Miscellaneous

Pavel Sakov

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Outline

Formulation and historical overview

Covariance inflation, or ensemble inflation refers to artificial increase of uncertainty in the state estimate. It is commonly applied as follows:

$$\mathbf{P} \leftarrow \rho^2 \mathbf{P}, \qquad \text{or}$$

$$\mathbf{A} \leftarrow \rho \mathbf{A}$$

where $\rho = 1 + \delta$.

- First use in the above form - probably in Anderson (2001)
- Has similarity with the "forgetting factor" used by Pham et al. (1998)
- Ott et al. (2004) use "enhanced ensemble inflation"
- Anderson (2007) introduces an adaptive algorithm
- Sacher and Bartello (2008) obtain theoretical inflation factor to compensate for finite ensemble size in the traditional EnKF
- Evensen (2009) introduces another adaptive scheme to compensate for sampling error

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$$(\mathbf{P}^a)^{-1} = \rho (\mathbf{P}^f)^{-1} + \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{H}, \qquad 0 < \rho \leq 1$$

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Possible reasons

Possible reasons for the covariance inflation:

- rank deficiency of the ensemble
- nonlinearity
- spurious correlations

Generally: compensate for over-optimistic analysis that does not take into account suboptimalities of the system

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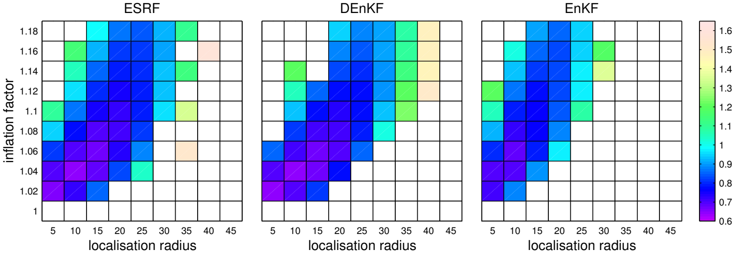
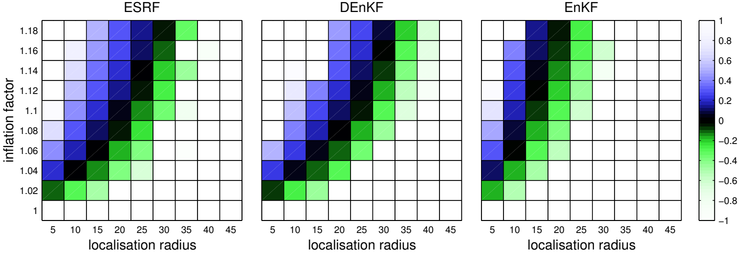
On tuning of a sub-optimal system

RMS error:



Ensemble spread:



For a system with perfect model: match ensemble spread and RMSE

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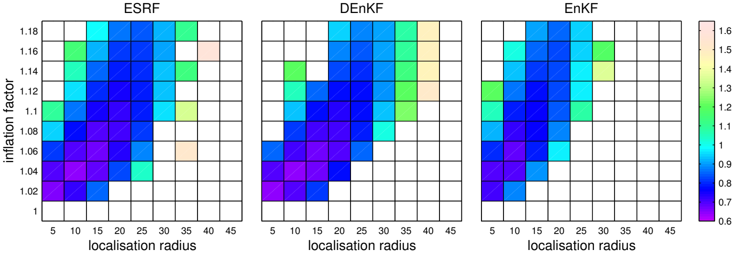

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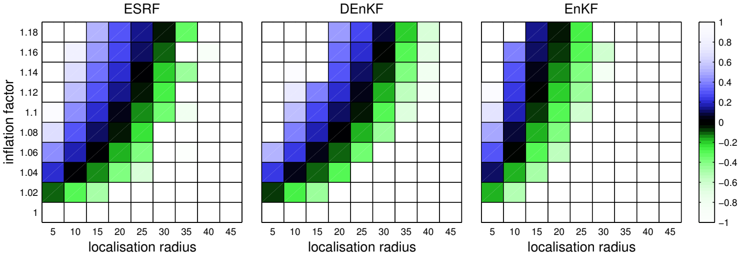

