Fermi and Luttinger liquids

Lecture notes for the 2011 Nordita winter school

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1 Fermi Liquids

The electric properties of most metals can be well understood by treating the electrons as non-interacting. This "free electron model" describes the electrons in the outermost shell of the atoms. In a crystal, these electrons are rather free to move around among the atoms, and can thus be described as an electron gas.[1] The deviation from ideal gas behavior can in many cases be captured by the concept of *Fermi liquid* which was introduced by the Russian physicist Lev Landau. This provides an effective description of interacting fermions which can be derived from microscopic physics either by resummation of quantum mechanical perturbation expansions, or by renormalization group (RG) techniques.

In the first of these four lectures we shall follow Landau's intuitive approach to establish the basic properties of Fermi liquids. The aim is to explain the underlying assumptions and to get an understanding of when the theory is applicable. For a more detailed exposition I recommend the first chapter in the book

- *Statistical Physics, Part 2* by E. M. Lifshitz and L. P. Pitaevskii (volume 9 of Landau and Lifshitz, Course of Theoretical Physics).

which gives a very good an concise presentation of the theory of Fermi liquids as developed originally by Landau. I will roughly follow the notation and logic in this reference.

In the second lecture I will outline the RG approach to Fermi liquids. Although the basic idea is quite simple, the actual calculations are technically somewhat involved, so the aim of the lecture is only to explain the principles, and present some of the results. You can view it as an introduction to the excellent paper

- R. Shankar, *"Renormalization-group approach to interacting fermions"*, Rev. Mod. Phys, **66**, 129 (1994).

which I strongly recommend to those who really want to learn the subject. Another good review article is

- H. J. Schultz, G. Cuniberti and P. Pieri, *Fermi liquids and Luttinger Liquids*, cond-mat/9807366.

which gives a very concise description of Landau's theory together with a discussion of the RG approach. In these lectures, I have borrowed several insights from this paper.

[1] The more detailed argument for why this is so in a crystal is rather sophisticated and can be found in *e.g.* reference [1].

1.1 Landau's approach

1.1.1 The basic idea

The starting point is a gas of free fermions with mass m_e[2] for which the zero temperature ground state is obtained by filling all single particle (plane wave) states up to the Fermi energy ϵ_F. The total ground state energy is calculated as[3]

$$E = V\mathrm{T}r\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m_e}\mathrm{n}_0(p) \equiv \mathrm{T}r\int d\tau \frac{p^2}{2m_e}(p)\mathrm{n}_0(p) = \frac{V}{5\pi^2}\frac{p_F^5}{2m_e} = \frac{Vp_F^2}{5\pi^2}\epsilon_F \quad (1)$$

where V is the volume of the system, $\epsilon_F = p_F^2/2m_e$ ($\hbar = 1$) and we introduced the notation

$$\int d\tau = V\int \frac{d^3p}{(2\pi)^3} = \sum_{\vec{k}_i} \quad (2)$$

for the phase space integration (the first expression is appropriate for continuum normalization and the second for quantization in a box). The distribution function n is a 2x2 matrix $n_{\alpha\beta}$ in spin space, and the trace is over these variables. In the absence of a magnetic field, both spin directions are equally probable and we have $n_{\alpha\beta} = \delta_{\alpha\beta}n(p)$. In the following we shall suppress the spin structure until discussing quasiparticle interactions.

The zero temperature *momentum distribution function* is given by $\mathrm{n}_0(p) = \theta(p - p_F)$ where where m_e is the fermion mass. The total number of particles is given by

$$N = \int d\tau\, \mathrm{n}_0(p) = V\frac{p_F^3}{3\pi^2}\,. \quad (3)$$

Excitations from the ground state is obtained by changing the occupation numbers of the single particle levels, *i.e.* by changing the distribution function,

$$n(p) = \theta(p - p_F) + \delta n(p) \quad (4)$$

It is simplest to think of δn for discrete momentum states, where it can only take the values ± 1 because of the Pauli principle.

[2] As condensed matter physicists we are interested in electrons, but the theory developed here is also relevant for liquid helium 3, and cold neutron matter.

[3] The description can be generalized to include an external potential, but in that case the integrals can in general not be calculated on a closed form.

Of particular importance is the distribution function for a state in thermal equilibrium with a reservoir kept at temperature T. This distribution is determined by maximizing the entropy,

$$S = -\int d\tau [n(p) \ln n(p) + (1 - n(p)) \ln(1 - n(p))] \tag{5}$$

under the constraints $\delta E = \delta N = 0$ to get

$$n_\beta(p) = \frac{1}{e^{\beta(\epsilon(p) - \epsilon_F)} + 1} \,. \tag{6}$$

where $\beta = 1/kT$, and $\epsilon(p) = p^2/(2m_e)$.

Landau's basic idea was that the interacting system can be thought of as "connected" to the free fermi gas by an adiabatic switching process. In particular, this means that there is an one-one correspondence between the excitations in the free and the interacting system, and that the latter also will have a Fermi surface.

Clearly the notion of adiabatic switching is a purely theoretical concept; in a real physical system, there is no way the interaction can be turned on and off. If we however follow Landau's reasoning we conclude that the excited states of the interacting system can also be *labeled* by the occupations numbers of the single particle states. For example, adding a single electron to the ground state of the free Fermi gas will give a state labeled by the conserved quantities charge q, momentum, $\vec{p}$, and spin, α. As interaction is turned on adiabatically, the state will still have the same spin and momentum, but the energy will change because of the interactions.

$$|\vec{p}, \alpha\rangle_{free} \rightarrow |\vec{p}, \alpha\rangle_{int.} \tag{7}$$

The resulting state can not be thought of as a single electron carrying momentum $\vec{k}$, while the other fill the Fermi sphere. Instead the momentum will be spread among many electrons. Pictorially we can think of the original electron as being "dressed" by the interaction - it has evolved into a *Landau quasielectron*. The same arguments can be used to deduce the existence of *quasiholes*, and combinations of quasiholes and quasielectrons. From this reasoning, it is also clear that the zero temperature ground state distribution function remains as $\theta(p - p_F)$, since only states with $p > p_F$ or holes with $p < p_F$ can be excited out of the non-interacting ground state. So as we already mentioned, the Fermi liquid at zero temperature still has a Fermi surface. We shall return to the finite T distribution of quasiparticles in a short while.

That the excited states are labeled by the quantum numbers of the excitations, or equivalently, by the distribution function $n(p)$ implies that the total

energy is a *functional* $E[n(p)]$ of the distribution function. It however *does not* mean that it can be written on the form (1) with some modified energy function $\epsilon(p)$ - this is possible only in the absence of electron-electron interaction. Landau's theory does not provide any methods for determining the energy functional, but it does give a *general framework for describing the low momentum excitations* of the ground state (or a thermal equilibrium state) in terms of the variation δn of the quasiparticle distribution. The mathematical expression of this idea is to make a *functional Taylor expansion*,

$$\delta E = \int d\tau \epsilon_0(p)\delta \mathrm{n}(p) + \frac{1}{2V}\int d\tau d\tau' f(\vec{p},\vec{p}')\delta \mathrm{n}(p)\delta \mathrm{n}(p') + \dots \tag{8}$$

where V is the volume of the system,

$$\epsilon_0(p) = \frac{\delta E[n]}{\delta \mathrm{n}(p)} \mid_{n=n_0} \tag{9}$$

and

$$f(\vec{p},\vec{p}') = \frac{\delta^2 E[n]}{\delta \mathrm{n}(p)\delta \mathrm{n}(p')} \mid_{n=n_0} \tag{10}$$

(Note that f is a tensor in spin space.) The total energy, $\epsilon(p)$, required to a add a quasiparticle is obtained by varying E with respect to n

$$\epsilon[n(p'),p] = \epsilon_0(p) + \frac{1}{V}\int d\tau' f(\vec{p},\vec{p}')\delta \mathrm{n}(p') + \dots \tag{11}$$

and it is natural to interpreat $\epsilon_0(p)$ as the "bare" single quasiparticle energy, and the second term as the interaction energy between the added quasiparticle, and those already present. This term is absent for the free Fermi gas.

The expression (5) for the entropy follows from a counting argument, and since there is an one-one correspondence between the states in the interacting and the non-interacting system, the calculation of the entropy will look precisely the same, except that the constraint $\delta E = 0$ now becomes,

$$\int d\tau' \epsilon[n,p]\delta n(p) = 0\,, \tag{12}$$

Minimizing the entropy under this constraint gives,

$$n_\beta(p) = \frac{1}{e^{\beta(\epsilon[n,p]-\epsilon_F)}+1}\,. \tag{13}$$

which is an in general very complicated functional equation for the distribution function $n_\beta(p)$. From the previous arguments we however know that it will approach $\theta(p-p_F)$ as T goes to zero.

