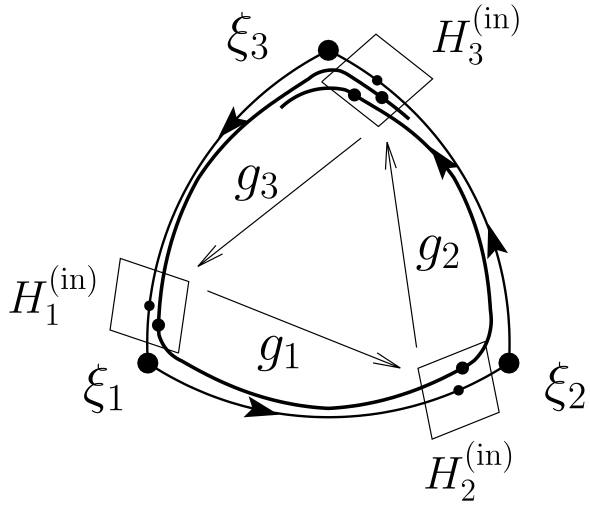
Global map $\psi_j : H_j^{(\mathrm{out})} \to H_{j+1}^{(\mathrm{in})}$



Denote $g_j = \psi_j \phi_j : H_j^{(\mathrm{in})} \to H_{j+1}^{(\mathrm{in})}$.

Poincaré map $g^{(j)} : H_j^{(\mathrm{in})} \to H_j^{(\mathrm{in})}$ is the superposition

$g^{(j)} = g_{j-1} \ldots g_1 g_m \ldots g_j$.



Near ξ_j, the equation $\dot{\mathbf{x}} = f(\mathbf{x})$ reduces to

$$\begin{aligned} \dot{v} &= -c_j v \\ \dot{w} &= e_j w \\ \dot{z}_s &= t_{j,s} z_s \end{aligned}$$

(g_j is contracting in radial directions $\Rightarrow$ radial coordinates are ignored), where c_j, e_j and $\{t_{j,s}\}$, $1 \leq s \leq n_t$, are contracting, expanding and transverse eigenvalues of $df(\xi_j)$.

$\Rightarrow$ The **local map** takes the form

$$\phi_j(w, \mathbf{z}) = (v_0 w^{c_j/e_j}, z_1 w^{-t_{j,1}/e_1}, ..., z_{n_t} w^{-t_{j,n_t}/e_{n_t}}).$$

The **global map** is predominantly linear:

$$\psi_j \begin{pmatrix} w \\ \mathbf{z} \end{pmatrix} = B_j \begin{pmatrix} w \\ \mathbf{z} \end{pmatrix}.$$

Definition.

A heteroclinic cycle is of **type Z**, if for all j

$$B_j = C_j D_j,$$

where

C_j is a **permutation matrix**;

D_j is a **diagonal matrix**.

Why this name?

Stability and bifurcations of simple heteroclinic cycles in particular systems:
Aguiar, Castro, *Physica D* **239**, 1598–1609, 2010;
Brannath, *Nonlinearity* **7**, 1367–1384, 1994;
Chossat, Krupa, Melbourne, Scheel, *Physica D* **100**, 85–100, 1997;
Driesse, Homburg, *J. Differential Equations* **246**, 2681–2705, 2009;
Kirk, Silber, *Nonlinearity* **7**, 1605–1621, 1994;
Krupa, Melbourne, *Proc. Roy. Soc. Edinburgh* **134A**, 1177–1197, 2004;
Podvigina, Ashwin, *Nonlinearity* **24**, 887–929, 2011;
Postlethwaite, Dawes, *Nonlinearity* **23**, 621–642, 2010.

Necessary and sufficient conditions for asymptotic stability of **type A** cycles: Krupa, Melbourne, *Ergodic Theory Dyn. Syst.*, **15**, 121–148, 1995.

All simple non-**type-A** cycles, studied so far, are of **type Z**.

New coordinates: $\boldsymbol{\eta} = (\ln|w|, \ln|z_1|, ..., \ln|z_{n_t}|)$.