For a system with perfect model: match ensemble spread and RMSE

Conclusions

- Covariance inflation is an ad-hoc modification of the EnKF
- The inflation factor ρ is usually empirically selected
- It is often essential for preventing the (quick) filter collapse as well as for the long-term stability of the system
- In practice, a small inflation can often have a substantial positive impact on the performance of the system

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EnOI

EnOI = ensemble optimal interpolation (Evensen, 2003) - uses static ensemble anomalies

$$\mathbf{P} = \alpha \frac{1}{m-1} \mathbf{A}\mathbf{A}^{\mathrm{T}}$$

- Cost effective (integrating only one instance of the model)
- Robust (no danger of ensemble collapse)
- Better than OI (krigging): anisotropic covariance; multivariate
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Hybrid systems

- Hamill and Snyder (2000) - a hybrid EnKF-3DVar system
- Wang et al. (2007) - a hybrid EnKF-EnOI ("ETKF-OI") system

Hybrid systems combine static and dynamic state error covariance matrices:

$$\mathbf{P} = \beta \mathbf{P}_d + (1-\beta)\mathbf{P}_s$$

Ensemble formulation:

$$\mathbf{A} = \sqrt{m-1}\left[\sqrt{\frac{\beta}{m_d-1}}\mathbf{A}_d, \sqrt{\frac{\alpha(1-\beta)}{m_s-1}}\mathbf{A}_s\right], \qquad \text{so that}$$

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in 4DVar:

$$\mathcal{L}_p \overset{conc.}{=} \beta(\mathbf{x}-\mathbf{x}^f)^{\mathrm{T}}(\mathbf{P}_b^f)^{-1}(\mathbf{x}-\mathbf{x}^f) + (1-\beta)(\mathbf{x}-\mathbf{x}^f)^{\mathrm{T}}(\mathbf{P}_d^f)^{-1}(\mathbf{x}-\mathbf{x}^f)$$

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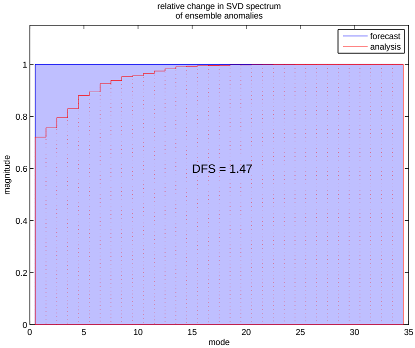
Observation impact: DFS and SRF

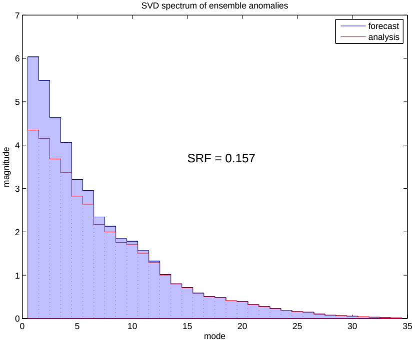
DFS $=$ "Degrees of Freedom of Signal" (Rodgers, 2000; Cardinali et al., 2004)

$$\mathrm{DFS} = \mathrm{tr}(\mathbf{KH})$$

SRF $=$ "Spread Reduction Factor"

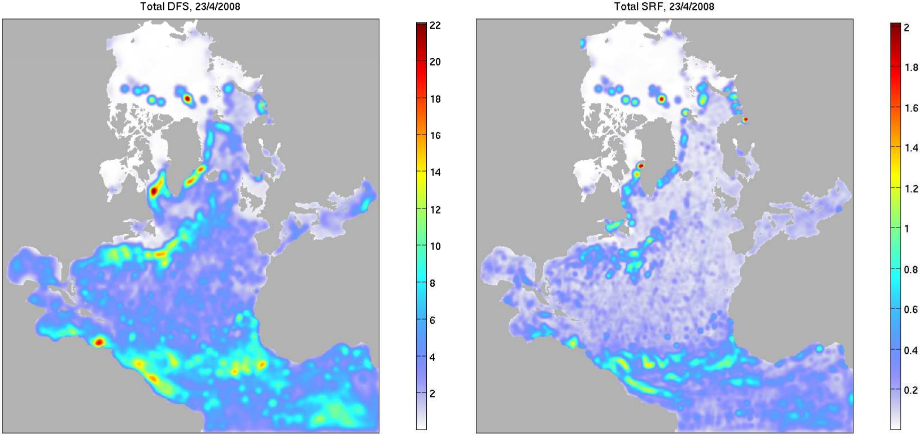
$$\mathrm{SRF} = \sqrt{\frac{\mathrm{tr}(\mathbf{HP}^f\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1})}{\mathrm{tr}(\mathbf{HP}^a\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1})}} - 1$$



