Asynchronous data assimilation with the EnKF

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Synchronous and asynchronous assimilation

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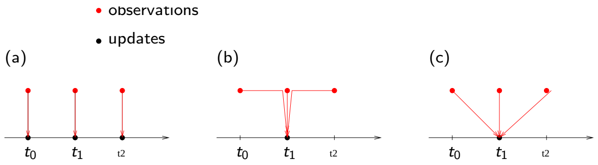


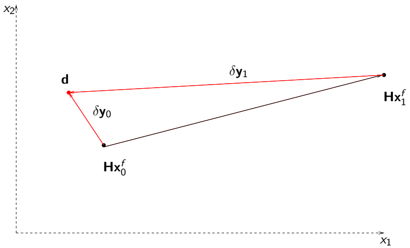
(a) Synchronous assimilation at each observation time

(b) Synchronous assimilation; asynchronous observations are assumed to be synchronous

(c) Asynchronous assimilation

Reasoning

0-order correction:



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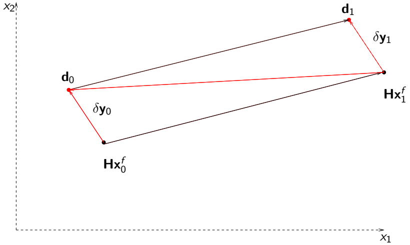

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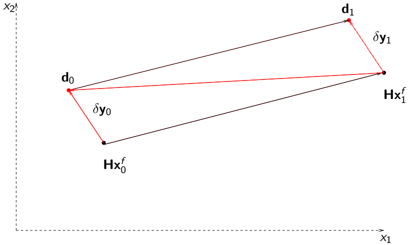

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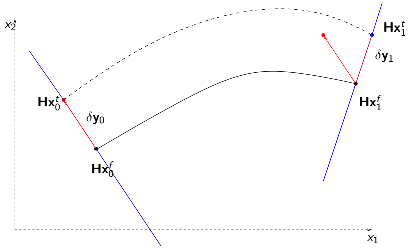

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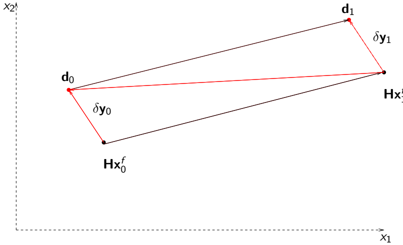

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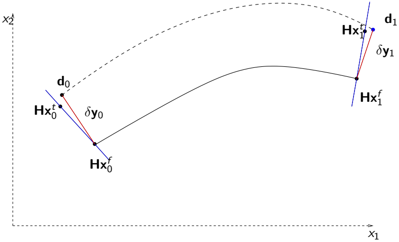

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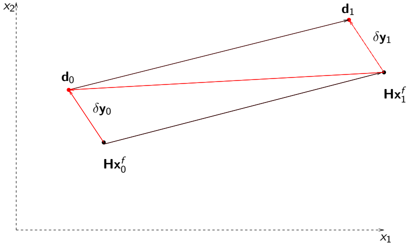

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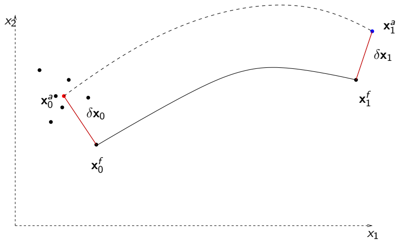

Historical overview

- Ensemble smoother (ES) (van Leeuwen and Evensen, 1996)
- Ensemble Kalman smoother (EnKS) (Evensen and van Leeuwen, 2000; Evensen, 2003, 2009)
- 4DEnKF (Hunt et al., 2004)
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Evolution of corrections

Let us assimilate observations at t_0

$$\delta \mathbf{x}_0 \equiv \mathbf{x}_0^a - \mathbf{x}_0^f = \mathbf{A}_0^f \boxed{\mathbf{c}_0}$$

$$\delta \mathbf{A}_0 \equiv \mathbf{A}_0^a - \mathbf{A}_0^f = \mathbf{A}_0^f \boxed{\mathbf{T}_0}$$

