Cold Atomic Fermi Gases

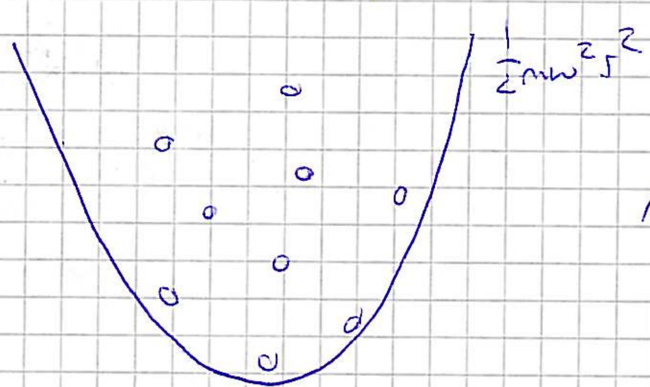
Outline:　- ~~What are they?~~

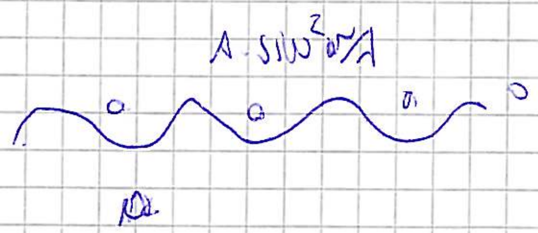
Few-body → - Atom-atom interaction. Feshbach resonances.

Many-body → - BEC-BCS cross-over & universality

Cold atomic gases:

- Trap and cool alkali atoms: H, Li, Na, K, Rb, Cs (Sodium, Potassium)
  in a vacuum chamber. Also He.
  Also molecules now!





- $N \sim 10 - 10^9$　- Bosons & Fermions: ^6Li: $3e + 3n + 3p$ = fermion, ^7Li: Boson

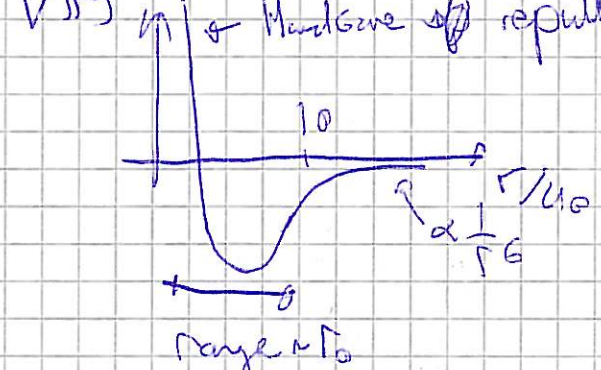
- Very dilute to prohibit 3-body loss. (Gas metastable state)
  $n \sim 10^{13} - 10^{15}\,cm^{-3}$　Air: $n \sim 10^{19}\,cm^{-3}$, Solid $n \sim 10^{23}\,cm^{-3}$

- Very cold: $T \sim 0{,}45\,nK$　$n\lambda^2 \sim 1$ to get quantum degeneracy.

Tailor made quantum system: - BEC physics
- Fermionic superfluidity
- Strong Coupling
- Macroscopic quantum phenomena.
- Optical lattice

②

Atom-Atom interaction



$r_0 \sim a_0 << n^{-1/3} \sim 10^{-6}\,m.$

So very short range. $\sim g\,\delta(\vec r)$

Will see we can manipulate this.

Basic scattering theory (Needed unfortunately, we need a):

$$\psi(\vec r_1, \vec r_2) = \psi_{cm}\left(\frac{\vec r_1 + \vec r_2}{2}\right) \cdot \psi_r(\vec r) \qquad \vec r = \vec r_1 - \vec r_2, \quad \psi_r = \frac{u(\vec r)}{r}$$

2 distinguishable with mass m_1 & m_2.

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

$$\cancel{-\frac{\hbar^2}{2m_r}\frac{\partial^2 u}{\partial r^2}} \quad u(\vec r) = u(r)\, Y_{lm}(\Omega)$$

$$\cancel{\left[-\frac{\hbar^2}{2m}\nabla_r^2 + V(\vec r)\right]\psi(\vec r) = E\,\psi(\vec r)}$$

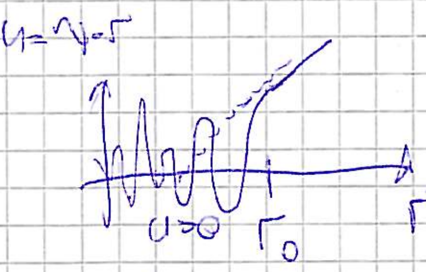
$$\left[-\frac{\hbar^2}{2m}\partial_r^2 + V(r) + \frac{l(l+1)}{r^2}\right]u(r) = E\,u(r)$$

$l = 0$ enough for $k \to 0$.
(s-wave scattering)

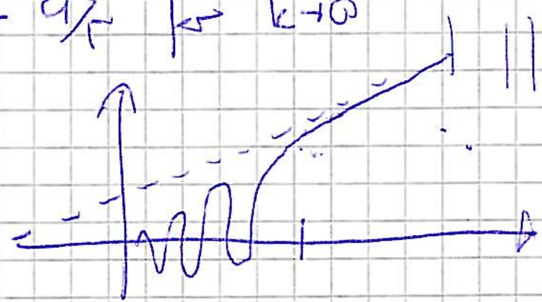
Scattering states: $\psi_{sc} = e^{i\vec k_0 \vec r} + \psi_{sc}(\vec r) = \underbrace{e^{i\vec k_0 \vec r} + f_k \cdot \frac{e^{ikr}}{r}}_{\text{Solution outside range of potential}}$

