for $\frac{\Omega}{2\epsilon_F} \cdot \frac{g^2}{\epsilon_F} \gg 1$. „Single channel model". (2)

BEC - BCS cross-over at T=0

Use $V_{eff}(k,k') = -\frac{g(k)g(k')}{2V_{bare}}$.

Understand qualitatively from BCS wave f^n:

$$|BCS\rangle = \prod_p (u_p + v_p a_p^\dagger a_{-p}^\dagger)\,|0\rangle \qquad p=0 \text{ molecules.}$$

$$u_p^2 = \tfrac{1}{2}(1+\xi_p/E_p) \qquad E_p = \sqrt{\xi_p^2+\Delta^2}$$
$$v_p^2 = \tfrac{1}{2}(1-\xi_p/E_p)$$

Gap eq^n: $$\Delta_p = -\int\frac{d^3p'}{(2\pi)^3}\, U_{eff} \cdot \frac{(1-2f(E_p))}{2E_p} \cdot \Delta_{p'}$$

$\Downarrow$

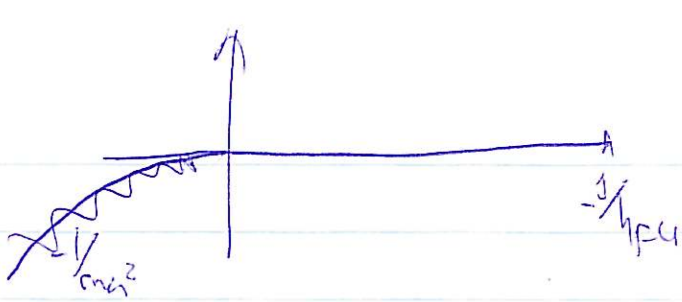
$$\frac{1}{U} = -\int\frac{d^3p}{(2\pi)^3}\frac{(1-2f(E_p))}{2E_p}$$

Lippmann-Schwinger: $T = U + U G_0 T$ ($G_0 = \frac{1}{E-p^2/m}$) $\qquad T(E=0) = \frac{4\pi a}{m}$

$\Downarrow$

(A) $$\frac{1}{T_{00}} = -\int\frac{d^3p}{(2\pi)^3}\left(\frac{1-2f(E_p)}{2E_p} - \frac{1}{p^2/m}\right) \quad \text{Gap eq}^n.$$

(B) Number eqⁿ: $n = \int \frac{d^3p}{(2\pi)^3} V_p^2$

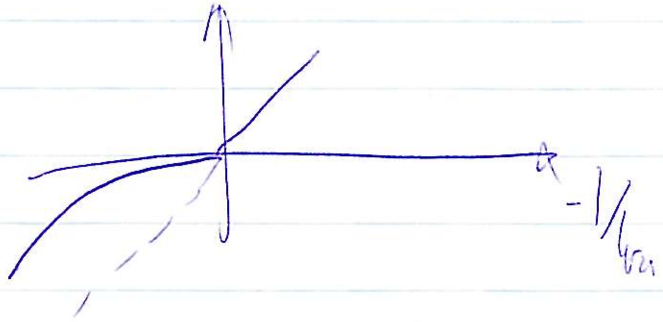


Weak Coupling BCS-side

$k_F a \to 0_-$

$\mu = \varepsilon_F$

$\Delta = \frac{8}{e^2} \cdot \varepsilon_F \cdot e^{-\pi/2k_F a}$.



Strong coupling BEC side: $k_F a \to 0_+$

Assume $\mu < 0$ & $|\mu| \gg \Delta$.

Solution:

$\mu \simeq -\frac{\hbar^2}{2ma^2} + \frac{\pi \cdot \hbar^2 a \cdot n}{m}$, $\Delta \simeq \frac{4\varepsilon_F}{\sqrt{3\pi}(k_F a)^{1/2}} \gg \varepsilon_F$

Interpretation:

(Not so surprising. dadder...)

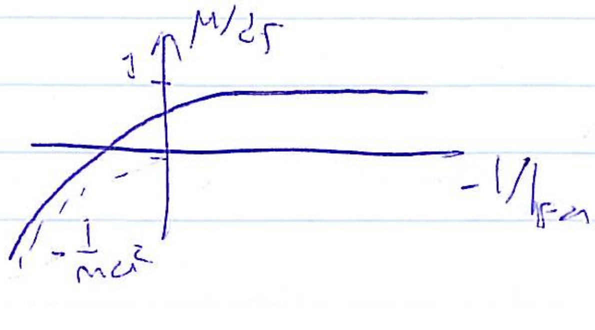
The energy to add two atoms is 2μ.