1.1.2 When is the theory applicable?

To proceed, we must find out when the expressions we just arrived at are applicable, and this will naturally lead to approximations that will allow us to get some handle on the unknown functions $\epsilon_0(p)$ and $f(p,p')$. The crucial notion is that a quasiparticles, as opposed to an electron, has a finite lifetime. The reason is very simple; if we excite say a quasielectron at a momentum $\vec{p}_1$ it can decay into two quasielectrons and one quasihole,

$$|\vec{p}_1\rangle_- \to |\vec{p}_2\rangle_- + |\vec{p}_3\rangle_- + |\vec{p}_2 - \vec{p}_1 - \vec{p}_3\rangle_+ \tag{14}$$

without violating any conservation law. (The subscript denote the charge with respect to the filled Fermi sea. Why can the quasielectron not decay into two quasiparticles?) Such a decay process however become less and less probable as $\vec{p}_1$ approaches the Fermi surface. This is a purely kinematic effect - if $\vec{p}_1$ is close to the Fermi surface, the phase space available for the process (14) is very tiny, $\sim |\vec{p} - \vec{p}_F|^2$ and thus,

$$\tau = 1/\Gamma \sim |\vec{p} - \vec{p}_F|^{-2} \tag{15}$$

where τ is the lifetime and Γ the decay rate.[4] We conclude that the concept of quasiparticle, and thus the idea to label (quasi)stationary states according to their quasiparticle content, only makes sense in the vicinity of the Fermi-surface. Since the finite T equilibrium state involves excitations with energies $\leq kT$, we conclude that our theory will work only at low temperatures. To give a quantitative estimate, we need the constant of proportionality in (15), but if interactions are strong in the sense that there is no small number in the theory (as *e.g.* the ratio between the interaction energy per particle and the Fermi energy), the only scale is the Fermi energy and the theory is expected to be valid for $T \ll T_F$. To give an idea about scales, the Fermi temperature for metallic sodium is about 40.000 K and for liquid Helium 3, it is about 7 K. (Can you roughtly explain this ratio without doing a calculation?)

At this point you should rightly be suspicious about the relation (3). As opposed to the expression (8), which depends only on the deviations from the equilibrium distribution, this relation, which expresses the density in terms of the Fermi momentum, involves an integral over all momenta and tacitly assumes that the ground state can be thought of as a collection of quasiparticles. Even though this is not true, one can nevertheless show that (3) holds for an interacting Fermi liquid, but the proof based on Greens function techniques (see *e.g.* §20 in Ref. [2]) goes beyond this presentation.

[4] A nice derivation of this result which only uses energy conservation, and thus applies also to disordered systems and finite systems, is given in section 5.3 of reference [5].

1.1.3 The effective mass

Now we assume that the temperature is low enough that we can put $n(p) = \theta(p - p_F)$ and expand around the Fermi momentum,

$$\epsilon[n(p'), p] \approx \epsilon[\theta(p - p_F), p] \approx \epsilon_F + v_F(p - p_F) + \dots \tag{16}$$

To discuss the physical meaning of the constants ϵ_F and v_F, recall that p_F is the same as in the noninteracting theory, but the Fermi energy is not. But, since according to (11), $\epsilon[n, p]$ is the energy cost for adding a single quasiparticle, we can still identify ϵ_F in (16) as the chemical potential μ. To understand the meaning of v_F consider the expansion (16) for the non-interacting gas,

$$\epsilon(p) = \epsilon_F + \frac{p_F}{m_e}(p - p_F) + \dots . \tag{17}$$

From this we conclude that v_F is the *Fermi velocity*, which however will differ from its value p_F/m_e in the non-interacting gas. if we introduce the *reduced mass* by $p_F = m^\star v_F$, we see that to lowest order, the only difference between the free fermi gas and the interacting Fermi liquid, is that the electron mass is renormalized.

One of the simplest observables that can be measured for a Fermi liquid is its specific heat, c_V. Using the approximation (17) we can now get the result directly from the one for the free electron gas by the substitution $m_e \to m^\star$, that is

$$c_V = \frac{m^\star p_F}{3} k_B^2 T \tag{18}$$

so the specific heat at low temperatures provides a direct determination of the effective mass. The ratio $m^\star/m_e$ can vary from around 3 for liquid ^{3}He to $10^2 - 10^3$ for "heavy fermion compunds" such as UPt_3.

1.1.4 The meaning of the Fermi surface

We already stressed that the picture of the Fermi liquid ground state as a filled Fermi sphere of quasiparticles is misleading in the sense that the states deep inside the sphere cannot be thought of as quasiparticles in momentum eigenstates. Another possible misconception relates to the momentum distribution $N(p)$ which for free fermions at zero T is just a step function at the Fermi momentum p_F. In the interacting case, the quasiparticle distribution function $n(p)$ has the same behaviour and, as discussed above, p_F is related to the density just as in the free case. The meaning of $n(p)$ is however not that of

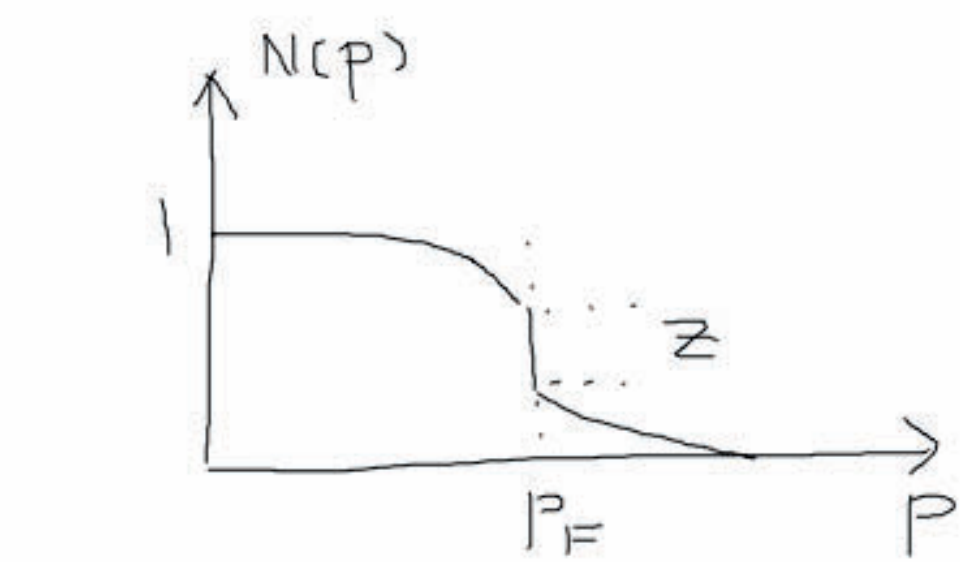

Figure 1: The $T = 0$ single particle momentum distribution $N(p)$ in a Fermi liquid. Z is the wave function renormalization constant, or the strength of the single particle pole.

the distribution of physical momentum in the ground state. This distribution does obviously not vanish for $p > p_F$ since scattering processes will populate the ground state with electron-hole pairs. This is not in contradiction with the notion that this state does not have any *quasi*electron or *quasi*holes in it. It is instructive to calculate the physical momentum distribution $N(p)$ for the Fermi liquid ground state, and those familiar with many-body perturbation theory can consult §10 in [2] for the details. The qualitative feature of the distribution is is shown in Fig. 1. $N(p)$ is no longer just a step function, but it does have a discontinuity at the Fermi momentum. It is this discontinuity that is characteristic for a Fermi liquid and which guarantees that there are low lying quasiparticle quasihole excitations just as in the free fermi gas. In a superconductor one can show that the discontinuity is not present and a finite energy 2Δ is required to excite a pair. The superconductor and the Fermi liquid represent different possible phases of fermionic matter. In the third lecture, we shall consider another case where the Fermi surface is destroyed by interactions, but the system remains gapless!

1.1.5 Quasiparticle interactions and the Fermi liquid parameters

We now turn to the second term in (8) and (11), which, as already mentioned, represent the interaction between two quasiparticles. A comment on the spin-structure is in order. Writing out the spin indices (11) reads

$$\epsilon[n(p'),p]_{\alpha\beta} = \epsilon_{0,\alpha\beta}(p) + \frac{1}{V}\int d\tau' f(\vec{p},\vec{p}')_{\alpha\gamma,\beta\delta}\delta n_{\gamma\delta}(p') + \dots . \tag{19}$$

Since all the momenta are close to the Fermi surface, the tensor f can only depend on the angle θ given by $\vec{p}\cdot\vec{p}' = pp'\cos\theta \approx p_F^2\cos\theta$. Next recall that f is the second functional derivative of the energy with respect to the matrix n, so it must be symmetric under a simultaneous exchange of spin and momentum variables. This fact, together with constraints from rotational and time-inversion symmetry implies the following general form for$f$,

$$f(\vec{p},\vec{p}')_{\alpha\gamma,\beta\delta} = f(\cos\theta)\delta_{\alpha\beta}\delta_{\gamma\delta} + g(\cos\theta)\vec{S}_{\alpha\beta}\cdot\vec{S}_{\gamma\delta}\,. \tag{20}$$

Recall that in the absence of an external magnetic field, there is no preferred direction for the spin and we have $n_{\alpha\beta} = n\delta_{\alpha\beta}$. In this case the spin structure becomes simple since the trace over $\vec{S}_{\alpha\beta}$ vanish, and only the first term in (20) contributes,

$$\epsilon[n(p'),p] = \epsilon_F + v_F(p-p_F) + \int d\tau' f(\cos\theta)\delta n(p') \tag{21}$$

It is conventional to define the dimensionless function $F(\cos\theta)$, and its partial wave components by,

$$F(\theta) = \frac{p_F m^\star}{\pi^2} f(\cos\theta) = \sum_{L=0}^{\infty}(2l+1)F_L P_L(\cos\theta) \tag{22}$$

where P_L are the Legendre functions, and the F_Ls are called Fermi liquid parameters

The important lesson to take home from all this is that a Fermi liquid is not characterized simply by a finite set of parameters, but by a *function* $F(\theta)$ or equivalently an infinite number of parameters F_L.[5] An infinite number of parameters sounds like bad news for any phenomenological model, but it turns out that important observables can be expressed in terms of just a few of the

[5]From (21) it looks like there is also a single extra parameter, $m^\star$, but an argument based on Galilean invariance gives the relation $\frac{m^\star}{m_e} = 1 + \frac{F_1}{3}$.