Let $\mathbf{M}_{01}$ be the tangent linear propagator along the forecast system trajectory between t_0 and t_1:

$$\delta \mathbf{x}_1 = \mathbf{M}_{01}\, \delta \mathbf{x}_0 + O\left(\|\delta \mathbf{x}_0\|^2\right)$$

At t_1 the corrections become:

$$\delta \mathbf{x}_1 \sim \mathbf{M}_{01}\, \delta \mathbf{x}_0 = \mathbf{M}_{01} \mathbf{A}_0^f \mathbf{c}_0 \sim \mathbf{A}_1^f \boxed{\mathbf{c}_0}$$

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Evolution of corrections (2)

Therefore,

- If the evolution of the ensemble anomalies within the assimilation window is linear...
- Then the increment from (i) assimilating an observation at time t_0 and (ii) propagating it to time t_1
- Is the same as applying the transform coefficients used at time t_0 at time t_1
- These transform coefficients depend only on the ensemble forecast observations at the time of observation (t_0)
- So we can simply use the ensemble forecast observations from time t_0 for assimilating at t_1 with the same effect as assimilating at t_0
- Serial and parallel assimilation of non-correlated observations are equivalent
- So one can just do parallel assimilation of observations from different times with the assimilation window by using the combined ensemble observations collected at the time of observations

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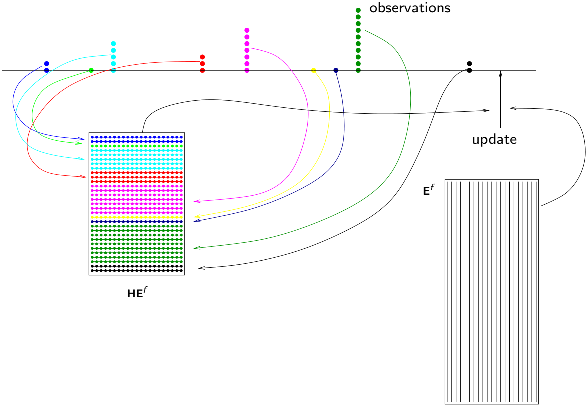
Parallel assimilation of asynchronous data with the EnKF

$$\mathbf{HE}^f = [(\mathbf{HE}_1)^{\mathrm{T}} \dots (\mathbf{HE}_k)^{\mathrm{T}}]^{\mathrm{T}}$$

or

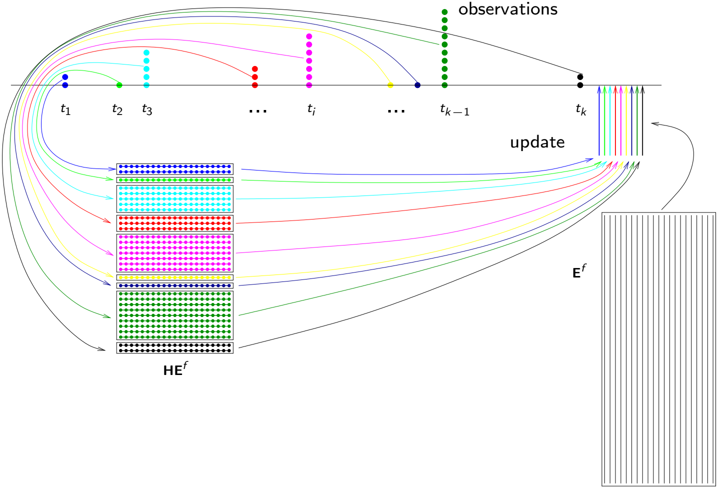
$$\mathbf{s} = [\mathbf{s}_1^{\mathrm{T}} \dots \mathbf{s}_k^{\mathrm{T}}]^{\mathrm{T}}$$

$$\mathbf{S} = [\mathbf{S}_1^{\mathrm{T}} \dots \mathbf{S}_k^{\mathrm{T}}]^{\mathrm{T}}$$



