Heisenberg XXZ:

$S = 1/2$ operators on a 1d lattice of N sites:

$$H = J \sum_{j=1}^{N} S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta \left(S^z_j S^z_{j+1} - 1/4 \right)$$

$$[S^\alpha_j, S^\beta_l] = i \delta_{jl} \varepsilon^{\alpha\beta\gamma} S^\gamma_j$$

$$S^\pm_j \equiv S^x_j \pm i S^y_j$$

$$[S^z_j, S^\pm_l] = \pm \delta_{jl} S^\pm_j \qquad [S^+_j, S^-_l] = 2 \delta_{jl} S^z_j$$

$$H = J \sum_{j=1}^{N} \frac{1}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + \Delta\left(S_j^z S_{j+1}^z - 1/4\right)$$

Basis of states:

at each site, $|\pm\rangle_j$ s.t.

$$S_j^z |\pm\rangle_j = \pm\frac{1}{2}|\pm\rangle_j$$

$$S_j^\pm |\mp\rangle_j = |\pm\rangle$$

$$S_j^\pm |\pm\rangle_j = 0$$

Hilbert space: $\bigotimes_{j=1}^{N} |\pm\rangle_j$

2^N states

Reference state:

$$|0\rangle = \bigotimes_j |+\rangle_j$$

Total S^z is conserved:

$$S^z_{tot} = \sum_{j=1}^{N} S^z_j, \quad [H, S^z_{tot}] = 0$$

For N sites with M overturned spins, $S^z_{tot} = \frac{N}{2} - M$

Subspaces of fixed S^z_{tot} have dim $= \binom{N}{M}$

$$\sum_M \binom{N}{M} = 2^N \quad \underline{\text{OK}}$$

Solve SE

$M=0$: easy

1 state, $|0\rangle$ $E=0$

$M=1$: use Ansatz

$$|\psi_1\rangle = \sum_{j=1}^{N} \psi_1(j)\,|j\rangle$$

only at j

$|j\rangle = |+\rangle_1 \otimes \cdots \otimes |-\rangle_j \otimes \cdots$

$$E_1 \langle j|\psi_1\rangle = \langle j|H|\psi_1\rangle \Rightarrow \qquad 1 \le j < N$$

$$\frac{J}{2}\left(\psi_1(j-1) + \psi_1(j+1)\right) = (E_1 + J\Delta)\,\psi_1(j)$$

Sol^n: $\psi_1(j) = e^{ikj}$

provided $E_1 = J(\cos k - \Delta)$

Periodicity: $\psi_1(j+N) = \psi_1(j)$

so $e^{ikN} = 1$ so $k = \frac{2\pi}{N} I$

Count states: $\in \mathbb{Z}$

can restrict to $I = 0, 1, \dots, N-1$

$\Rightarrow$ N states

$M=2$

~~Ansatz~~: $|\psi_2\rangle = \sum_{j_1<j_2} \psi_2(j_1,j_2)|j_1 j_2\rangle$

SE: $\langle j_1 j_2|(H-E_2)|\psi_2\rangle = 0$

$\Rightarrow$

$$\frac{J}{2}\{\psi_2(j_1-1,j_2)+\psi_2(j_1+1,j_2) + \psi_2(j_1,j_2-1)+\psi_2(j_1,j_2+1)\}$$

$$=(E_2+2J\Delta)\psi_2(j_1,j_2) \quad *$$

for $2<j_1+1<j_2<N$

For $2<j_1+1=j_2<N$,

$$\frac{J}{2}\{\psi_2(j_1,j_2+1)+\psi_2(j_1-1,j_2)\}$$

$$=(E_2+J\Delta)\psi_2(j_1,j_2)$$

Ansatz:

$$\psi_2(j_1, j_2) = A_{12} e^{ik_1 j_1 + ik_2 j_2} + A_{21} e^{ik_2 j_1 + ik_1 j_2}$$

$A_{ij} \in \mathbb{C}$

$$\circledast \Rightarrow E_2 = J(\cos k_1 + \cos k_2 - 2\Delta)$$

Ex: show that

$$\frac{A_{12}}{A_{21}} = \frac{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_2}}$$

$$\equiv -e^{-i\phi(k_1, k_2)}$$

$$\phi(k_1, k_2) = 2\,\mathrm{atan}\left[\frac{\Delta \sin\frac{k_1 - k_2}{2}}{\cos\frac{k_1+k_2}{2} - \Delta\cos\frac{k_1-k_2}{2}}\right]$$