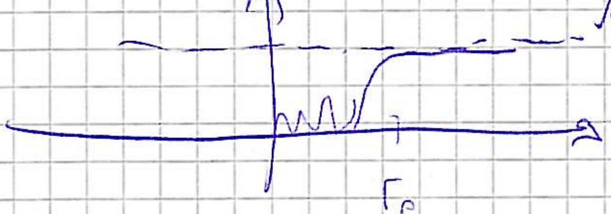
Scattering length: $f_k \to -a$ for $k \to 0$

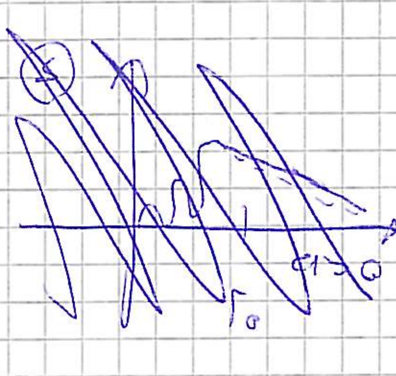
$\Downarrow$ $\psi_{sc} \to 1 - a/r$ for $k \to 0$



repulsive







Formulate in momentum space:

$$\psi(\vec r) = e^{i\vec k_0 \vec r} + \psi_{sc}(\vec r) = \frac{1}{V}\sum_{k} e^{i\vec k \vec r} \cdot \psi(\vec k)$$

$\Downarrow$

$$\psi(\vec k') = \int d^3r\, e^{-i\vec k' \vec r} \cdot \psi(\vec r) = V \cdot \delta_{\vec k_0, \vec k'} + \psi_{sc}(k)$$

Schrödinger eq.:

$$\frac{\hbar^2}{2m_r} k^2$$

$$-\frac{\hbar^2}{2m_r}\nabla^2 \psi(\vec r) + V(\vec r)\psi(\vec r) = E\,\psi(\vec r)$$

$\Downarrow$

$$-\frac{\hbar^2}{2m_r}\nabla^2 \frac{1}{V}\sum_{\vec k'} e^{i\vec k' \vec r}\psi(\vec k') + \frac{1}{V^2}\sum_{\vec k', \vec k''} V(\vec k')\psi(\vec k'') e^{i(\vec k' + \vec k'')\cdot \vec r} = \frac{\hbar^2 k^2}{2m_r} \cdot \frac{1}{V}\sum_{\vec k'} e^{i\vec k' \vec r} \cdot \psi(\vec k')$$

$\Downarrow$

$$\cancel{+\frac{\hbar^2 k^2}{2m_r} e^{i\vec k \vec r}} + \frac{\hbar^2}{2m_r}\frac{1}{V}\sum_{\vec k'} k'^2 e^{i\vec k' \vec r} \cdot \psi_{sc}(k') + \frac{1}{V}\sum_{k'} V(\vec k') \cdot e^{i(\vec k' + \vec k)\vec r} + \frac{1}{V^2}\sum_{k', k''} e^{i(\vec k' + \vec k'')\vec r} \cdot V(k')\psi_{sc}(\vec k'')$$

$$= \cancel{\frac{\hbar^2 k^2}{2m_r} e^{i\vec k \vec r}} + \frac{\hbar^2 k^2}{2m}\frac{1}{V}\sum_{k'} e^{i\vec k' \vec r} \cdot \psi_{sc}(k')$$

④

$$\Downarrow \int d^3r\, e^{-i\bar{k}'\cdot\bar{r}}$$

$$\frac{\hbar^2}{2m_r}(k^2-k'^2)\psi_{sc}(k') = V(\bar{k}-\bar{k}') + \frac{1}{V}\sum_{k''} V(\bar{k}'-\bar{k}'')\psi_{sc}(k'')$$

$$\Downarrow$$

$$\psi_{sc}(k') = \frac{1}{\frac{\hbar^2}{2m_r}\cdot(k^2-k'^2)+i\delta}\cdot\left[V(\bar{k}',\bar{k}) + \frac{1}{V}\sum_{k''} V(\bar{k}',\bar{k}'')\psi_{sc}(k'')\right]$$

outgoing waves.

Integral eq^n.

The solution can be written in terms of the T-matrix:

$$\psi_{sc}(\bar{k}') = \left(\frac{\hbar^2}{2m}k^2 + i\delta - \frac{\hbar^2 k'^2}{2m}\right)^{-1}\cdot T\left(\bar{k}',\bar{k};\frac{\hbar^2 k^2}{2m_r}\right)$$

with Defined for arbitrary energy $\frac{k^2}{2m} \neq E$.

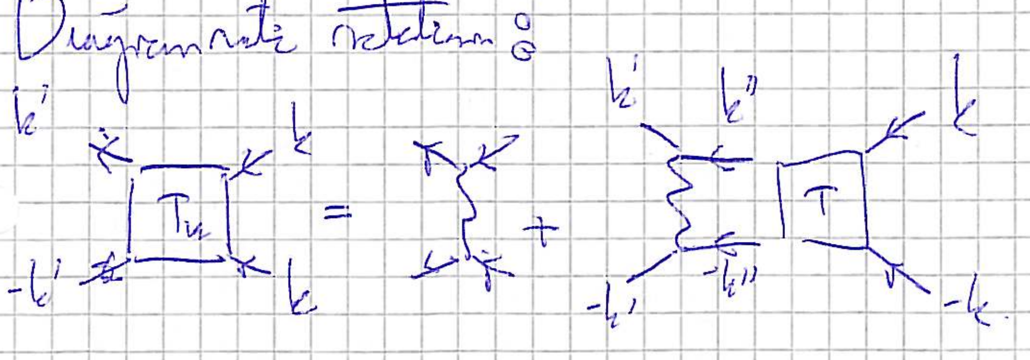
$$T(\bar{k}',\bar{k},E) = V(\bar{k}',\bar{k}) + \frac{1}{V}\sum_{k''} V(\bar{k}',\bar{k}'')\frac{1}{E-\frac{\hbar^2 k''^2}{2m}+i\delta}\cdot T(\bar{k}'',\bar{k},E)$$