F_Ls. We already mentioned the specific heat that only depends on $m^\star$, and another example is the compressibility given by,

$$\kappa = \frac{m^\star p_F}{\pi^2 \rho^2 (1 + F_0)} \tag{23}$$

where ρ is the density. To calculate the response to a magnetic field, we must include the term in (20) that involves the function $g(\cos\theta)$, and the susceptibility is determined by the parameter G_0, where G_L is defined analogously to F_L

1.2 The Renormalization Group approach

Although Landau's line of arguments is very suggestive, it cannot be true in general. We know that many metals at sufficiently low temperatures become superconductors which are very different from metals. Of particular importance for our discussion is that in superconductors there are no low energy quasiparticle excitations, which is related to that the Fermi surface is destroyed by interactions. Since we have already argued that Landau's theory should be applicable at sufficiently low temperature, the presence of superconductors is a real problem.

The first go at this problem was to start from the free electron gas and take the electron-electron interaction into account by many-body perturbation theory. The first aim of this program, which was to derive Landau's theory from first principle, was rather successful, and it is described in many textbooks[2]. Since the interaction is strong, it is not sufficient just to calculate a couple of terms, but one must perform (infinite) sums over many terms in the perturbation expansion - this makes the calculations cumbersome, and it is not always easy to keep track on what is included, what is neglected, and why that is so. The case of superconductivity is rather interesting. The first microscopic explanation - the BCS theory - was based on a rather simplified model for the electron-electron interaction, which however captured the essential physical mechanism, namely that of Cooper pair formation. Only later was this theory formally derived using the perturbation theory machinery.

The second approach, which we shall discuss very briefly in this lecture, is based on Ken Wilson's concept of the renormalization group (RG). We first shortly review the basic idea of the RG method, and then show how it can be adapted to deduce the possibility of a Fermi liquid state, but also why the superconductor is an alternative low temperature state.

1.2.1 The basic idea of the RG method

The goal of the RG method is to find an effective action, or Hamiltonian, that captures the low energy/low momentum properties of a theory.[6] The technique is that of sequential elimination of high momentum modes which starting from a microscopic action $S = S_\Lambda$ gives to a sequence S_{Λ_n} of (in general increasingly complicated) actions. Here Λ_n is a cutoff and the action S_{Λ_n} describes the physics for momenta $p < \Lambda_n$. The first Λ can be thought of as a physical cutoff such as a lattice spacing in a crystal, or the magnetic lenght in strong magnetic field. Mathematically this elimination procedure is most easily decribed using path integrals, and the object to study is the partition function $Z[T, \mu, \ldots]$, that depends on some number of control parameters such as the temperature T, the chemical potential μ *etc.*. Z can be used to calculate thermodynamic observables. To get correlation functions, or Greens functions in a quantum mechanical theory, one must also couple sources for the various fields. In the Euclidean formulation the correlation functions are directly related to retarded response functions, but by analytical continuation one can also, at least in simple cases, retrieve real time correlation functions. In the following we will supress the dependence on external parameters and sources and just write

$$Z = \int \mathcal{D}[\phi, \phi^\star]\, e^{-S[\phi]}\,. \tag{24}$$

For simplicity you can think of ϕ as a bosonic field that describes for instance the density in a gas or the spin density in some direction in a magnet. In general there will be many fields, and, importantly, some of them will be fermionic. We shall assume that the action is some local function of the field ϕ and its derivatives. As an (important) example we take

$$S = \int d^d x\, \left[\frac{1}{2}\phi^\star(-\vec{\nabla}^2 + g_2)\phi + g_4|\phi|^4\right] = S_0 + S_{int} \tag{25}$$

which for for $d = 4$ could describe a cloud of cold atoms or a part of the Higgs sector of the standard model.

Next we fourier transform to momentum space and decompose the field as

$$\begin{aligned} \phi(\vec{x}) &= \sum_{0\le|\vec{p}|\le\Lambda} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) = \sum_{0\le|\vec{p}|\le\Lambda_1} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) + \sum_{\Lambda_1\le|\vec{p}|\le\Lambda} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \\ &\equiv \phi_<(\vec{x}) + \phi_>(\vec{x}) \end{aligned} \tag{26}$$

[6] When dealing with quantum mechanical problems, we talk about actions, but we shall all the time use the Euclidian formulation where $i\int dt \to \int d\tau$, and where τ is the "imaginary time" variable. In this way, the generating functional for the quantum system in d spatial dimension, is mapped into a $d+1$ dimensional classical statistical mechanics system.

The action becomes

$$\begin{aligned} S &= \sum_{0\le|\vec{p}|\le\Lambda_1} [p^2+g_2]\,\phi^\star(\vec{p})\,\phi(\vec{p}) \;+\; \sum_{\Lambda_1\le|\vec{p}|\le\Lambda} \frac{1}{2}[p^2+g_2]\,\phi^\star(\vec{p})\,\phi(\vec{p}) \;+ S_{int} \\ &= S_0[\phi_<] + S_0[\phi_>] + S_{int}[\phi_<,\phi_>] \end{aligned} \tag{27}$$

and we define the effective action S^{eff} at scale $\Lambda_1 = \Lambda/s$ by the relation

$$e^{-S^{eff}[p_<]} = e^{-S_<[p_<]} \int \mathcal{D}[\phi_>]\, e^{-S_>[\vec{p}_>]-S_{int}[\vec{p}_<,\vec{p}_>]} \tag{28}$$

We shall not go into the details for how S^{eff}, is calculated, but take for granted that this can be done, and just state that after the two rescalings,

$$\vec{p}_< \;\rightarrow\; \frac{1}{s}\,\vec{p}\,' \tag{29}$$

$$\phi_< \;\rightarrow\; \zeta\,\phi' \tag{30}$$

it can be written in the form

$$S^{eff} = \int d^dx\, \left[\phi^\star(-\vec{\nabla}^2+g_2')\phi + g_4'|\phi|^4 + g_6|\phi|^6 + g_{22}|\vec{\nabla}^2\phi|^2 + ...\right]. \tag{31}$$

where we skipped the primes on the new field variables. The scaling (29) is picked so that the original cutoff Λ is restored, and (30) so that the kinetic term in S and S^{eff} are the same.[7] This is necessary for making a meaningful comparison between the original coupling constants g_2 and g_4 and the new coupling constants g_2' and g_4'. By studying the "flow" of the coupling constants as we change the scale parameter s, we learn whether a particular interaction term becomes stronger or weaker as we study phenomena at lower and lower momenta. At first this might seem like an impossible endeavor since the dots in (31) denotes a, in general infinite, number of terms of higher order in both ϕ' and derivatives, which are generated when evaluating the functional integral in (28). What saves the day, is that the operators can be classified in three groups, *relevant, irrelevant, and marginal*, according to wheter they increase, decrease, or remain the same under the *renormalization group transformation* which consists of the elimination of high momentum modes according to (28) and then performing the two rescalings (29) and (30). In most cases there are only a few relevant and marginal operators, and these are the only ones that will survive at low momenta. Which class a particular operator belongs to often

[7] In a free field theory simple dimensional analysis gives $\zeta = s^{1-d/2}$, but for interacting field theories this changes and the scaling dimension of the field becomes "anomolous".

follows from dimensional analysis. In the action (25), the (mass) dimension of g_2 is that of $\vec{\nabla}^2$, *i.e.* $[g_2] = 2$ while $[g_4] = 4-d$ since $[\phi] = d/2-1$ and the action itself is dimensionless. Thus under the scale change (29), $g_2 \to g_2' = s^2 g_2$ and $g_4 \to g_4' = s^{4-d} g_4$. Thus we expect g_2 to be relevant while g_4 is relevant or irrelevant depending on whether we are below or above four dimensions.[8]

Mathematically the flow of coupling constants are determined by a set of coupled first order differential equations of the type

$$s\frac{dg_i}{ds} = \frac{dg}{dt} = \beta_i(g_1, g_2, \ldots) \tag{32}$$

where we introduced the parameter t by $s = e^t$. There is no universal method for calculating the beta functions, but in many cases one can get important information from various perturbative expansions. In our case, a one loop calculation gives

$$\begin{aligned} \frac{dg_2}{dt} &= 2g_2 + ag_4 \\ \frac{dg_4}{dt} &= -bg_4^2 \end{aligned} \tag{33}$$

where a and b are positive constants. The resulting flow pattern in the (g_2, g_4) plane is shown in the figure 2. A *fixed point* is a set of values, here $(g_2^\star, g_4^\star)$, for the couplings, which are unchanged under renormalizations, and from (32) we see that this occurs at zeros of the beta functions. The interesting fixed points are those at $s \to \infty$, since these determine the low energy theory.[9] A particularly important example is the *gaussian fixed point* where $g_2^\star = g_4^\star = 0$, as seen in the figure. In particle physics this describes a free massless theory, and in statistical mechanics a critical point signalling a phase transition. A famous example of a non-gaussian fixed point is the Wilson-Fisher fixed point which appear in the real version of the model (25) in three dimensions.

