Thermodynamic limit

$N \to \infty$

Use analytics:

Intuitive picture:

$\{I\}$ $\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

$$\Big\Downarrow \text{B.E.}$$

$\{\lambda\}$ $\cdot\ \cdot\ \cdot\cdots\cdot\ \cdot\ \cdot\ \cdot\ \cdot$

Rewrite BE: as $N \to \infty$, treat $\dfrac{I_\alpha}{N} \to x$

$x \in \mathbb{R}$

Define a density $\rho(x) \equiv \frac{1}{N}\sum_{a=1}^{M} \delta\left(x - \frac{I_a}{N}\right)$

Rewrite BE using ρ:

$$\left[\Theta_1(\lambda(x)) - \int_{-x_-}^{x_+} dy\, \Theta_2(\lambda(x)-\lambda(y))\,\rho(y) - 2\pi x\right]\Bigg|_{x=\frac{I_a}{N}} = 0$$

$$\lambda\left(x = \frac{I_a}{N}\right) \equiv \lambda_a$$

Let $\lambda(x)$ be defined on whole x axis:

$$\Theta_1(\lambda(x)) - \int_{-x_-}^{x_+} dy\, \Theta_2(\lambda(x)-\lambda(y))\,\rho(y) = 2\pi x$$

or $\Theta_1(\lambda) -$ " " $= 2\pi x(\lambda)$

To go further: define hole & total densities:

given a set $\{I_a\}$,

$$\rho(x) = \frac{1}{N} \sum_{n \in \{I\}} \delta\left(x - \frac{n}{N}\right) \qquad \rho_h(x) = \frac{1}{N} \sum_{n \notin \{I\}} \delta\left(x - \frac{n}{N}\right)$$

$$\rho_T(x) = \rho(x) + \rho_h(x) \xrightarrow[N \to \infty]{} 1$$

Rewrite everything in rapidity space:

$$\rho(\lambda) = \rho(x(\lambda)) \frac{dx(\lambda)}{d\lambda} \quad \& \quad \rho_h(\lambda) \quad \rho_T(\lambda) = \frac{dx(\lambda)}{d\lambda}$$

BE

$$\Rightarrow \Theta_1(\lambda) - \int_{-\lambda_-}^{\lambda_+} d\lambda' \, \Theta_2(\lambda - \lambda') \rho(\lambda') = 2\pi x(\lambda)$$

Take $\frac{d}{d\lambda}$ of this:

$$a_1(\lambda) - \int_{-\lambda_-}^{\lambda_+} d\lambda' \, a_2(\lambda-\lambda')\,\rho(\lambda') = \rho(\lambda) + \rho_h(\lambda)$$

$$a_n(\lambda) \equiv \frac{1}{2\pi}\frac{d}{d\lambda}\Theta_n(\lambda) = \frac{1}{2\pi}\frac{n}{\lambda^2 + n^2/4} \qquad n = 1, 2, \ldots$$

Energy of a state: $$\frac{E}{N} = -J\pi \int_{-\lambda_-}^{\lambda_+} d\lambda \, a_1(\lambda) \quad (\lambda)$$

See notes for further calculations: – GS ρ

– simple excitations

Naive def$^{\underline{n}}$ of QI: $\exists$ set $\{\hat{I}\}$ of conserved charges

$$[H, \hat{\underset{\sim}{I}}] = 0 \quad \& \quad [\hat{I}_n, \hat{I}_m] = 0 \quad \forall n, m$$

Define a function $\tau(\lambda)$ taking operator values for $\lambda \in \mathbb{C}$, i.e.: transfer matrix

$$\tau(\lambda) = \exp \sum_{n=0}^{\infty} \frac{i_n}{n!} \hat{I}_n (\lambda - \xi)^n$$

conserved charges

some extra parameter

i_n: parameters

$$\hat{I}_n = i_n^{-1} \frac{d^n}{d\lambda^n} \ln \tau(\lambda) \Big|_{\lambda = \xi}$$

Since $[I, I] = 0$, we have

$$[\tau(\lambda), \tau(\mu)] = 0 \qquad \forall \lambda, \mu.$$

monodromy matrix

Introduce auxiliary ~~space~~ A s.t. $\tau(\lambda) = \mathrm{Tr}_A T(\lambda)$

$$\left[\left(\mathrm{Tr}_A T(\lambda)\right), \left(\mathrm{Tr}_A T(\mu)\right)\right] = 0$$

Hilbert ~~space~~

View this as a relation in $A_1 \otimes A_2 \otimes \mathcal{H}$

Define $T_1(\lambda) = T(\lambda) \otimes \mathbb{1}_2$

$$T_2(\mu) = \mathbb{1}_1 \otimes T(\mu)$$

$$\Rightarrow \mathrm{Tr}_{A_1 \otimes A_2} [T_1(\lambda), T_2(\mu)] = 0$$

or $\mathrm{Tr}_{A_1 \otimes A_2} T_1(\lambda) T_2(\mu) = \mathrm{Tr}_{A_1 \otimes A_2} T_2(\mu) T_1(\lambda)$

General sol$\underline{^n}$: $\exists$ similarity tr$\underline{^n}$ $R_{12}(\lambda,\mu)$ in $A_1 \otimes A_2$

s.t. $R_{12}(\lambda,\mu) T_1(\lambda) T_2(\mu) R_{12}^{-1}(\lambda,\mu) = T_2(\mu) T_1(\lambda)$

or $R_{12}(\lambda,\mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda,\mu)$

Intertwining relation or Yang-Baxter rel$\underline{^n}$