Matrix notation:

$$T_{V_{rel}}(E) = V + V\cdot G^{(2)}_{V_{rel}}(E)\cdot T_{V_{rel}}(E)$$

$$\hookrightarrow \frac{\delta_{k,k'}}{E+i\delta-k^2/2m_r}$$ Pair propagator.

Diagrammatic notation:



Keep.

Check that it solves:

$$\left(\frac{\hbar^2}{2m}k^2 + i\delta - \frac{\hbar^2 k'^2}{2m_r}\right)\psi_{sc}(k') = V(\bar{k},\bar{k}') + \frac{1}{V}\sum_{\bar{k}''} V(\bar{k}',\bar{k}'')\psi_{sc}(\bar{k}'')$$

$\Downarrow$

$$T(\bar{k}',\bar{k},\overset{\hbar^2k^2/2m_r}{E}) = V(\bar{k}',\bar{k}) + \frac{1}{V}\sum_{\bar{k}''} V(\bar{k}',\bar{k}'')\cdot\psi_{sc}(k'') \quad \text{Correct.}$$

Connection between T & a:

$$\psi_{sc}(\bar{k}') = G^{(2)}_{vac}(k', \hbar^2k^2/2m_r)\cdot T(\bar{k}',\bar{k};\hbar^2k^2/2m_r)$$

$\Downarrow$

$$\psi_{sc}(\bar{r}) = \int\frac{d^3k'}{(2\pi)^3}e^{i\bar{k}'\bar{r}}\cdot\psi_{sc}(\bar{k}') = \int\frac{d^3k'}{(2\pi)^3}\cdot e^{i\bar{k}'\cdot\bar{r}}\cdot\frac{1}{\frac{\hbar^2k^2}{2m_r}+i\delta-\frac{\hbar^2k'^2}{2m_r}}\cdot T(\bar{k}',\bar{k};E)$$

Take $E = \frac{\hbar^2k^2}{2m_r}\to 0 \Leftrightarrow k\to 0.$ (look for zero energy solution to s-wave Schrödinger eq^n.)

$\Downarrow$

$$\psi_{sc}(\bar{r}) = \int\frac{d^3k'}{(2\pi)^3}\frac{e^{i\bar{k}'\cdot\bar{r}}}{i\delta-\hbar^2k'^2/2m_r}\cdot T(\bar{k}',0;0)$$

$\bar{k}' \lesssim \frac{1}{r}$

$$\underset{r\to\infty}{\simeq} -T(0,0;0)\cdot\int\frac{d^3k'}{(2\pi)^3}\frac{e^{i\bar{k}\bar{r}}}{\frac{\hbar^2k^2}{2m}} = -\frac{T(0,0;0)\cdot 2m_r}{\hbar^2\cdot 4\pi\cdot r}$$

$$= 1 - \frac{a}{r}$$

$$\left(\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\cdot e^{i\bar{k}\cdot\bar{r}} = \frac{1}{4\pi r}\right)$$

$\Downarrow$

$$\boxed{T(0,0;0) = \frac{2\pi\hbar^2 a}{m_r}}$$

Important. a is an observable. $\quad \sigma = 4\pi a^2$

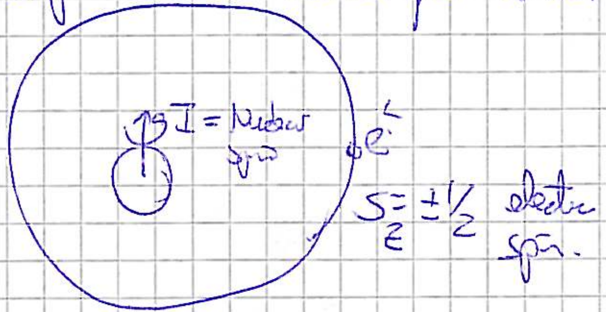
Realistic Interaction between atoms
Multichannel Scattering

Zeeman: $C = 2\mu_B \cdot B$

Spin-Hamiltonian for alkali-atoms: $H_{spin} = A\,\vec{I}\cdot\vec{S} + C\cdot S_z + D\cdot I_z$

Hyperfine Constant

$D = -\mu_n \cdot \frac{B}{I}$

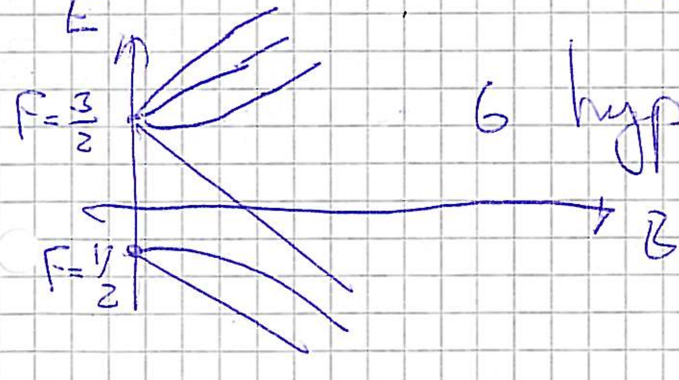


Gives hyperfine states:

$\vec{F} = \vec{I} + \vec{S} \Leftrightarrow F^2 = I^2 + S^2 + 2\vec{I}\cdot\vec{S}$

$\Rightarrow [H_{spin}, F_z] = 0.$

Example: ^{6}Li (Fermion) $I = 1$, $S = 1/2$

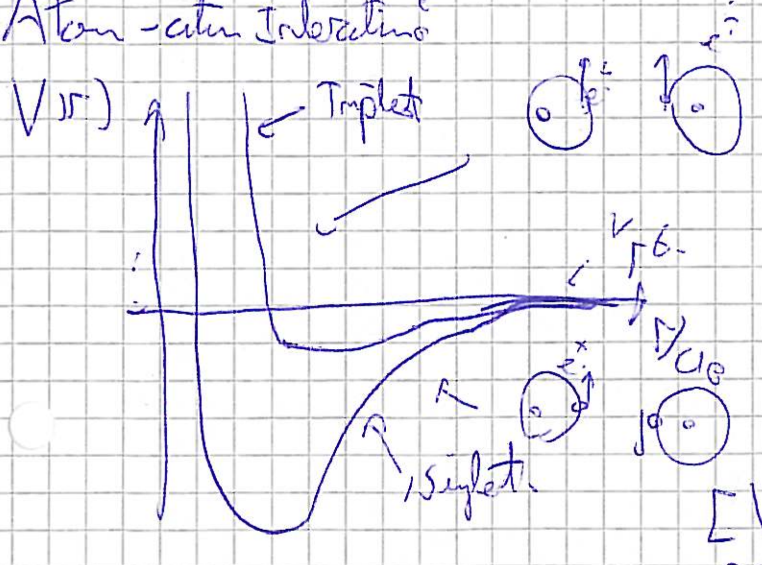


6 hyperfine states

$P_t = \frac{1}{2} S^2 = \frac{1}{2}(S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2) = \frac{3}{4} + \vec{S}_1\cdot\vec{S}_2$

$P_s = 1/4 - \vec{S}_1\cdot\vec{S}_2$

Atom-atom Interaction:



$V(r) = V_S \cdot P_S + V_t \cdot P_t$

$= \frac{V_S(r) + 3V_t(r)}{4} + (V_t(r) - V_S(r))\,\vec{S}_1\cdot\vec{S}_2$

$[V(r);\, H_{spin1} + H_{spin2}] \neq 0.$

$[V(r);\, F_z] = 0.$

Multichannel scattering: $H = H_0 + V$

$H_{spin}(1) + H_{spin}(2) + p^2/2m_r$

Here, V mixes different $|\alpha\beta;k\rangle$.

Here $H \simeq H_0$ and eigenstates are $|\alpha\beta;k\rangle$

$H_{spin}|\alpha\rangle = \varepsilon_\alpha$.

$H_0|\alpha\beta;k\rangle = \varepsilon_\alpha + \varepsilon_\beta + \hbar^2/2m_r$

Different „channels"

$$\psi(\bar r) = e^{i\bar k_{\alpha\beta}\cdot\bar r}\,|\alpha\beta\rangle + \sum_{\alpha'\beta'} f^{\alpha'\beta'}_{\alpha\beta}(\bar k_{\alpha\beta}, \bar k'_{\alpha'\beta'})\,\frac{e^{ik_{\alpha'\beta'}\cdot r}}{r}\,|\alpha'\beta'\rangle$$

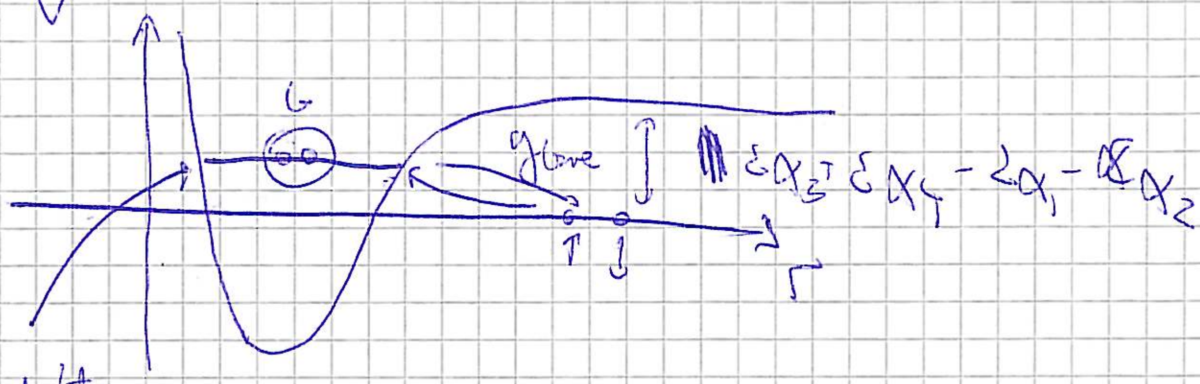
$$\varepsilon_{\alpha'} + \varepsilon_{\beta'} + \frac{\hbar^2}{2m_r} k'^2_{\alpha'\beta'} = \varepsilon_\alpha + \varepsilon_\beta + \frac{\hbar^2 k^2_{\alpha\beta}}{2m_r}$$

T-matrix: $T(k'_{\alpha'\beta'}; k_{\alpha\beta})$

$$T = V + V\cdot G_0\cdot T$$

Example: One open channel $|o\rangle = |\alpha_1 \alpha_2\rangle$

One closed channel $|c\rangle = |\alpha_3 \alpha_4\rangle$.