1.2.2 Renormalization group approach to Fermi systems

We now apply the above method to a system of interacting fermions. The first step is to construct a path integral for fermions. How to do this is by no means obvious, but it is described in many textbooks, and with a few

[8]The marginal case, $d = 4$ is more tricky and it takes a more careful analysis to find out whether the interaction is truly marginal, *marginally relevant* or *marginally irrelevant*

[9]In high energy physics it is also of interest to study "ultraviolet fixed points" corresponding to $1/s \to \infty$, which govern the short distance behavior of the theory.

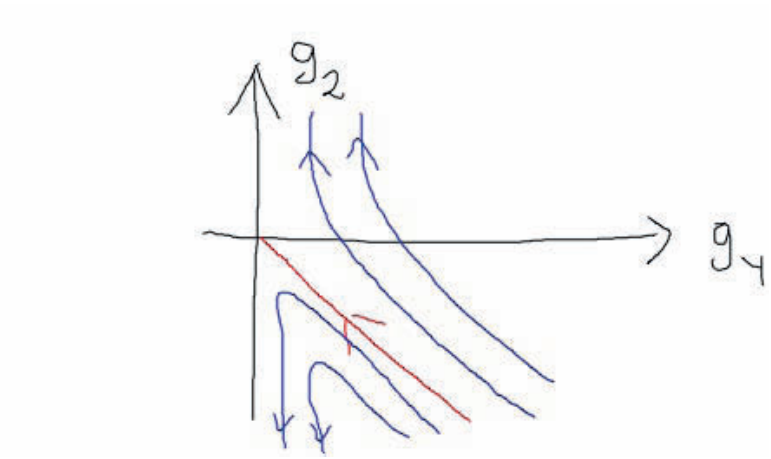

Figure 2: The RG flow generated by the equations (33). The gaussian fixed point is at the origin

changes most expressions parallel those for bosons. There is however a crucial qualitative difference between bosons and fermions in how the renormalization group transformations are defined. This difference is illustrated in figure 3 where we show how the cutoff changes if we do not perform any rescaling - this is the adequate way to illustrate which physical momenta are described by the effective theory as we follow the RG flow. The crucial point is that for fermions, the fixed point theory is not defined by the values of the couplings as the sphere in momentum space shrinks to a point, but as the the values of the couplings on the Fermi sphere. Thus we do not have a set of fixed point coupling *constants*, but rather several fixed point coupling *functions* of the position on the Fermi surface.

1.2.3 The RG flow equations

To see a little bit more in detail how this comes about, we now study the scattering between two particles close to the Fermi surface, $1+2 \rightarrow 3+4$. We take two spatial dimensions to simplify the geometry (three spatial dimensions is discussed in detail in [3]). The Fermi surface is labeled by the polar angles $\theta_i \quad i = 1, \ldots 4$, and an arbitrary two momentum can be parametrized as $\vec{p} = (p_F + p)\vec{\Omega}$, where $\vec{\Omega} = (\cos\theta, \sin\theta)$. As in the previous section we now consider a four particle interaction, which in momentum space can be written as,

$$S_{int} = \int \prod_{i=1}^{4} d\omega_i \sum_{\vec{p}_i} \psi^\dagger_{\vec{p}_3} \psi^\dagger_{\vec{p}_4} \psi_{\vec{p}_2} \psi_{\vec{p}_1} \, f(\omega_i, k_i, \theta_i) \, \delta^4(\sum P_i) \, . \tag{34}$$

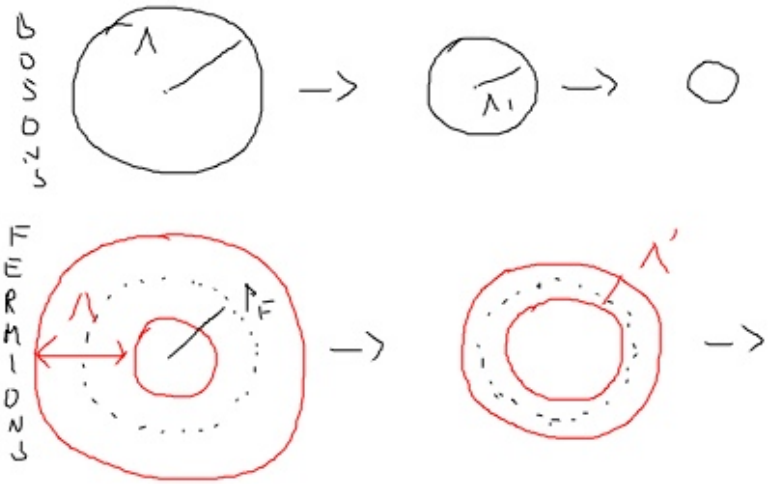

Figure 3: For bosons the RG transformation restricts to momenta to a smaller and smaller ball around $p = 0$ while for fermions the momenta are restricted to a smaller ands smaller *shell* around the Fermi momentum p_F.

Here we distinguished between energy and momentum, used ψ instead of ϕ to indicate that we are dealing with fermions, and f instead of g_4 to anticipate the connection to Fermi liquid theory. The last factor is the delta function that imposes energy and momentum conservation. Under the RG transformation, $k_i \to k_i/s$ and $\omega_i \to \omega_i/s$, while the angles θ_i remain unchanged. Thus we expect that any low momentum fixed point, corresponding to $s \to \infty$ should be characterized not by a coupling constant $f^\star$, but by a coupling *function* $f^\star(\theta_i)$. To get some handle on whether there are any such fixed point functions, and what properties they have, we must find the RG flow equations. The details of the perturbative calculations you find in [3], but to understand the results we must know a few things about how they are derived. The idea is to perform the integration in (28) using perturbation theory to one loop. We should thus evaluate the scattering graphs i figure 4. The important point is now that for all the momenta to lie close to the Fermi circle, there are only three possibilities for the vectors $\vec{\Omega}_i$, as illustrated in figure 5. The first two are,

$$\vec{\Omega}_1 = \vec{\Omega}_3 \quad \text{and} \quad \vec{\Omega}_2 = \vec{\Omega}_4 \qquad (I) \tag{35}$$

$$\vec{\Omega}_1 = \vec{\Omega}_4 \quad \text{and} \quad \vec{\Omega}_2 = \vec{\Omega}_3 \qquad (II) \tag{36}$$

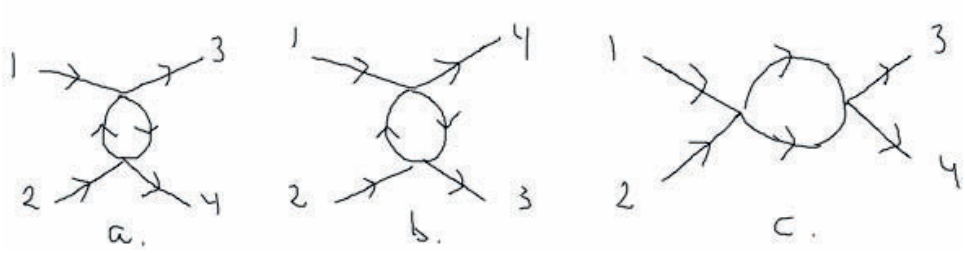

Figure 4: One loop contribution to electron-electron scattering

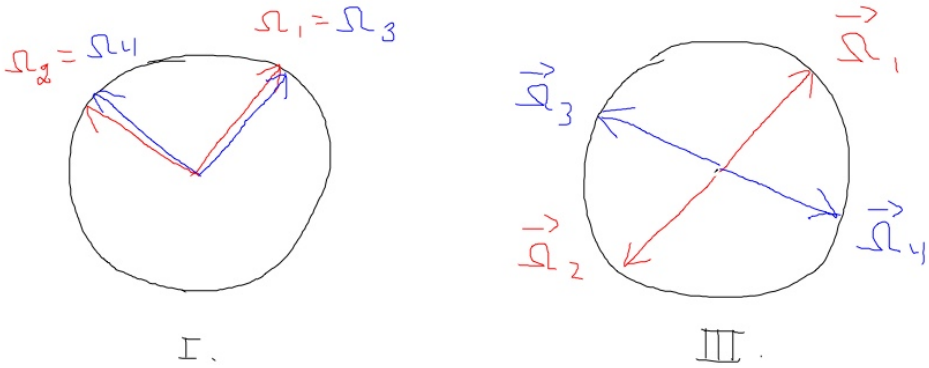

Figure 5: The direction of the momenta of particle scattering close to the Fermi surface. I and III refers to the labels in (35) and (38).