The map g_j reduces to $\mathcal{M}_j \boldsymbol{\eta} = M_j \boldsymbol{\eta} + F_j$, where

$$M_j := C_j A_j = C_j \begin{pmatrix} a_{j,1} & 0 & 0 & \dots & 0 \\ a_{j,2} & 1 & 0 & \dots & 0 \\ a_{j,3} & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ a_{j,N} & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$a_{j,1} = -c_j/e_j, \quad a_{j,l+1} = -t_{j,l}/e_j, \quad 1 \le l \le n_t, \quad 1 \le j \le m.$$

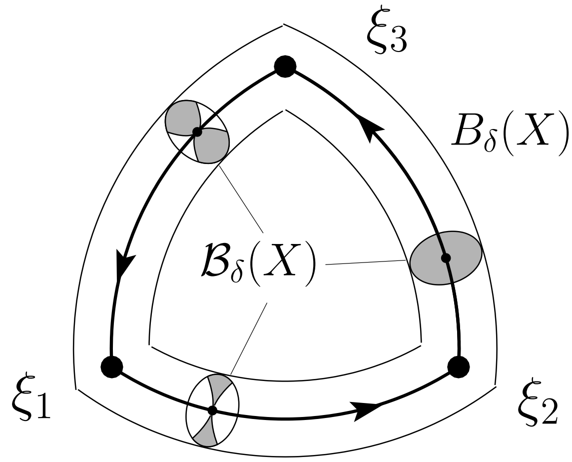
Finite F_j are ignored $\Rightarrow$ the **Poincaré map** $g^{(j)} : H_j^{(\mathrm{in})} \to H_j^{(\mathrm{in})}$ is now

$$M^{(j)} = M_{j-1} \dots M_1 M_m \dots M_j.$$

Theorem 1.
Suppose all transverse eigenvalues of $df(\xi_j)$ are negative for all j.
Let $\lambda_{\max}$ denote the largest in absolute value eigenvalue of $M^{(1)} = M_m \ldots M_1$.
Then

- If $|\lambda_{\max}| > 1$, then the heteroclinic cycle is asymptotically stable;
- If $|\lambda_{\max}| \leq 1$, then it is unstable.

Example of a set, which is not asymptotically stable,
but is **fragmentarily asymptotically stable**:



$B_\delta(X) = \{\mathbf{x} \in R^n : \; d(\mathbf{x}, X) < \delta\}$

$\mathcal{B}_\delta(X) = \{\mathbf{x} \in \mathbb{R}^n : \; d(\Phi_t(\mathbf{x}), X) < \delta \; \forall \; t \geq 0 \text{ and } \lim_{t\to\infty} d(\Phi_t(\mathbf{x}), X) = 0\}$

Definition.

A compact invariant set X is **fragmentarily asymptotically stable**, if $\mu(\mathcal{B}_\delta(X)) > 0$ for any $\delta > 0$.

Denote

$\lambda^j_{\max}$ the largest in absolute value eigenvalue of

$$M^{(j)} = M_{j-1} \ldots M_1 M_m \ldots M_j,$$

$\mathbf{v}^j$ the associated eigenvector.

Theorem 2.

Suppose some transverse eigenvalues of $df(\xi_j)$ are positive for some j. Then

- If $\lambda^j_{\max}$ is real, $\lambda^j_{\max} > 1$ and $v^j_s v^j_k > 0$ for all j, s and k, then the heteroclinic cycle is **fragmentarily asymptotically stable**.
- Otherwise it is **unstable**.

Bifurcations of a heteroclinic cycle in the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \alpha), \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

We assume that the heteroclinic cycle exists for $\alpha_- \leq \alpha \leq \alpha_+$,
where $\alpha_- < 0$ and $\alpha_+ > 0$.

Theorem 3.
Suppose transverse eigenvalues of $df(\xi_j)$ are negative for all j,
$|\lambda_{\max}| > 1$ for $\alpha < 0$, and $|\lambda_{\max}| < 1$ for $\alpha > 0$.
Then **a periodic orbit bifurcates from the heteroclinic cycle**.
If the orbit exists for $\alpha > 0$, then it is **asymptotically stable**.
If the orbit exists for $\alpha < 0$, then it is **unstable**.

Theorem 4.

Suppose for some j some transverse eigenvalues of $df(\xi_j)$ are positive,
for $\alpha < 0$ conditions of Theorem 2 are satisfied,
and for $\alpha > 0$ they are not satisfied.
Then

- if $|\lambda_{\max}| < 1$ for $\alpha > 0$, then **a periodic orbit bifurcates from the heteroclinic cycle**. If the orbit exists for $\alpha > 0$, then it is **asymptotically stable**. If the orbit exists for $\alpha < 0$, then it is **unstable**.
- if $|\lambda_{\max}| > 1$ for $\alpha > 0$, then **no periodic orbits or other invariant sets bifurcate from the cycle**.