Doublette (of singlet) potential.

Could set up formulation for full multichannel scattering problem.

Use simplified model $\Rightarrow$ ("Effective theory"):

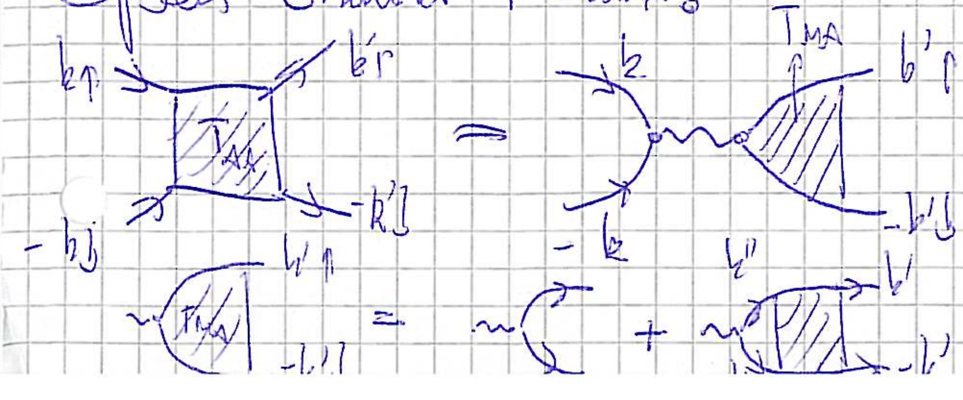
$$\hat{H} = \sum_{\bar{k}\sigma} \varepsilon_k a^{\dagger}_{k\sigma} a_{k\sigma} + \sum_{\bar{p}} E^{\text{bare}}_p \cdot b^{\dagger}_p b_p$$

$\varepsilon_k = k^2/2m$, $\sigma = \alpha_1, \alpha_2 = \uparrow \downarrow$, $E^{\text{bare}}_p = 2V_{\text{bare}} + p^2/4m$ Point boson.

$$+ \frac{1}{\sqrt{V}} \sum_{\bar{p},\bar{k}} g_{\text{bare}}(k) \left[b_p a_{\frac{p}{2}+k\uparrow} a_{\frac{p}{2}-k\downarrow} + h.c. \right]$$

$\sim$(

Open channel T-matrix:



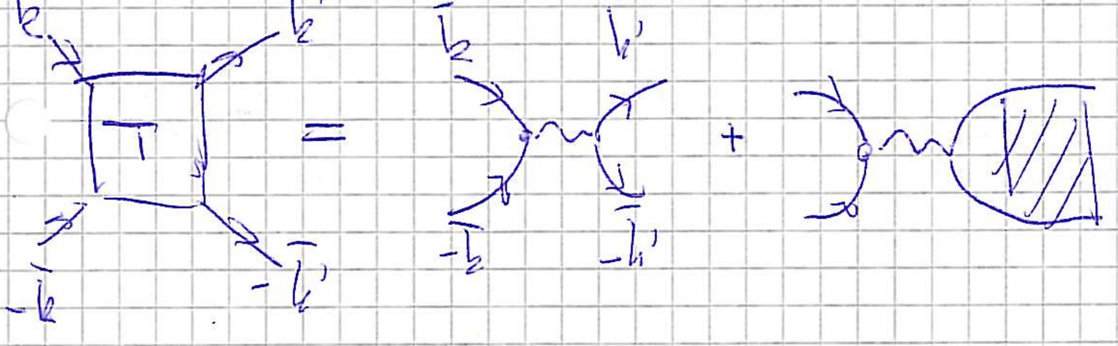
In math:

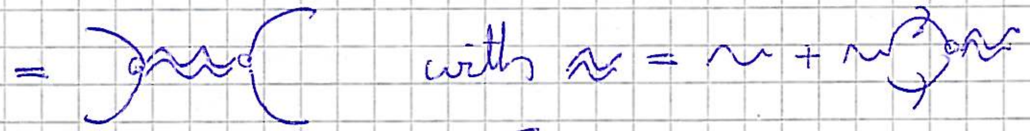
$$T_{AA}(\bar{k},\bar{k}',E) = g_{bare}(\bar{k})\cdot \overset{\frac{1}{E+i\delta-2\nu_{bare}-(h^2/4m)}}{D_0}\cdot T_{MA}(\bar{k}')$$

$$T_{MA}(\bar{k}') = g_{bare}(k') + g_{bare}(k'')\cdot \overset{\frac{1}{E+i\delta-k''^2/2m_r-(h^2/4m)}}{G_A(k'',E)}\cdot T_{AA}(k'',k',E)$$

$\Downarrow$

$$T_{AA}(\bar{k},\bar{k}',E) = g_{bare}(\bar{k})D_0\, g_{bare}(\bar{k}') + g_{bare}(\bar{k})D_0\cdot g_{bare}(\bar{k}'')\,G_A(\bar{k}'',E)\cdot T(\bar{k}'',\bar{k}')$$





$$= g_{bare}(\bar{k})\, D_0\, g_{bare}(\bar{k}')$$

$$T(\bar{k},\bar{k}',E) = g_{bare}(\bar{k})\, D(\bar{h},E)\, g_{bare}(\bar{k}')$$

$$D(\bar{h},E) = D_0 + D_0\cdot \underbrace{g_{bare}(\bar{k})\, G_A(k,h,E)\, g_{bare}(k)}_{\int \frac{d^3k}{(2\pi)^3}\, g^2_{bare}(k)\cdot \frac{1}{E+i\delta-k^2/2m_r-(h^2/4m)} \;\equiv\; \Pi(h;E)}\cdot D(k,E)$$

Molecule self-energy

$\Downarrow$

$$D(\vec{k},E) = \frac{1}{D_0^{-1} - \Pi(\vec{k},E)} = \frac{1}{E+i\delta - k^2/4m - 2\nu_{bare} - \Pi(k,E)}$$

Problem solved if we know $2\nu_{bare}$ and $g_{bare}(k)$.