which are identical up to an exchange of the final particles. This process is possible for arbitrary angles θ_1 and θ_2, and because of rotational invariance we expect this scattering to be characterized by a fixed point function,

$$F(\theta_1 - \theta_2) = f(\theta_1, \theta_2, \theta_1, \theta_2) = -f(\theta_1, \theta_2, \theta_2, \theta_1) \tag{37}$$

where the sign comes from the particles being fermions. The third possibility is

$$\vec{\Omega}_1 = -\vec{\Omega}_2 \quad \text{and} \quad \vec{\Omega}_3 = -\vec{\Omega}_4 \qquad (III)\,, \tag{38}$$

that is scattering between particles at opposite positions on the Fermi circle. Here the coupling function can depend only on the angle $\theta_1 - \theta_3$,

$$V(\theta_1 - \theta_3) = f(\theta_1, -\theta_1, \theta_3, -\theta_3)\,. \tag{39}$$

Here we just used geometric intuition to put in the constraints on the angles "by hand", but a carful evaluation of the diagrams in figure 4, taking the cutoff

into account, will give the same result. Furthermore it will give the following RG flow equations for the functions $F(\theta)$ and $V(\theta)$,

$$\frac{dF(\cos\theta)}{dt} = 0 \tag{40}$$

$$\frac{dV(\cos\theta)}{dt} = -\frac{1}{8\pi^2}\int \frac{d\theta'}{2\pi} V(\theta-\theta')V(\theta')\,. \tag{41}$$

Noting that the RHS of (41) is a convolution, we easily find a solution for the Fourier components V_L,

$$V_L(t) = \frac{V_L(0)}{1+V_L(0)t/(4\pi)}\,, \tag{42}$$

where $V_L(0)$ are the starting values for the RG evolution.

1.2.4 Physical interpretation

We are now finally in the position to to draw some conclusions about the low energy theory. First assume that all the $V_L(0)s$ are positive, as would be the case for a reasonably behaved repulsive potential, then $V(\theta)$ will renormalize to zero,[10] while the function $F(\theta)$ remains marginal and characterizes the fixed point theory which can be identifed as a Fermi liquid! If instead at least one of the $V_L(0)$ is negative, as would be the case for any attractive interaction, the RG flow will hit a singularity for some t. At this point the coupling constant, V_L grows out of control, and the perturbative treatment can no longer be trusted. From the details of the calculations, one can however see, that the contribution to the renormalization of $V(\theta)$ comes from the "BCS" diagram in figure 4c. We can thus interpret this singularity as the emergence of a pole in the L^{th} partial wave of the particle - particle scattering amplitude, corresponding to the formation of a Cooper pair with angular momentum L.

We have glossed over several important technical points such as how to properly include spin and and the constraints imposed by the Pauli principle. It also takes some work and care to establish the flow equations (40) from perturbation theory, and to make sure that the resulting putative Fermi liquid, actually coincide with the phenomenological theory discussed in the previous lecture. Another question of conceptual interest namely how to consistently impose the cutoff to increasingly thinner shells around the Fermi surface, is discussed in detail in section V of reference [3].

[10] This is not strictly true. At sufficiently low temperatures, an additional contribution, that was neglected in (41) becomes important, and the system is driven towards superconductivity. This is the RG manifestation of the Kohn-Luttinger effect[6].

2 Luttinger liquids

Fermion systems in one dimension can be realized in various ways, such as quantum wires, carbon nano-tubes or QH edge states. In this lecture you will not learn about the fascinating phenomenology of any of these systems, but get a glimpse of the rich theoretical machinery that has been developed to describe them. The simplest model, and the perhaps most studied, is the Tomonaga-Luttinger model which describes the low lying excitation around the two Fermi points in a one-dimenstional Fermi gas with density-density interactions. In figure 6 you see two different realizations of this model. The left, and more realistic, represents a single parabolic band in some one-dimensional crystal. It should however be clear that the lowlying excitations in this model are the same as in the one shown on the right, which is an 1+1 dimensional Dirac theory. That the former has a finite number of electrons filling the band, and the other an infinite "Dirac sea" should not change the physics close to the Fermi points. The second model, that was originally studied by Luttinger, is given by the Hamiltonian

$$H = \sum_{\alpha=\pm 1} v_F \int dx\, \psi_\alpha^\dagger (i\alpha\partial_x - k_F)\psi_\alpha - \frac{1}{2}\int dx dx'\, \rho(x) V(x-x')\rho(x') \qquad (43)$$

where α lables the two Fermi points, v_F is the Fermi velocity, and ρ the total density, $\rho = \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-$. This model can be studied using perturbation theory or RG methods, but in these lectures we shall describe a method that is specific to one dimension, namely bosonization. The basic observation is that in 1+1 dimensions there is no real difference between fermions and bosons. A strong hint that this is the case comes from noticing the partition sum for free bosons and free fermions quantized on a circle are in fact identical. This means that all energy levels, and the degeneracies of these levels, are the same for fermons and bosons.[11] This does not necessarily mean that all observables in the fermionic theory can be calculated in the boson language, since it is not obvious how to describe a single fermion in terms of bosons. That this can indeed be done is the essence of the *bosonization* scheme, which allows one to express local fermion field operators in terms of a (non-local) function of boson fields. The last lecture will be on some aspects of the bosonized theory, and in particular contrast it with the Fermi liquids we studied earlier. At the end we shall also discuss the chiral Luttinger liquids that describe the edges of quantum Hall samples. To set the scene for all this, we shall in this third

[11] You can easily convince yourself that this is true for the low lying states. The general analytic proof for the partition functions involves some identities for theta functions and can be found *e.g.* in [7]. An elementary proof based on counting is given in chapt. 14 of [8].

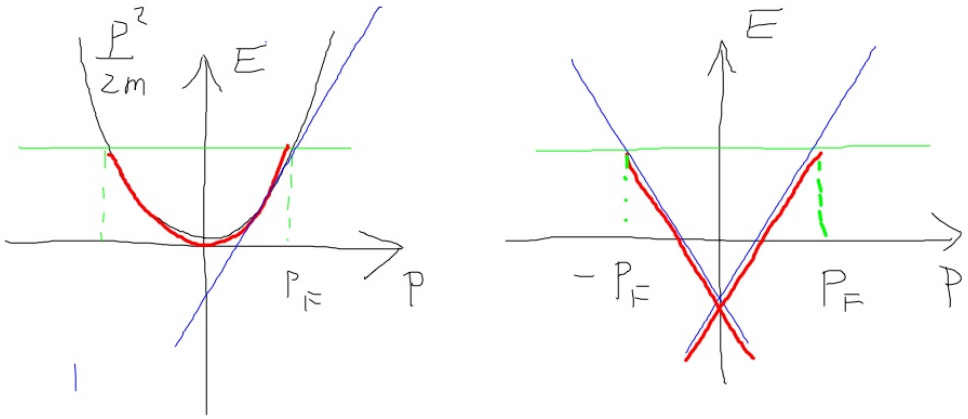

Figure 6: On the left, the dispersion relation for a parabolic band, with the Fermi level and Fermi momentum indicated in green, and the red denotes states filled in the ground state. The blue line indicate the approximate linearized dispersion relation. On the right, the dispertion relation for the Luttinger model, which consists of two branches of right and left moving chiral fermions. Note that in this model there is an unphysical infinite "Dirac sea" of filled negative energy states.

lecture describe a very simple heuristic method developed by Haldane[9], which captures the essence of the low energy dynamics of 1+1 dimensional particles with quartic interactions. En route, it will be clear why fermions and bosons are so similar.

2.1 Haldane's description of 1D harmonic fluids

Haldane has formulated a general description of one-dimensional fluids based on a low-momentum expansion around a constant density[9]. The starting point is a bosonic field theory in 1+1 dimensions,

$$S = \int dtdx\, \phi^\star i\partial_t \phi - \int dt\, H \tag{44}$$

which satisfies the periodic boundary condition $\phi(x+L) = \phi(x)$. Varying $\phi^\star$ gives the equation of motion

$$\partial_t \phi(x,t) = \frac{\delta H}{\delta \phi^\star(x,t)}\,. \tag{45}$$

and taking

$$H = \frac{1}{2m}\int dx\, \phi^\star(-\partial_x^2)\,\phi + \frac{1}{2}\int dxdx'\, \phi^\star(x)\phi^\star(x')V(x-x')\phi(x)\phi(x')\,, \quad (46)$$

we get the non-linear Schrödinger equation

$$i\partial_t\phi(x,t) = -\frac{1}{2m}\partial_x^2\phi + \int dx'\, \phi^\star(x')V(x-x')\phi(x)\phi(x) \quad (47)$$

appropriate for spin-less bosons with mass m interacting via a density-density interaction.