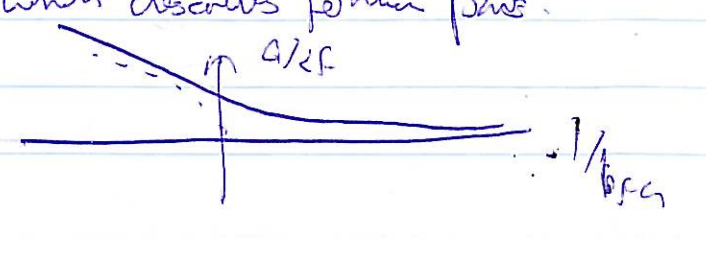
$2\mu = -\frac{\hbar^2}{ma^2} + \frac{2\pi\hbar^2 a}{m} \cdot n = -\frac{\hbar^2}{ma^2} + \frac{4\pi\hbar^2}{2m} \cdot 2a \cdot \frac{n}{2}$

Molecule-molecule scattering length.

(ff) $\overset{2a}{\sim}$ (ff) Remarkable

Comes out of BCS wavefⁿ which describes fermion pairs.





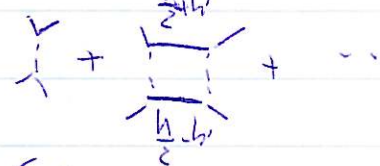
BCS eqn:

$$= \frac{V}{1 - V\Pi} = \ldots \qquad T\left(k, k', \frac{K}{2}, E\right)$$

Medium effects.

~~$1 - V\Pi = 0 = \frac{1}{V} = \Pi = \int \frac{d^3k}{(2\pi)^3} \frac{1 - f - f'}{\omega - \ldots}$~~

$$T(k, k', K, E) = \frac{V}{1 - V\Pi(K)}$$



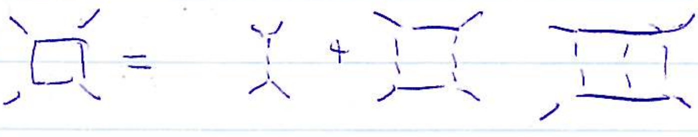
← Medium polarization

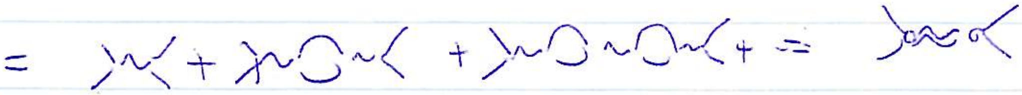
$$1 - V\Pi(K) = 0 \Rightarrow \frac{1}{V} = \Pi(K) = \int \frac{d^3k}{(2\pi)^3} \frac{1 - f_{\frac{K}{2}+k} - f_{\frac{K}{2}-k}}{E - \epsilon_{\frac{K}{2}+k} - \epsilon_{\frac{K}{2}-k}}$$

$K = 0, E = 0$

$$\frac{1}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2f}{-\xi_k 2} \quad \text{Gap eqn.}$$

so gap-eqn is $p=0$ molecules which cause Feshbach.





We put in the same physics.

Minimum excitation energy:

$$\mathrm{Min}\, 2\sqrt{\xi_p^2+\Delta^2} = \begin{cases} 2\Delta & \text{for } \mu>0 \\ \sim 2|\mu| & \text{for } \mu<0 \text{ binding a molecule.} \end{cases}$$

$$\sim \frac{1}{ma^2}$$

E Laboratory

Pair wave f^n:

$$A^+ = A$$
$$A^*_{ji} = A_{ij}$$

Two particle density matrix:

$$\rho_2(\bar{r}_1,\bar{r}_2;\bar{r}_1'\bar{r}_2') \equiv \langle \psi^+_\uparrow(\bar{r}_1)\psi^+_\downarrow(\bar{r}_2)\,;\,\psi_\downarrow(\bar{r}_2')\psi_\uparrow(\bar{r}_1')\rangle$$

~~Hermitian: $\rho^*(\bar{r},\bar{r}') = \rho(\bar{r}',\bar{r})$~~ ↑ Probability amplitude.