Express in terms of observables (Landau Fermi-liquid theory)

(not our energy effective theory)

To do this:

$$\Pi_V(0,E) = \Pi_V(0,0) + \int \frac{dk^3}{(2\pi)^3} g^2_{bare}(k) \left(\frac{1}{E+i\delta - k^2/2m_r} + \frac{1}{k^2/2m_r} \right)$$

$$= \Pi_V(0,0) + \int \frac{dk^3}{(2\pi)^3} g^2_{bare}(k) \cdot \frac{E}{(E+i\delta - k^2/2m_r)\, k^2/2m_r}$$

$\Downarrow$

$$\Pi(0,E) = \Pi(0,0) + \underbrace{g^2_{bare}(0) \cdot \int \frac{dk^3}{(2\pi)^3} \frac{E}{(E+i\delta - k^2/2m_r)} \frac{2m_r}{k^2}}_{g^2_{bare}(0) \cdot \frac{m^{3/2}}{4\pi} \cdot \sqrt{-E}} + \underbrace{\int \frac{dk^3}{(2\pi)^3} (g^2_{bare}(k) - g^2_{bare}(0)) \cdot \ldots}_{\Delta\Pi(E)}$$

„High energy dressing".

$\sqrt{-1} = -i$

„Dressing" of molecule

$\Downarrow$

Energies low compared to range $\frac{1}{r_0^2}$

„low energy effective theory".

$$\Pi(0,E) \simeq \Pi(0,0) + g^2 \frac{m^{3/2}}{4\pi} \sqrt{-E} + \frac{\partial \Delta\Pi}{\partial E} \cdot E$$

So „dressed" molecule is (Exact solution) (11)

$$D(E) = \frac{1}{\omega - 2\nu_{bare} - \Pi(0,0) - \partial_\omega \Delta\Pi \cdot \omega - g_{bare}^2 \frac{m^{3/2}}{4\pi} \cdot \sqrt{-\omega}}$$

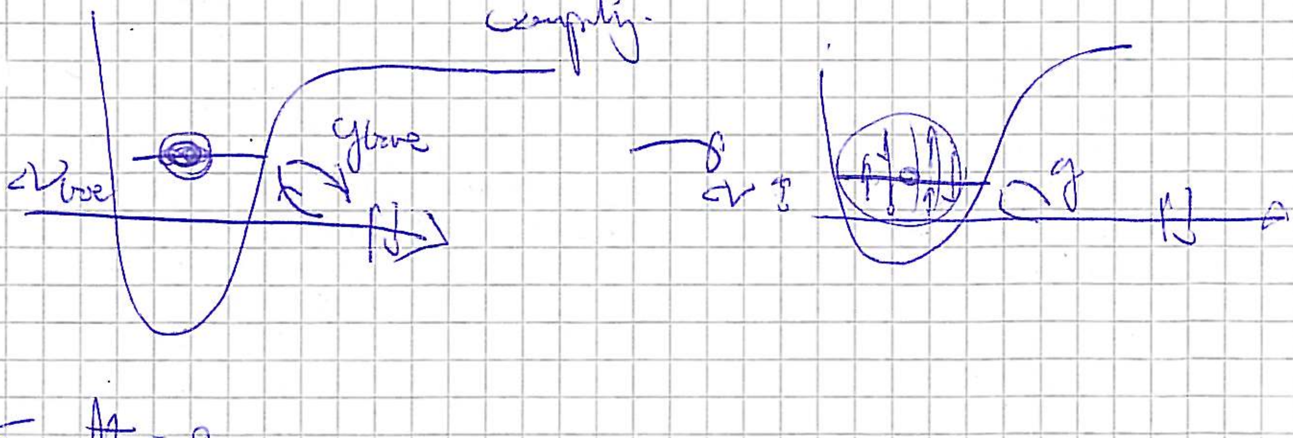
$$= \frac{1}{\underbrace{(1-\partial_\omega \Delta\Pi)}_{\tilde{Z}^{-1}} \cdot \omega - 2\nu_{bare} - \Pi(0,0) - g_{bare}^2 \frac{m^{3/2}}{4\pi} \sqrt{-\omega}}$$

Threshold screening

$$= \frac{\tilde{Z}}{\omega - 2\nu - g^2 \frac{m^{3/2}}{4\pi} \sqrt{-\omega}}$$

$2\nu = \tilde{Z}\,[2\nu_{bare} + \Pi(0,0)]$ shifted molecule energy due to high-energy screening.

$g^2 = g_{bare}^2 \cdot \tilde{Z}$: screened molecule-atom coupling.



Scattering:

$$T(\bar{k},\bar{k}',E) = g_{bare}(\bar{k})\, D(E) \cdot g_{bare}(\bar{k}') \approx \frac{g_{bare}^2 \cdot \tilde{Z}}{\omega - 2\nu - g^2 \frac{m^{3/2}}{4\pi} \sqrt{-\omega}}$$

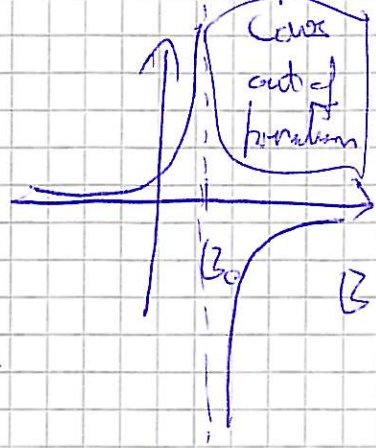
$$= \frac{g^2}{\omega - 2\nu - g^2 \frac{m^{3/2}}{4\pi} \sqrt{-\omega}}$$

Connection to scattering length:

Diverges.