To quantize this theory, which is linear in the time derivative, we notice that the action is already on the Hamiltonian form.[10] To make this point clear, assume that the x is discrete and use the notation $\phi(x_i,t) = \phi_i(t)$. The action can now be written

$$S = \int dt\, \left[\sum_i \phi_i^\star i\partial_t\phi_i - H\right] \quad (48)$$

but this we recognize as the action of a collection of canonical variables that is usually written as $S = \int[p_i\dot{q}_i - H(p_i,q_j)]$. Thus the action (48) implies the canonical comutation relations $[\phi_i(t), i\phi_j^\dagger(t)] = i\delta_{ij}$, or in the continuum model (44)

$$[\phi(x,t),\phi^\dagger(x',t)] = \delta(x-x')\,, \quad (49)$$

where the delta funtion is understood to be periodic with period L. In the following, we shall use a phase-density representation,

$$\phi^\dagger = \sqrt{\rho}\, e^{i\varphi}\,. \quad (50)$$

This is however a bit tricky to use in the continuum, so we start with a lattice version of (49)

$$[\phi_i(t),\phi_j^\dagger(t)] = \delta_{ij}\,. \quad (51)$$

Assuming that variables on different sites commute, we immediately gets the equal time comutation relations

$$[\rho_i, e^{i\varphi_j}] = e^{i\varphi_j}\delta_{ij} \quad (52)$$

or[12]

$$[\varphi_i,\rho_j] = i\delta_{ij}\,. \quad (53)$$

[12]The below relation has to be interpreted with care even in the discrete case. The problem is that while the *phase factor operator* is a well defined, it is not unitary, so the phase is *phase operator*, φ, as naively defined, is not hermitian. Usually one assumes that this problem is unimportant for states with high mean occupation number. For an exposition of this problem see *e.g.* chapt. 10 in [11].

2.1.1 The mean field description

Since we are going to study the low momentum, or long wave length theory, we can no go back to a continuum notation and write,

$$[\varphi(x), \rho(x')] = i\delta(x - x') \tag{54}$$

Note that the field φ is an angle, *i.e.* it is only well defined modulo 2π. In particular this means that the boundary condition on ϕ implies, $\varphi(x + 2\pi) = \varphi(x) + J\pi$, with J an even integer, corresponding to non-trivial winding of the phase field. We shall give the physical interpretation of this winding below. Integrating (52) relation over x we have

$$[\hat{N}, e^{i\varphi}] = e^{i\varphi} \tag{55}$$

where $\hat{N}$ is the total number operator, or the total charge, and thus quantized in units of 1. The relation (55) shows that the "soliton" operator $e^{i\varphi(x)}$ creates a unit charge at x, just as on the lattice.

We next parametrize the small, low momentum deviations from the constant density $\rho_0 = N/L$ as

$$\rho(x) = \rho_0 + \Pi(x)\,, \tag{56}$$

and of course have,

$$[\varphi(x), \Pi(x')] = i\delta(x - x')\,. \tag{57}$$

showing that is φ and Π are conjugate variables, but only as long as we confine ourselves to the low momentum part of the Hilbert space. The cutoff in wave vector is set by the mean density ρ_0, so our approximation is valid for $|k| \ll k_F = \pi\rho_0$.

2.1.2 Putting the particles back

You might think that all effects having to do with the microscopic particle content of the theory is irretrievably lost in this coarse-grained approximation. The basic insight of Haldane, was that this is not necessarily true - certain aspects of the particle nature of the fluid can be built into the effective low-momentum description. The trick is to introduce a new low momentum field θ by,

$$\rho(x) = \rho_0 + \Pi(x) = \frac{1}{\pi}\partial_x\theta(x) \tag{58}$$

To get some intuition for the new field, let us consider the charge operator $\hat{N}$,

$$\hat{N} = \int_0^L dx \, \frac{1}{\pi} \partial_x \theta(x) = \frac{1}{\pi}[\theta(L) - \theta(0)] \tag{59}$$

which is half the winding number of the map from the circle in real space, parametrized by the polar angle $\alpha = 2\pi x/L$, to the circle in the space of field configurations, parametrized by the angular field variable θ. Since the charge in the system is large (otherwise it is not meaningful to consider fluctuations around a mean density) the phase θ will wind many times when we go around the circle. Given this, we expand the field θ as

$$\theta(x) = \hat{\theta} + \frac{\pi x}{L} \hat{N} + \tilde{\theta}(x) \tag{60}$$

where $\tilde{\theta}(x + L) = \tilde{\theta}(x)$ and the operator $\hat{\theta}_0$ is x independent. Recall that the winding number of the phase field φ was defined as the quantum number J, so in analogy we write

$$\varphi(x) = \hat{\varphi} + \frac{\pi x}{L} \hat{J} + \tilde{\varphi}(x) \tag{61}$$

The "zero mode" operators $\hat{\theta}$ and $\hat{\varphi}$ are defined by the relations

$$[\hat{\varphi}, \hat{N}] = [\hat{\theta}, \hat{J}] = i \, , \tag{62}$$

which imply that $e^{im\hat{\varphi}}$ shifts the eigenvalue N of the charge operator $\hat{N}$ by m steps, consistent with the relation (55). To interpret the operator $\hat{J}$, recall that the spatially constant part of the current density is just $\rho_0 \partial_x \varphi$, from which we can infer that J is the zero mode of the total current.

With this background we are prepared to understand Haldane's ansatz for the *granular density operator*, ρ_g,

$$\rho_g(x) = \partial_x \theta(x) \sum_{n=1}^{N} \delta(\theta(x) - n\pi) \, . \tag{63}$$

The interpretation of this central formula should now be clear: when we go around the circle, the integrated charge $\int dx \, \rho$ is not changing continuously, but jumps one unit at each of the N positions determined by $\theta(x_n) = n\pi$. It is thus natural to think of these as the positions of N point-like charges. To understand the prefactor, first consider the case of a constant density so that $\theta(x) = \pi \rho_0 x = \pi N x/L$, and (63) becomes

$$\rho_{0,g} = \frac{\pi N}{L} \sum_{n=1}^{N} \delta(\frac{\pi N}{L} x - n\pi) = \sum_{n=1}^{N} \delta(x - \frac{n}{N} L) \tag{64}$$

i.e. a sum over evenly spaced delta functions which we can interpret as the particle positions. That the normalization is correct also when the mean density varies, follows by integrating ρ over an interval around x, to get the smeared density ρ_s,

$$\begin{aligned} \rho_s(x) &= \frac{1}{\epsilon}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dx\,\rho_g(x) = \frac{1}{\epsilon}\sum_{n=1}^{N}\int_{\theta(x-\frac{\epsilon}{2})}^{\theta(x+\frac{\epsilon}{2})} d\theta\,\delta(\theta-n\pi) \\ &\approx \frac{1}{\epsilon}\frac{\theta(x+\frac{\epsilon}{2})-\theta(x-\frac{\epsilon}{2})}{\pi} \approx \frac{1}{\pi}\partial_x\theta(x) = \rho(x) \end{aligned} \tag{65}$$

where the interval ϵ is large enough for the delta functions peaks at the "particle positions" $x_n = \theta^{-1}(n\pi)$ to be smoothened. Note that it is crucial that $\theta(x)$ is a monotonously increasing function which is guaranteed since the density is positive. It is only then that the inverse θ^{-1} and thus the particle positions are well defined. We see that the smeared density $\rho(x)$ can be identified with $\rho_s(x)$ and is thus the smeared version of the granular $\rho_g(x)$.

Since the sum in (63) defines a periodic delta function, ρ_g can be rewritten as,

$$\rho_g(x) = \frac{1}{\pi}\partial_x\theta(x)\sum_{m=-\infty}^{\infty} e^{i2m\theta(x)} \tag{66}$$

so we see that the low-momentum approximation (56) amounts to retaining only the $m = 0$ term in the sum.

2.1.3 Bosonization

We shall now be more ambitious, and try to construct creation and annihilation operators in terms of the low momentum boson fields φ and θ. You might think that this is going in a circle, just as the particles we are studying - after all we started from a bosonic field theory - but the point is that basically the same procedure will also allow us to express *local Fermi fields* in terms of the low momentum bosons. Or in other words, it will allow us to *bosonize the fermions*. From (50) we see that we must somehow take the square root of the expression (64). Clearly, whatever this distribution is, it must have support only at the points x_n, so up to an undetermined normalization constant we

can again take it as the same delta function.[13] Thus, we are led to define,

$$\Psi_B^\dagger = A\sqrt{\rho_0 + \Pi(x)}\sum_m e^{i2m\theta(x)}e^{i\varphi(x)} \tag{67}$$

where A is a normalization constant that will depend on detailed short distance properties. From the above it is clear that this operator creates one unit of charge, but is a linear combination of creating different amount of current.

It is important to check that $\Phi_B(x)$ and $\phi_B^\dagger(x'))$ so defined commute for $x \neq x'$ as required by Bose statistics. This is not obvious, since it depends on the fields $\theta(x)$ and $\varphi(x)$ which satisfy the commutation relation,

$$[\varphi(x), \theta(x')] = \frac{i\pi}{2}\epsilon(x - x') \tag{68}$$

where $\epsilon(x)$ is the sign function. Using the notation $\phi_m = e^{i2m\theta(x)}e^{i\varphi(x)}$ we have

$$\phi_m(x)\phi_m(x') = e^{im\pi\epsilon(x-x')-im\pi\epsilon(x'-x)}\phi_m(x')\phi_m(x) = \phi_m(x')\phi_m(x) \tag{69}$$

and the other combinations of $\phi_m^\dagger(x)$ and $\phi_m(x')$ can be checked in the same manner.[14] (This calculation, works for $x \neq x'$ and we shall here not worry about the delta function in (49).)