$$\propto N^2 \int d\bar{r}_3 \cdots d\bar{r}_N\, \psi^*(\bar{r}_1,\bar{r}_2,\bar{r}_3\cdots\bar{r}_N)\,\psi(\bar{r}_1',\bar{r}_2';\bar{r}_3\cdots\bar{r}_N)$$

If $\psi(\bar{r}_1\cdots\bar{r}_N) = \mathcal{A}\prod_i \phi(\bar{r}_{i\uparrow}-\bar{r}_{i\downarrow})$

then $\rho_2(\bar{r}_1\bar{r}_2;\bar{r}_1'\bar{r}_2') \propto \phi(\bar{r}_1-\bar{r}_2)^*\,\phi(\bar{r}_1'-\bar{r}_2')$,

More precisely: $\rho(\bar{r}_1\bar{r}_2;\bar{r}_1'\bar{r}_2') = \sum_i \lambda_i\, \chi_i^*(\bar{r}_1\bar{r}_2)\,\chi_i(\bar{r}_1'\bar{r}_2')$

with $\int d\bar{r}_1 d\bar{r}_2\, \rho(\bar{r}_1\bar{r}_2;\bar{r}_1'\bar{r}_2')\,\chi(\bar{r}_1''\bar{r}_2'') = \chi(\bar{r}_1'\bar{r}_2')$

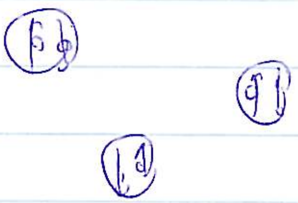
Pair wave f^n: The χ connected with the macroscopic eigenvalue

Putting in the BCS wave f. we get for the macroscopic eigenvalue:

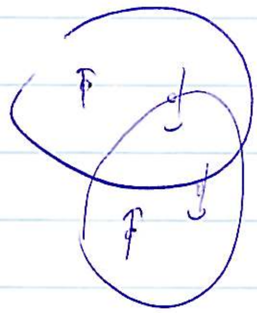
$$F(\vec{r} = \vec{r}_\uparrow - \vec{r}_\downarrow) = \int \frac{d^3k}{(2\pi)^3} v_p u_p \cdot e^{i\vec{k}\cdot\vec{r}} \qquad \lambda \sim N \cdot \frac{\Delta}{\epsilon_F}$$

$$= \begin{cases} \dfrac{\sin k_F r}{r^{3/2}} \cdot e^{-r/\xi_{BCS}} \\ \dfrac{e^{-r/a}}{r} \end{cases} \qquad \xi_{BCS} = \frac{\hbar k_F}{m\Delta} \quad k_F a \to 0_-$$

BEC limit



BCS limit



Cross-over at finite T

Critical temperature $k_B T_c = \frac{\Delta(T=0)}{1{,}76} \sim \varepsilon_F \cdot e^{-\pi/k_F a}$

BCS limit.

BEC limit: $k_F a \to 0_+$.

Would expect $k_B T_c = \frac{\pi}{[2\zeta(3/2)]^{2/3}} \cdot \frac{\hbar^2 n^{2/3}}{m} = 0{,}218\, T_F$

↖ Riemann Zeta

Weakly coupled BEC of molecules.

BCS theory yields:

$$n = 2\cdot \int \frac{d^3p}{(2\pi)^3}\left[v_p^2 (1-f_p) + u_p^2 f_p\right]$$

$\mu < 0 \wedge \Delta = 0$ at T_c

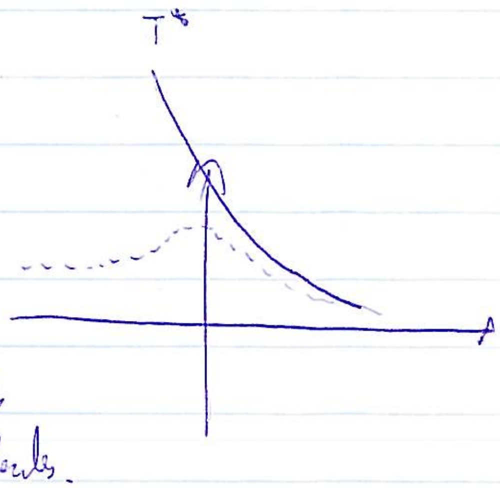
⇓

$v_p^2 = \frac{1}{2}(1 - \xi_p/E_p) = 0$

⇓

$n = 2\cdot \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta\xi_p}+1}$ at T_c

⇓

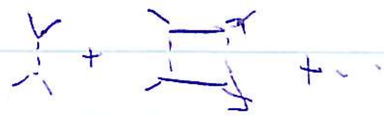


$k_B T_c \simeq \frac{2}{3} \cdot \frac{|\mu|}{\ln|\mu|/\varepsilon_F}$. Way too big. Energy to break molecules.