Important:
$a < 0$ for $\nu > 0$
$a > 0$ for $\nu < 0$
Diverges at $\nu = 0$!

$$T(0,0,0) = \frac{4\pi a}{m} = -\frac{g^2}{2\nu}$$



Phenomenology: $a = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$

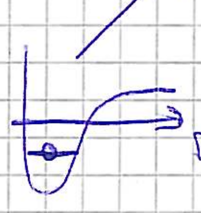
Observable. Observable. Observable.

Write $2\nu = \Delta\mu\,(B - B_0)$

Magnetic moment of molecule.

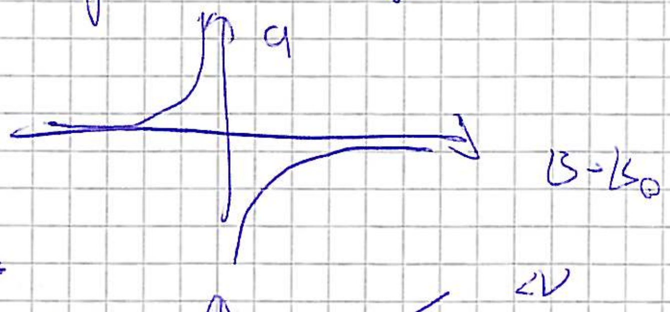
!!

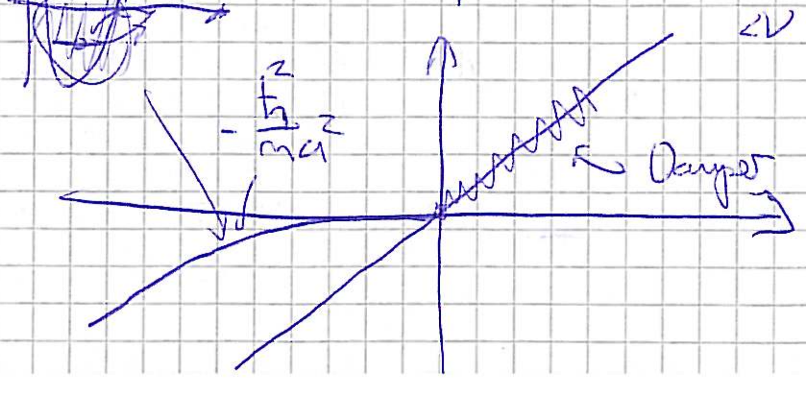
$$\frac{4\pi a_{bg}}{m} \cdot \frac{\Delta B}{B - B_0} = \frac{g^2}{\Delta\mu\,(B - B_0)}$$



So if we know ΔB, B_0, a_{bg} & $\Delta\mu$,
we have effective theory in terms of observables.

Molecule energy:





Described by eq:

$$\omega_M = 2\nu + \frac{g^2 m^{3/2}}{4\pi}\sqrt{-\omega_M} \qquad 2\nu > 0 \Rightarrow \omega \text{ complex}$$

$2\nu < 0$: $\quad 2\nu \to 0$ g:

$$\frac{g^2 m^{3/2}}{4\pi}\sqrt{-\omega_M} = -2\nu$$

$$\Updownarrow$$

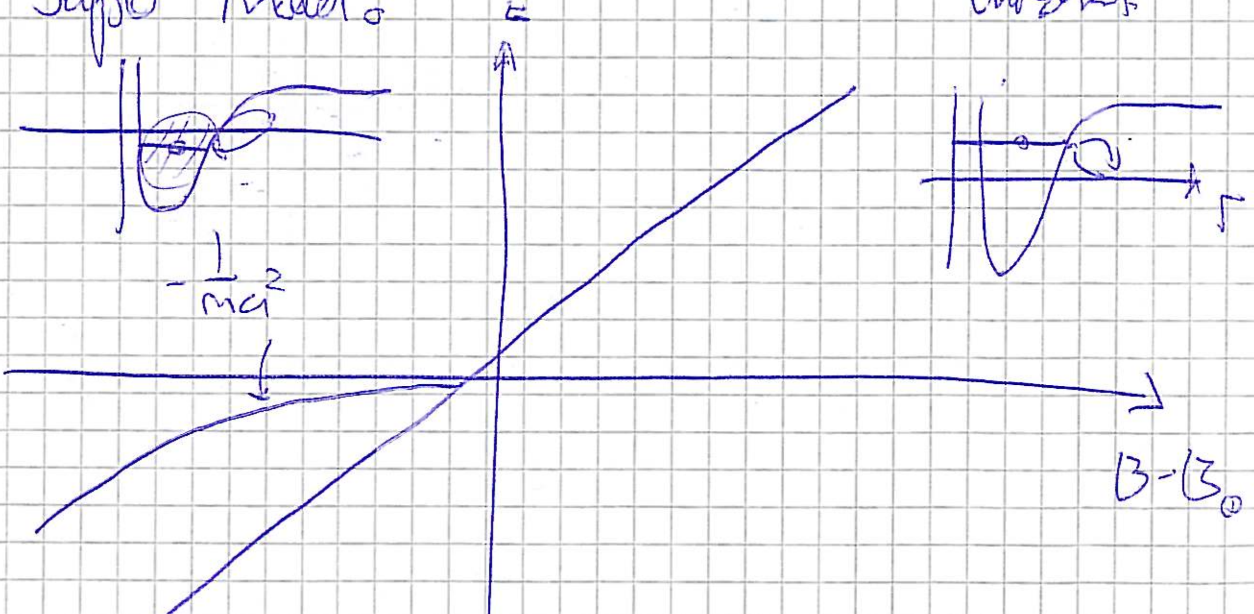
$$-\omega_M = \left(\frac{4\pi\cdot 2\nu}{g^2 m^{3/2}}\right)^2 \qquad \frac{4\pi a}{m} = -g^2/2\nu$$