From the expression (69), we can now easily construct Fermi operators, just by shifting $2m \to 2m + 1$,

$$\Psi_F^\dagger = A\sqrt{\rho_0 + \Pi(x)}e^{i(2m+1)\theta(x)}e^{i\varphi(x)} \tag{70}$$

which will satisfy,

$$\{\Psi_F(x), \Psi_F(x')\} = 0 \tag{71}$$

etc.. For bosons the lowest order approximation is simply $\Psi_B^\dagger \sim \sqrt{\rho_0 + \Pi}\, e^{i\varphi}$ while for fermions it is

$$\Psi_F^\dagger \sim \sqrt{\rho_0 + \Pi}\left(e^{i\theta} + e^{-i\theta}\right)e^{i\varphi} \tag{72}$$

[13] Instead of worrying what it could mean to take the square root of an distribution, let us define the delta function as the $\epsilon \to 0$ limit of $\delta_\epsilon(x) = \sqrt{\frac{1}{\pi\epsilon}}e^{-\frac{x^2}{\epsilon}}$ and thus $\sqrt{\delta_\epsilon(x)} = \left(\frac{1}{\pi\epsilon}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2\epsilon}} = (4\pi)^{\frac{1}{4}}\sqrt{\epsilon}\,\delta_{2\epsilon}(x)$ which shows that up to a normalization factor the square root of a delta function is again a delta function.

[14] Strictly speaking this does not demonstrate that the commutator $[\Phi_B(x), \Phi_B(x')]$ vanish, since there are cross terms where $m + m'$ is not even. Such terms will however come with oscillatory factors $\exp[i\pi(m \pm m')k_F(x + x')/2]$.

Since θ contains the term $k_F x$, the to terms will have the behaviour $e^{ik_F x}$ and $e^{-ik_F x}$ corresponding to adding a particle at the right or left Fermi point respectively, it turns out that this gives an *exact* description of the Luttinger model (43). For bosons, on the contrary, a charge can be added without any current, corresponding to a $k = 0$ state. Also note that in the fermionic case the condition to make $\Psi_F^\dagger$ single valued, is no longer that J is an even integer but that $N \pm J$ is an even integer.

2.1.4 The harmonic approximation

To proceed, we must turn our attention to the Hamiltonian. Since we are interested in small fluctuations around the constant density ρ_0, we expand the interaction energy to second order in the deviation to get,

$$H = \frac{1}{2\pi} \int dx \, \left[v_J (\partial_x \varphi)^2 + v_N (\partial_x \theta - \pi \rho_0)^2 \right] \tag{73}$$

where $v_J = \pi \rho_0 / m$ and v_N is a velocity parameter that depends on the interaction.[15] The Hamiltonian (73) together with the commutation relations (68) and (62) defines the theory. We start by finding the spectrum, and in the next section we shall concentrate on the fermion case and calculate a typical fermion-fermion correlation function. Substituting the expansions (60) and (61), and introducing the parameters

$$v_s = \sqrt{v_N v_J} \qquad g = \sqrt{\frac{v_J}{v_N}} \tag{74}$$

we can rewrite the the Hamiltonian as

$$H = \frac{v_s}{2\pi} \int dx \, \left[g (\partial_x \tilde{\varphi})^2 + \frac{1}{g} (\partial_x \tilde{\theta})^2 \right] + \frac{\pi}{2L} [v_J \hat{J}^2 + v_N (\hat{N} - N_0)^2] \tag{75}$$

where $\rho_0 = N_0 / L$. As we already noticed, the operators $e^{\pm i \hat{\varphi}}$ and $e^{\pm i \hat{\theta}}$ provide creation and annihilation operators for the quantum numbers of total charge, N, and total current, J, respectively. This allows us to move between different superselection sectors, and in the following we shall assume that we fixed both N and J. Typically N is large $\sim N_0$ since we are at finite density, while J is small since it measures the imbalance between the population of the right and left movers. From now one we concentrate on the fluctuation part of H.

[15] More specifically, it is related to the compressibility κ by $v_N = \kappa / (\pi \rho_0^2)$.

At this point it is convenient to switch to a Lagrangian formalism

$$\mathcal{L} = \frac{1}{\pi}\partial_x\tilde{\theta}\partial_t\tilde{\varphi} - \frac{v_s}{2\pi}\left[g(\partial_x\tilde{\varphi})^2 + \frac{1}{g}(\partial_x\tilde{\theta})^2\right] \tag{76}$$

and you should make sure that you understand how the first term reproduce the commutation relation (57). To decouple the two fields $\tilde{\theta}$ and $\tilde{\varphi}$, we define,

$$\chi_\pm = \sqrt{g}\tilde{\varphi} \mp \frac{1}{\sqrt{g}}\tilde{\theta} \tag{77}$$

to get

$$\mathcal{L} = \pm\frac{1}{4\pi}\sum_\pm \chi_\pm\partial_x(\partial_t \pm v_s\partial_x)\chi_\alpha \tag{78}$$

which shows that our action is that of *two decoupled chiral bosons*. The term "chiral" refers to the dispersion

$$(\partial_t \pm v_s\partial_x)\chi_\pm = 0 \tag{79}$$

which shows that χ_+ describes a right-moving, and χ_- a left moving wave. It is now also clear that for given values of N and J, the spectrum is that of a free boson, with the two chiralities described by $\chi_\pm$.

2.2 One-dimensional Fermions

You might not find the above result very impressive - from the point of view of bosons, it merely says that a bose gas with sufficiently strong repulsive forces to keep a constant density, will have harmonic sound waves, which is why we used the subscript s on the velocity. The really interesting thing comes when we consider the properties of the fermionic operators,

$$\Psi^\dagger_{F\pm} \sim \sqrt{\rho_0}\, e^{\mp i\theta + i\varphi} \tag{80}$$

which is the simplest version of (70). We are interested in correlations functions of these operators, since they will tell us whether of not we are dealing with a Fermi liquid.[16]

[16] In (80), we shall ignore the the zero mode parts which are of no importance for the x-dependence of the correlation functions.

2.2.1 Correlation functions

The first step in the calculation of the correlations functions is to express $\Psi^\dagger_{F\pm}$ in terms of the independent fields $\chi_\pm$,

$$\Psi^\dagger_{F\pm} \sim \sqrt{\rho_0}\, e^{\mp i\pi\rho_0 x} e^{\frac{i}{2}\left(\sqrt{\frac{1}{g}}\pm\sqrt{g}\right)\chi_+ + \frac{i}{2}\left(\sqrt{\frac{1}{g}}\mp\sqrt{g}\right)\chi_-} e^{i\varphi} \tag{81}$$

where we restored the term $\sim x$ from (60). Note that for the free theory corresponding to $g = 1$, $\Psi^\dagger_{F\pm} \sim e^{i\chi_\pm}$. To evaluate correlators, we introduce the imaginary time variable $\tau = it$ and use the formula

$$\langle e^{ia\chi_\pm(x,\tau)} e^{ib\chi_\pm(x',\tau')}\rangle = e^{ab\,G_\pm(x,t;x',t')} \tag{82}$$

where we defined the Euclidean Greens function

$$G_\pm = \frac{\mp 2\pi}{\partial_x(-i\partial_\tau \mp v_s\partial_x)}\,. \tag{83}$$

or more explicitly,

$$G_\pm(x,\tau) = \mp 2\pi \int \frac{dk}{2\pi}\int \frac{d\omega}{2\pi}\frac{e^{i\omega\tau+ikx}}{ik(\omega \mp iv_s k)}\,. \tag{84}$$

A direct evaluation of this integral gives

$$G_\pm(x,\tau) = \ln(x \pm iv_s\tau)\,. \tag{85}$$

and the corresponding Eucledian correlation functions

$$\langle e^{ia\chi_\pm(x,\tau)} e^{ib\chi_\pm(x',\tau')}\rangle = (x \pm iv_s\tau)^{ab} \tag{86}$$

Combining (81), (82) and (86), we can now calculate the Euclidean correlators of any number of fermion fields. Here we just give the simplest example,

$$\langle \Psi_{F\pm}(x,\tau)\Psi^\dagger_{F\pm}(0,0)\rangle \sim \frac{e^{\pm i\pi\rho_0 x}}{(x+iv_s\tau)^{\gamma_+}(x-iv_s\tau)^{\gamma_-}}\,, \tag{87}$$

where $\gamma_\pm = (1/g + g \pm 2)/4$. For $\tau = 0$ we get the simple result,

$$\langle \Psi_{F\pm}(x)\Psi^\dagger_{F\pm}(0,0)\rangle_{g=1} \sim \frac{e^{\pm i\pi\rho_0 x}}{x^{\frac{1}{2g}+\frac{g}{2}}} \tag{88}$$

From (74) we see that for $g = 1$ we have $v_N = v_J = v_s = v_F$ where v_F is the Fermi velocity, and, recalling that $k_F = \pi\rho_0$, (87) becomes

$$\langle \Psi_{F\pm}(x)\Psi^\dagger_{F\pm}(0,0)\rangle_{g=1} \sim \frac{e^{\pm ik_F x}}{(x \pm iv_s\tau)}$$

which is the correct result for free chiral fermions! This shows that we can recover the correct low energy physics for the fermions starting from the harmonic liquid that originally knew nothing about quantum statistics. The crucial extra input was of course the ansatz (63) for the granular density operator.