Miss thermal molecules with $p \neq 0$.

- We need to include finite momentum molecules in number eq^n.

- Instability ~~...~~ still given by formation of $p=0$ molecules:

$$T = \frac{V}{1-V\Pi}$$ 

$$1 - V\Pi = 0 \Rightarrow \frac{1}{V} = \Pi = \int \frac{d^3k}{(2\pi)^3} \frac{1-f-f'}{\omega - \epsilon - \epsilon'} = \int d^3k \, \frac{1-2f}{-2\epsilon_k}$$

Thermodynamic potential:

$$\Omega = U - T \cdot S - \mu N = pV = -k_B T \ln \mathrm{Tr}\left(e^{-\beta(H_0 - \mu N - H_{int})}\right)$$

Non-interacting limit:

$$\Omega_0 = -2 k_B T \cdot \sum_p \ln\left(1 + e^{-\beta\left(\frac{p^2}{2m} - \mu\right)}\right)$$ Fermions, 2 spin states.

(2 ↑ for spin sum.)

$$\Omega_0 = k_B T \cdot \sum_p \ln\left(1 - e^{-\beta(\epsilon_M(p) - \mu_M)}\right)$$ single species bosons

Include interactions in Ω:

BCS and Feshbach use for ladder.

(Exact in a vacuum. Approximate in a medium)

But use app ladder approximation in a medium for Ω.

Coupling constant integration:

$$H_\lambda = H_0 - \mu N + \lambda H_{int}$$

$$\Omega(\lambda) = -\frac{1}{\beta}\ln\left(\mathrm{Tr}\, e^{-\beta(H_0 - \mu N + \lambda H_{int})}\right)$$

$$\frac{\partial \Omega}{\partial \lambda} = -\frac{(-\beta)}{\beta}\,\frac{\mathrm{Tr}\, H_{int}\, e^{-\beta(H_0 - \mu N + \lambda H_{int})}}{\mathrm{Tr}\, e^{-\beta(\dots)}} = \langle H_{int} \rangle$$

$\Downarrow$

$$\Omega(1) - \Omega(0) = \int_0^1 d\lambda\, \frac{1}{\lambda} \underbrace{\langle \lambda H_{int} \rangle}_{\Delta\Omega(\lambda)}$$

$\uparrow$ Calculate this in ladder approx.

Diagrammatic perturb. theory for $\langle \lambda H_{int} \rangle$:

$\langle \lambda H_{int} \rangle =$  $+ \dots$

Given diagram with r interaction lines: $\Delta\Omega_r(\lambda)$

$U_{eff} \sim g(k)\, g(k')$

$$i\omega_n = i(2n+1)\pi k_B T$$

Pair propagator: $$\Pi(\bar{k}, i\omega_\nu) = \int \frac{d^3k}{(2\pi)^3}\, \underbrace{g(k)^2}_{-2L_{bare}} \cdot \frac{1 - f_{\frac{k}{2}+k} - f_{\frac{k}{2}-k}}{i\omega_\nu - \xi_{\frac{k}{2}+k} - \xi_{\frac{k}{2}-k}} \cdot \lambda$$

Medium generalization of pair propagator.

Diagram gives

$$\Delta\Omega_r(\lambda) = -\frac{1}{\beta}\sum_{i\omega_n}\int\frac{d^3k}{(2\pi)^3}\left(\lambda\,\Pi(\vec{k}, i\omega_n)\right)^r$$

Sum over all ladder diagrams:

$$\Delta\Omega(\lambda) = \sum_{r=1}^{\infty}\Delta\Omega_r(\lambda) = -\frac{1}{\beta}\sum_{\omega_n}\int\frac{d^3k}{(2\pi)^3}\sum_{r=1}^{\infty}(\lambda\Pi)^r$$

$$= -\frac{1}{\beta}\sum_{\omega_n}\frac{\lambda\,\Pi(\vec{k}, i\omega_n)}{1-\lambda\,\Pi(\vec{k}, i\omega_n)}$$