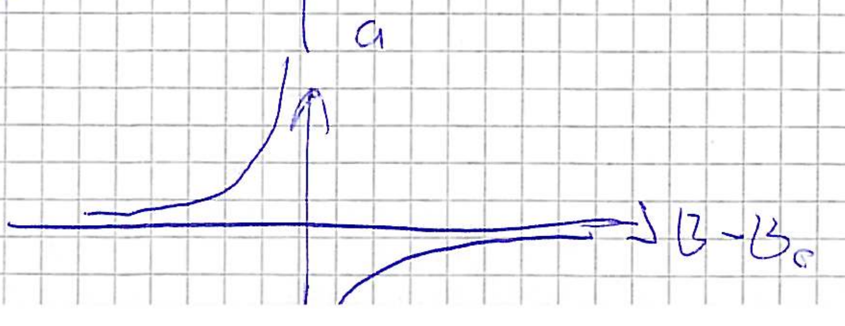
$$\Updownarrow$$

$$\omega_M = -\left(\frac{2\nu}{g^2}\cdot\frac{4\pi}{m^{3/2}}\right)^2 = -\frac{1}{ma^2}$$

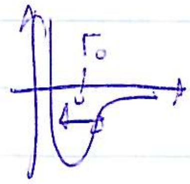
Due to strong coupling at threshold

From single model:



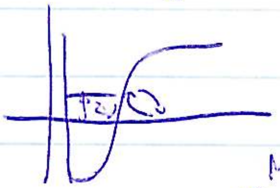


BEC-BCS cross-over & Universal Behavior

yesterday: - Atom-atom interaction short range. 

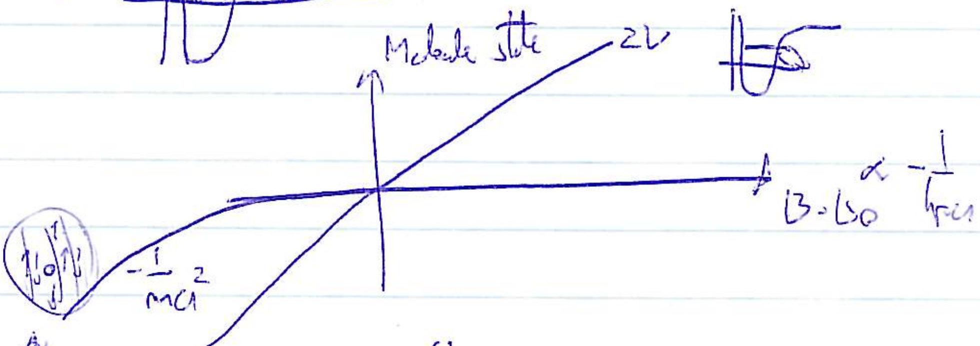
$r_0 \ll d$.

- Characterized by one parameter: a.

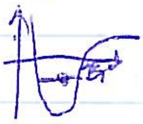
- Multi-channel scattering: 

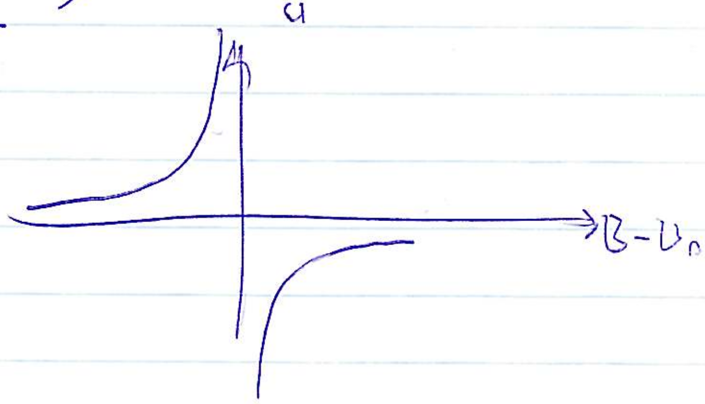
⇓

Feshbach resonance.



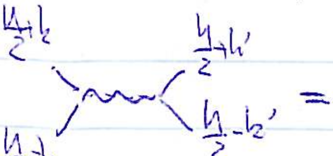
$$\frac{4\pi a}{m} = -\frac{g^2}{2\nu}.$$





$$\hat{H} = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma} + \sum_q \left(2\nu_{bare} + q^2/4m\right) d^\dagger_q d_q$$

$$+ \frac{1}{\sqrt{V}} \sum_{k,p} \left[g_{bare}(k)\, b^\dagger_p a_{\frac{p}{2}+k\uparrow} a_{\frac{p}{2}-k\downarrow} + c.c. \right]$$



$$= \frac{g_{bare}(k)\, g_{bare}(k')}{E - 2\nu_{bare} - q^2/4m} \simeq \frac{-g_{bare}(k)\, g_{bare}(k')}{-2\nu_{bare}}$$

$\succ\!\!\prec$: V_{eff}