Note that in the expression (88) the exponents $\gamma_\pm$, are no longer integer, so the pole that was present in the free propagator becomes a cut! But in a Fermid liquid, there is always a pole describing the long-lived quasiparticles close to the Fermi surface. Thus we suspect that the Fermi surface has disappeared, and this can be verified by calculating the Fourier transform of the propagator to see that the pole in the ω-plane is also replaced by a cut and that the momentum distribution function $N(p)$, discussed in section 1.1.4, no longer has a discontinuity at p_F. For more on how to calculate correlation functions, and how they relate to observable quantities, you can consult the rather detailed review by Voit[12]. You should also note that although there are no gapless fermions in the spectrum, the bosonic excitations are gapless. In this respect the Luttinger model differs qualitatively from a superconductor. In an RG treatment one can in fact see that there is a very delicate balance between a tendency towards superconducting and charge density wave ordering. Since there can be no spontaneous symmetry breaking of a continuous symmetry in 1+1 dimension even at zero temperature, neither of these tendencies win, and the system remains unordered. Which is the dominating tendency is easily understood from the Hamiltonian (76). For large g, corresponding to attractive interactions, the phase $\tilde{\varphi}$ tend to be slowly fluctuating, as in a superconductor, while for small g corresponding to strong repulsion, $\tilde{\theta}$ tend to be uniform, corresponding to the particles sitting at fixed positions with only small fluctuations, as in a charge density wave.

2.2.2 The concept of Luttinger liquid

As stressed at the start, the above derivations are heuristic. To rigorously (or at least rigorously enough for a physicist!) establish that the spectrum and correlation functions is really that of interacting fermions, one starts from the Luttinger Hamiltonian, (43), and by a series of manipulations derive the theory we studied in this section. A detailed account of this procedure is given in [7].[17] In this paper two other facts are stressed, first that the above

[17]The history of bosonization is dates back much further, to Jordan and Wigner, and with important contributions from among others Mattis, Lieb, Luther, Peschel and Mandelstam. The 1983 paper by Haldane is however the first to properly treat the zero modes, to stress the universality of 1+1 dimensional liquids and it is also where the name "Luttinger liquid" was first proposed.

results would be essentially unchanged in a more realistic system where the dispersion is no longer linear, corresponding to non-harmonic terms in the energy, secondly that the many other systems, *e.g.* certain spin chains, show the same qualitative behavior in the low energy region. The term "Luttinger liquid" is used to describe all such system which has low energy characteristics in common with the simple Luttinger model. The advantage of the heuristic derivation given in the previous section is that although it does not rely on any microscopic details - it is never defined whether the "particle positions" in (63) are the locations of fermions, bosons, spin excitations, solitons or something else - it still captures the physics of many different systems. The simplest example of this we have already seen; by choosing m differently in (66) we can describe either bosons or fermions. The strength of the formalism does however go beyond this. As discussed in [9] we can use the more general form (67), and the corresponding fermion expression, to calculate correlation functions that will include oscillatory terms of the form $e^{i2mk_F x}$. These terms are not universal, and the coefficients are not determined, but nevertheless they provide an expansion of the correlation function that can be successfully compared with results derived by other methods, such as Bethe ansatz, that can be applied to specific models. Finally, to connect to the second lecture, from RG perspective, the Luttinger liquid is a fixed point for electron and spin systems in 1+1 dimensions. But just as the Fermi liquid fixed point in higher dimensions is not the only possibility, neither is the Luttinger liquid. A simple counter example are chains of integer spin where the spectrum is gapped, while those of half integer spin are gapless and can show Luttinger liquid behaviour.

2.2.3 Chiral Luttinger liquids and quantum Hall edges

So far we only treated the simplest example of a single Luttinger liquid that is appropriate for describing interacting spin-less fermions. Things become much more interesting for fermions with spin, *i.e.* real electrons. Here one can show that the charge and spin degrees of freedom are described by two different Luttinger liquids characterized by different velocity parameters. This "spin-charge" separation in particular mean that there are spin excitations, called spinons, that do not carry any charge. You can learn more about this fascinating subject in the review by Schultz *et al.*[4]. Here we shortly address another interesting topic, that of chiral Luttinger liquds.

In (78) the Luttinger liquid was expressed as the sum of two independent pieces, that in the free limit $g = 1$, just described the free electrons and holes at the two Fermi points. When $g \neq 1$? the physics at the Fermi points were

coupled, so that a right moving excitation is a superposition of a rightmoving particle, and a left moving hole. (This interpretation might not be completely clear from the above, but follows form a more detailed analysis, and is discussed in some detail *e.g.* in [13].) You might now ask if there is any way to separate the two liquids to get particles that move only in one direction, and furthermore whether such a system could still be interacting in the sense of having $g \neq 1$. It turns out the answer to both these questions are positive.

The quantum Hall effect (QHE) is seen in high quality two-dimensional electron gases at low temperature and high magnetic field.[18] Theoretically, the observation of quantized Hall resistance and several other features, is explained by the formation of incompressible liquids. These quantum hall liquids are topological states of matter, as discussed by J. Moore in this school. The topological field theory describing the simplest QH liquids, are Abelian Chern-Simons gauge theories,

$$\mathcal{L} = \frac{k}{4\pi} ada + \frac{e}{2\pi} adA - ja\,. \tag{89}$$

where we used the notation $ab = a_\mu b^\mu$ and $abc = \epsilon^{\mu\nu\sigma} a_\mu b_\nu c_\sigma$. Here A is the external electromagnetic potential, and a a "statistical" gauge field. Integrating out the a field we get the effective action for the electromagnetic field

$$\mathcal{L} = -\frac{e^2}{4\pi k} AdA + \frac{e}{k} jA - \frac{\pi}{k} j \frac{1}{d} j\,, \tag{90}$$

where we put $\nu = 1/k$. The first term in (89) gives the quantum Hall conductance, $\sigma_H = \nu e^2/h$, the second shows that the quasiparticles have charge νe, and the third encodes the statistical interaction making them $\theta = \nu\pi$ anyons. This shows that the theory (89) correctly captures the low energy pheomenology of a quantum Hall liquid with filling fraction $\nu = 1/k$. For the following analysis we shall take $j = 0$, *i.e.* the system does not contain any quasiparticles.

We now consider our Hall liquid on a bounded region Ω that has a one dimensional boundary $\partial\Omega$—the edge of the system. A proper specification requires that we pick a boundary condition which in a concrete case should follow from the microscopic physics. Here we follow Wen and take $a_0 = 0$ at the boundary[14]. With this choice, and the absence of background fields, the action corresponding to (89) can be reorganized as,

$$S = \frac{k}{4\pi} \int_\Omega d^3x \left[a_2 \dot{a}_1 - a_1 \dot{a}_2 + 2 a_0 b \right] . \tag{91}$$

[18] In graphene, it is enough with high magnetic field; the QH effect is observed at room temperature.

to exhibit a_0 as a Lagrange multiplier field that imposes the constraint $b = da = 0$. This can be solved as

$$a_j = -\frac{1}{k}\partial_j \chi \tag{92}$$

and on substituting this back in (91) we find that

$$S = -\frac{1}{4\pi k}\int_{\partial\Omega} d^2x\, \partial_0\chi\partial_1\chi \tag{93}$$

where we have chosen to parametrize the edge by the coordinate labelled 1. We see, consequently, that for a bounded region the action depends only upon the field χ at the boundary, *i.e.* the only physical degrees of freedom live at the boundary. The remaining degrees of freedom are purely gauge ones and should be eliminated by a suitable choice of gauge for the a field.

The main observation is now that (91) is precisely the kinetic part of one of the components in (78), *i.e.* it is our sought after chiral Luttinger liquid![19] There is however one very important difference between (93) and (78) in that the coefficients in front of the actions differ by the factor $1/k$. Naively one would think that this would only amount to an unimportant renormalization of the field χ, but this is not the case since this coefficient will change the Greens function in (85) to $k \ln x$, and thus change the exponent in the correlation function. Note, however, that while in the ordinary Luttinger liquid, this exponent depended on the compressability, here it is fixed by topology! A lot of experimental effort has gone into trying to verify that the edge states of quantum Hall samples really have the expected chiral Luttinger liquid behavior, but the situation is still not clear.

A more detailed analysis also shows that the edge supports excitations with fractional charge ne/k, corresponding to fractional winding numbers. This is also required by consistency, since there are gapped, fractionally charged vortex like excitations in the bulk. One might ask whether such fractional winding numbers are allowed also in the usual Luttinger liquid? Here the situation is more subtle since the full system is gapless and the definition of charge has to be carefully examined, as discussed in [13].

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[19] By taking the instead the boundary condition $a_0 + va_1 = 0$, or, more physically, introducing an edge potential, we can also reproduce the velocity dependent term.

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