Conditions for R:

Invertible: $R_{12}(\lambda,\mu)\, R_{21}(\mu,\lambda) = \mathbb{1}$

Consistency of algebra:

look at product of 3 monodromy matrices

$T_1(\lambda)\, T_2(\mu)\, T_3(\nu)$

2 paths

$T_3(\nu)\, T_2(\mu)\, T_1(\lambda)$

leads to requirement

Yang-Baxter

$$R_{12}(\lambda,\mu)\, R_{13}(\lambda,\nu)\, R_{23}(\mu,\nu) =$$

$$= R_{23}(\mu,\nu)\, R_{13}(\lambda,\nu)\, R_{12}(\lambda,\mu)$$

Searching for R matrices

So-called 'Fundamental' models

where $\mathcal{H} = \bigotimes_j \mathcal{H}_j$ & $\mathcal{H}_j \sim \mathcal{A}$

$j \leftarrow$ site index

Simplest case: $\mathcal{H}_j \sim \mathcal{A} \sim \mathbb{C}^2$ rep: 2×2 matrices

Start with 4×4 R-matrices.

Say we try $R_{12}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b, c: f^{n}'s to be determined.

c.f. notes, eq$^{\underline{n}}$s 26 - 29 of ABA chapter

Simplest sol$^{\underline{n}}$: $\quad b(\lambda) = \dfrac{\lambda}{\lambda + i} \qquad c(\lambda) = \dfrac{i}{\lambda + i}$

Monodromy matrix: $T(\lambda) \subset \mathcal{A} \otimes \mathcal{H}$

Represent it as $T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

$A, B, C, D(\lambda) \subset \mathcal{H}$

notes 32-45

Had $R_{12} T_1 T_2 = T_2 T_1 R_{12}$

$\rightarrow$ commutation rel$^{\underline{n}}$s between A, B, C, D

Transfer matrix: $\tau(\lambda) = \mathrm{Tr}_A T(\lambda) = A(\lambda) + D(\lambda)$

Interpret $B(\lambda)$ as a raising/creation op.

Assume $\exists$ state $|0\rangle$ (reference state)

s.t. $A(\lambda)|0\rangle = a(\lambda)|0\rangle$

$D(\lambda)|0\rangle = d(\lambda)|0\rangle$

a, d: free f$^{\underline{n}}$s.

Consider state $\prod_j B(\lambda_j)|0\rangle$,

can check this is an eigenstate of $\tau(\lambda)$

provided

$$\frac{a(\lambda_j)}{d(\lambda_j)} \prod_{l \neq j} \frac{b(\lambda_j - \lambda_l)}{b(\lambda_l - \lambda_j)} = 1$$

Notes:

Finding an explicit model related to the simplest R-matrix

For $R_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$ get identity

$$i \frac{d}{d\lambda} \ln \tau(\lambda) \Big|_{\lambda = i/2} = \sum_{j=1}^{N} \frac{1}{2} \left[\sigma_j \cdot \sigma_{j+1} - 1 \right]$$

so $$H_{xxx} = \sum_{j=1}^{N} \underset{\sim}{S}_j \cdot \underset{\sim}{S}_{j+1} - \frac{1}{4} = \frac{i}{2} \frac{d}{d\lambda} \ln \tau(\lambda) \Big|_{\lambda = i/2}$$

This is an example of a <u>trace identity</u>