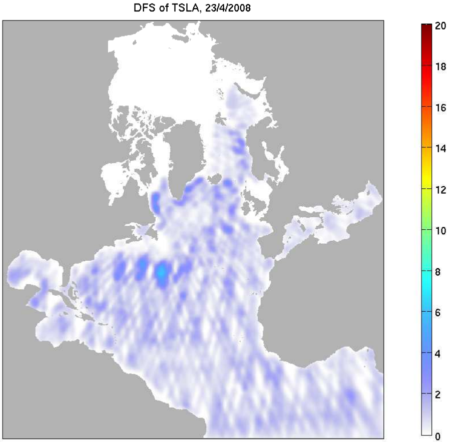


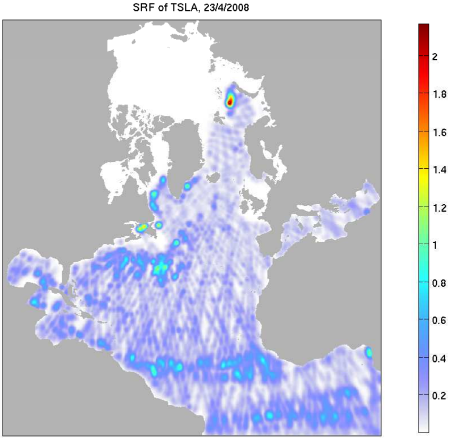
Total DFS and SRF

(PilotTOPAZ reanalysis, 24 April 2008)

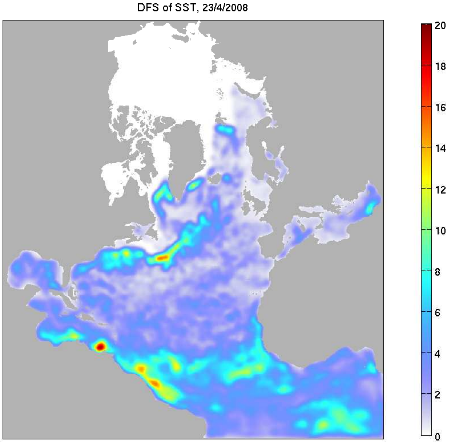


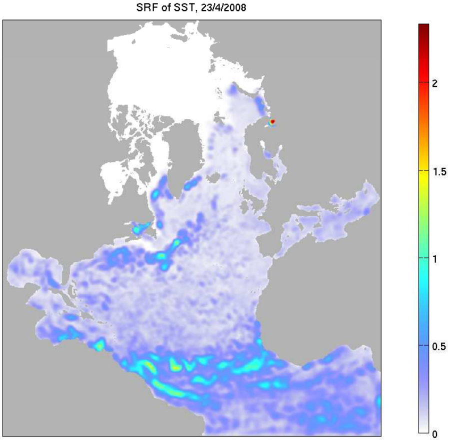
DFS and SRF for TSLA





DFS and SRF for SST





Some system design issues

- Assimilation strength:
  model e-folding time τ ? assimilation cycle length T
- Constraining the model:
  number of observations p ? degrees of freedom D_{mod}
- Ensemble rank:
  dependence on localisation radius r_{loc}

Some system design issues

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$$\frac{\|\mathbf{A}^f\|}{\|\mathbf{A}^a\|} \sim e^{T/\tau}$$

  $T \ll \tau$ – "weak" assimilation
  $T \gg \tau$ – "strong" assimilation

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$$\frac{D_{obs}}{T} \quad \gtrsim \quad \frac{D_{mod}}{\tau}$$

$$\mathrm{tr}(\mathbf{KH})\frac{\tau}{T} \quad \gtrsim \quad D_{mod}$$

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$$r_{loc} \gtrsim v_{adv} T$$

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