Integration of coupling constant

$$\frac{\Delta\Omega}{V} = \Omega(1) - \Omega(0) = \frac{1}{V}\int d\lambda\,\frac{1}{\lambda}\Delta\Omega(\lambda)$$

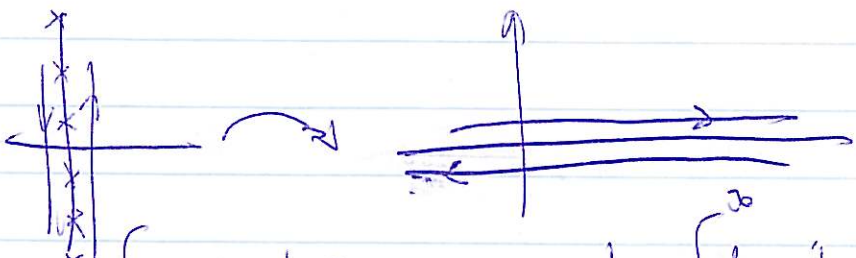
$$= -\frac{1}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\int_0^1 d\lambda\,\frac{\Pi(k, i\omega_n)}{1-\lambda\Pi(k, i\omega_n)}$$

$\Downarrow$

$$\frac{\Delta\Omega}{V} = k_BT\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left[1-\Pi(k, i\omega_n)\right]$$

Matsubara sum:

$$S = +\frac{1}{\beta}\sum_n \ln\left(1-\Pi(k, i\omega_n)\right)$$



$$= \frac{1}{2\pi i}\cdot\oint\frac{1}{e^{\beta z}+1}\cdot\ln\left[1-\Pi(k,z)\right] = \frac{1}{2\pi i}\cdot\int_{-\infty}^{\infty}d\omega\,\frac{1}{e^{\beta\omega}-1}\cdot\left[\ln\left(1-\Pi(k,\omega+i\delta)\right) - \ln\left(1-\Pi(k,\omega-i\delta)\right)\right]$$

$$\Delta\Omega = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \mathrm{Im} \ln \left(1 - \Pi(k, \omega + i\delta)\right) \cdot \frac{1}{e^{\beta\omega} - 1}$$

$$\Pi(k, \omega + i\delta) = \frac{-1}{2U_{bare}} \cdot \int \frac{d^3k}{(2\pi)^3} \frac{1 - f(\xi_{k/2+k}) - f(\xi_{k/2-k})}{\omega + i\delta - \xi_{k/2+k} - \xi_{k/2-k}} \cdot g^2_{bare}(k)$$

Unknown parameters.

Again, express in terms of scattering length:

$$T = U_{eff} + U_{eff} \cdot G_V^{(2)} \cdot T$$

$U_{eff} \to -\frac{g(k)g(k')}{2V_{bare}}$, $\quad G_V^{(2)} \to -\frac{1}{p^2/m}$

Plug in. We get

$$\Delta\Omega = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{\pi} \, \mathrm{Im} \ln \left(1 - T_{00} \cdot (G - G_V)\right) \cdot \frac{1}{e^{\beta\omega} - 1}$$

$T_{00} \to \frac{4\pi a}{m}$

$$\frac{4\pi a}{m} \cdot \int \frac{d^3k}{(2\pi)^3} \left[\frac{1 - f_{k/2+k} - f_{k/2-k}}{\omega + i\delta - \xi - \xi} - \frac{1}{p^2/m} \right]$$

Only scattering length enters.
Universality.

[illegible]

Show that we recover Bose number eq^n when there is deep bound state: $\mu \to -\infty$.

$$\Delta\Omega = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{1}{e^{\beta\omega}-1} \cdot \operatorname{Im} \ln(1 - T \circ \Pi_r)$$

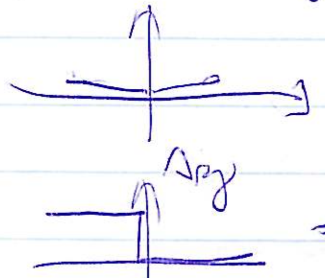
A bound state with momentum k:

$$\frac{\Pi}{1 - T\circ\Pi} \sim \frac{Z}{\omega + i\delta - \varepsilon_M(k) + 2\mu} \qquad \underline{I} + \underline{II} + \cdots$$

