$$E_2 = J\,(\cos k_1 + \cos k_2 - 2\Delta)$$

$$P = k_1 + k_2$$

$M=2$

$$\psi_2(j_1, j_2) = e^{ik_1 j_1 + ik_2 j_2}\, e^{-\frac{i}{2}\phi(k_1,k_2)}$$

$$- e^{ik_2 j_1 + ik_1 j_2}\, e^{+\frac{i}{2}\phi(k_1,k_2)}$$

Periodicity:

$$\psi_2(j_2, j_1+N) = \psi_2(j_1, j_2) = \psi_2(j_2 - N, j_1)$$

$$\rightarrow\; e^{ik_1 N} = -e^{-i\phi(k_1,k_2)} \qquad e^{ik_2 N} = -e^{+i\phi(k_1,k_2)}$$

Take logs:

$$N k_1 + \phi(k_1,k_2) = 2\pi \tilde{I}_1 \qquad N k_2 - \phi(k_1,k_2) = 2\pi \tilde{I}_2$$

$$\tilde{I} \in \mathbb{Z} + 1/2$$

General $M \leq N/2$ for <u>any</u> set $\{j_1, \dots, j_M\}$

$$\Psi_M(j_1, \dots, j_M) = \prod_{M \geq a > b \geq 1} \mathrm{sgn}(j_a - j_b) \times$$

$$\sum_{P_M} (-1)^{[P]} e^{i \sum_{a=1}^{M} k_{P_a} j_a + \frac{i}{2} \sum_{M \geq a > b \geq 1} \mathrm{sgn}(j_a - j_b) \phi(k_{P_a}, k_{P_b})}$$

This is the Bethe Ansatz

$$|\Psi_M\rangle = \sum_{\{j\}} \Psi_M(\{j\}) S^-_{j_1} \dots S^-_{j_M} |0\rangle$$

$$E_M = J \sum_{a=1}^{M} (\cos k_a - \Delta) \qquad P = \sum_a k_a$$

Bethe equations:

$$e^{i k_a N} = (-1)^{M-1} \, e^{-i \sum_{b=1}^{M} \phi(k_a, k_b)} \qquad a = 1, \dots, M$$

$$N k_a + \sum_{b=1}^{M} \phi(k_a, k_b) = 2\pi \tilde{I}_a$$

$$\tilde{I}_a \in \begin{cases} \mathbb{Z} + 1/2 & M \text{ even} \\ \mathbb{Z} & M \text{ odd} \end{cases}$$

XXX antiferromagnet ($J>0$, $\Delta=1$)

Parametrization: $k = \pi - 2\operatorname{atan} 2\lambda$

rapidities $\to$ $\lambda = \frac{1}{2}\cot\frac{k}{2}$

Why? $\phi(k(\lambda_1), k(\lambda_2)) \to \Theta(\lambda_1 - \lambda_2)$

$$\Theta(\lambda) = 2\operatorname{atan}\lambda$$

Bethe eq$^{\text{ns}}$: $$\left[\frac{\lambda_a + i/2}{\lambda_a - i/2}\right]^N = \prod_{b\neq a=1}^{M} \frac{\lambda_a - \lambda_b + i}{\lambda_a - \lambda_b - i}$$

or $$\Theta_1(\lambda_a) - \frac{1}{N}\sum_{b=1}^{M}\Theta_2(\lambda_a - \lambda_b) = \frac{2\pi}{N} I_a$$

$\Theta_n(\lambda) \equiv 2\operatorname{atan}\frac{2\lambda}{n}$ $I_a \in \begin{cases} \mathbb{Z} + 1/2 & N-M \text{ even} \\ \mathbb{Z} & \text{ odd} \end{cases}$

$$E = J \sum_{a=1}^{M} \frac{-2}{4\lambda_a^2+1} \qquad P = \sum_{a=1}^{M} \frac{1}{i} \ln\left[\frac{\lambda_a + i/2}{\lambda_a - i/2}\right] = \pi M - \frac{2\pi}{N} \sum_{a=1}^{M} I_a$$

$$\text{mod } 2\pi$$

Proper solutions to the Bethe equations: must have distinct rapidities.

(most cases: this translates to choosing different quantum numbers in Bethe equations)

Ground state: at fixed M, N even

quantum numbers $I_j^0 = -\frac{M+1}{2} + j \quad j = 1, \dots, M$

$M=5$:

Allowable quantum nos: careful (4 notes)

For real λ, I cannot be arbitrarily large.

$$\lim_{\lambda_a \to \infty} 2\,\mathrm{atan}\, 2\lambda_a - \frac{1}{N} \sum_{b=1}^{M} 2\,\mathrm{atan}(\lambda_a - \lambda_b)$$

$$= \pi - \frac{1}{N}\pi(M-1) = \frac{2\pi}{N} I_M^\infty$$

$$I_M^\infty = \frac{N-M+1}{2}$$

$$I_{M=\frac{N}{2}}^\infty = \frac{N}{4} + \frac{1}{2}$$

$|\lambda_a|$ is $< \infty$ if $|I_a| \leq I_M^{MAX} \equiv I_M^\infty - 1$

$= I_{M=\frac{N}{2}}^{MAX}$

Simple case: $M = N/2$. For GS:

$$\{I\} = \left\{-\frac{M-1}{2}, -\frac{M-1}{2}+1, \ldots, \frac{M-1}{2}\right\} = \left\{-\frac{N}{4}+\frac{1}{2}, \ldots, \frac{N}{4} - 1/2\right\}$$