EnKS

$$\mathbf{E}^a = \mathbf{E}^f \prod_{i=1}^{k} \mathbf{X}_5(t_i),$$



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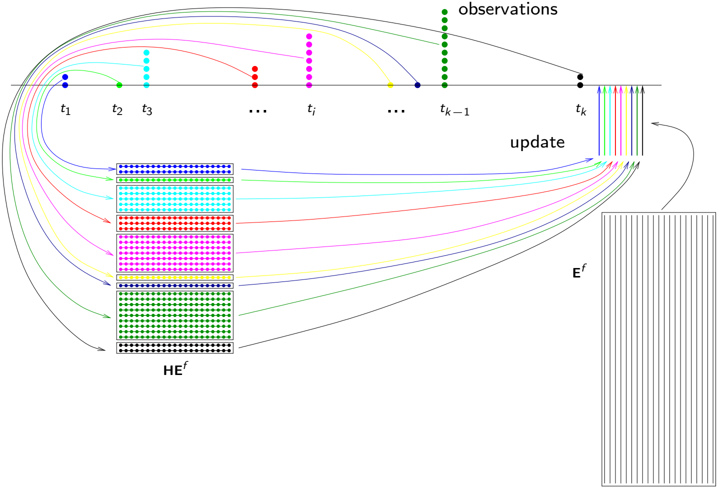

4DEnKF

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$$\mathbf{H}\mathbf{x}_0 \;\rightarrow\; \mathbf{H}\mathbf{E}_0(\mathbf{E}_1^{\mathrm{T}}\mathbf{E}_1)^{-1}\mathbf{E}_1^{\mathrm{T}}\mathbf{x}_1,$$
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What is $(\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1}\mathbf{E}\,\mathbf{x}$ doing? - It is a vector of coefficients of projection of $\mathbf{x}$ onto the range of $\mathbf{E}$

$$(\mathbf{E}_1^{\mathrm{T}}\mathbf{E}_1)^{-1}\mathbf{E}_1^{\mathrm{T}}\mathbf{x}_1 = \mathbf{1}/m \quad \rightarrow \quad \mathbf{E}_0(\mathbf{E}_1^{\mathrm{T}}\mathbf{E}_1)^{-1}\mathbf{E}_1^{\mathrm{T}}\mathbf{x}_1 = \mathbf{x}_0$$
$$(\mathbf{E}_1^{\mathrm{T}}\mathbf{E}_1)^{-1}\mathbf{E}_1^{\mathrm{T}}\mathbf{A}_1 = \mathbf{I} - \mathbf{1}\mathbf{1}^{\mathrm{T}}/m \quad \rightarrow \quad \mathbf{E}_0(\mathbf{E}_1^{\mathrm{T}}\mathbf{E}_1)^{-1}\mathbf{E}_1^{\mathrm{T}}\mathbf{A}_1 = \mathbf{A}_0$$

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Conclusions

- Recipe: use ensemble observations stored at observation times, as in "4D-LETKF" description (Hunt et al., 2007)
- This method is scheme-independent
- And it is compatible with localisation
- It is formally equivalent to the EnKS solution (Evensen and van Leeuwen, 2000)
- But it is better suited than the EnKS for assimilating observations from multiple times within the assimilation window
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Appendix: TLM and AM from ensemble clouds

By definition, tangent linear model (propagator) is the map $\mathbf{M}_k$ such as

$$\delta \mathbf{x}_{k+1} = \mathbf{M}_k \, \delta \mathbf{x}_k ,$$

and the adjoint model is its transposition.
It is easy to verify that for perturbation belonging to the subspace spanned by ensemble anomalies

$$\mathbf{M}_k = \mathbf{A}_{k+1} \left[(\mathbf{A}_k)^{\mathrm{T}} \mathbf{A}_k \right]^{+} (\mathbf{A}_k)^{\mathrm{T}} .$$

Accordingly,

$$(\mathbf{M}_k)^{\mathrm{T}} = \mathbf{A}_k \left[(\mathbf{A}_k)^{\mathrm{T}} \mathbf{A}_k \right]^{+} (\mathbf{A}_{k+1})^{\mathrm{T}} .$$

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