Partial integration:

$$\Delta\Omega = \int \frac{d^3k}{(2\pi)^3} \left(\left[k_B T \ln(1 - e^{-\beta\omega}) \operatorname{Im}\ln(\ldots) \right]_{-\infty}^{\infty} - k_B T \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \ln(1 - e^{-\beta\omega}) \partial_\omega \operatorname{Im}\ln(1 - T\Pi_r) \right)$$

$$\ln z = \ln|z| + i \operatorname{Arg} z \Leftrightarrow \operatorname{Im}\ln(1 - T\Pi_r) = \operatorname{Arg}(\omega - \varepsilon_M(k) + 2\mu + i\delta)$$



$\omega - \varepsilon_M + 2\mu$

$$\Rightarrow \partial_\omega \operatorname{Im}\ln(1 - T\Pi) = -\pi\,\delta(\omega - \varepsilon_M + 2\mu)$$

$\Downarrow$

$$\frac{\Delta\Omega}{V} = k_B T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta(\varepsilon_M - 2\mu)}). \text{ Molecule partition f}^n$$

~~Gives number eq^n: $N = -\frac{\partial\Omega}{\partial\mu} =$~~

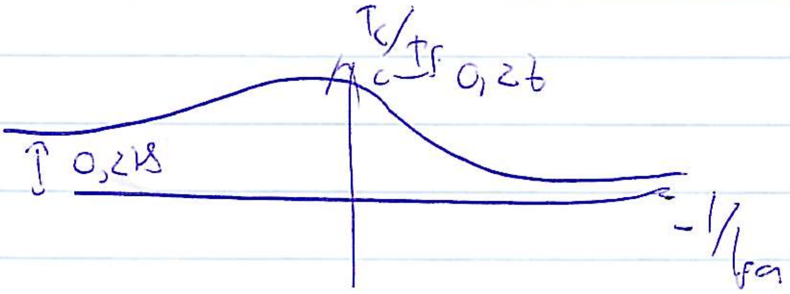
In Effect

$$\Omega = \Omega_0 + \Delta\Omega$$

$$n = -\partial_\mu \Omega = 2\int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta \xi_k}+1} + \int \frac{d^3h}{(2\pi)^3} \frac{2 - \partial_\mu \varepsilon_m}{e^{\beta(\varepsilon_m - 2\mu)}-1}$$

$$\xrightarrow[\mu \to -\infty]{} 2 \cdot \int \frac{d^3h}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_m - 2\mu)}-1}$$

We get



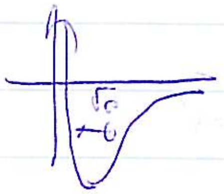
MC: 0,15.

Exp. 0,15.

Viscous slides.

A universal limit

- We saw that for broad resonances, only a matters.



$r_0 << d$.

- „Smooth" development from weak to strong coupling as a of $\frac{1}{k_F a}$.

- $|a| \to \infty$. „Unitarity limit". Resulted molecule state

$$\mathcal{E}(n, T, a, r_0, \ldots) = \mathcal{E}_0(n, T) \text{ for } |a| \to \infty.$$

Details of interaction have disappeared.

Just n and T left.

Example:

$$\frac{E}{N} = \frac{3}{5}\varepsilon_F$$

$$= \xi \cdot \frac{3}{5}\varepsilon_F \qquad \xi = \begin{cases} 0{,}595 & \text{BCS} \\ 0{,}4 & \text{cross-over} \\ 0{,}42 & \text{Exp.} \end{cases}$$

0,4 MC.

$$\frac{E}{N} = \frac{3}{5} \cdot \varepsilon_F \cdot \xi \cdot f(T/T_F)$$

↳ Universal f°

All thermodynamics universal

Relevance to neutron stars:

n-n scattering $a \sim -18{,}5$ fm
range $r_0 \sim 1$ fm.

Also Transport

Shear Viscosity:

$$T_{xy} = -\eta \cdot \partial_y u_x$$ 

$$\eta = n \cdot \hbar\, \alpha(T/T_F)$$

link to string theory and AdS/CFT

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \qquad \frac{\eta}{s} = \begin{cases} 0{,}1 - 0{,}2 & \text{QG.} \\ 0{,}5 & \text{cold gases} \\ 0{,}9 & \text{Li He} \end{cases}$$