Kalman filter

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Outline

KF and EnKF: historical overview

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  propagation and update of state error covariance and mean for a linear stochastic system
- Serial square root filter (Potter, ∼1963)

  propagation and update of state error covariance in a square root form
- Extended Kalman Filter (Smith et al., 1962)

  Propagation of state error covariance with linearised model
- Parallel square root filter (Andrews, 1968)

  formally equivalent to the ETKF solution
- Ensemble Kalman filter (Evensen, 1994; Burgers et al., 1998)

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Some conventions

Fonts

capital, bold, as in $\mathbf{X}$	–	a matrix
small, bold, as in $\mathbf{x}$	–	a vector
small, normal, as in x	–	a scalar
capital, fancy, as in $\mathcal{M}$	–	a function

Indices

lower, without brackets, as in $\mathbf{X}_i, \mathbf{x}_2, x_n$	–	ith object in a sequence
upper, without brackets, as in $\mathbf{X}^a, \sigma^{obs}$	–	used to mark the object
after a bracketed object, as in$(\mathbf{X})_i, (\mathbf{X})^{\mathrm{T}}, (\mathbf{X})^2$	–	an operation

in particular:

$(\mathbf{X})_i$	–	ith column of $\mathbf{X}$
$(\mathbf{X})_{i,:}$	–	ith row of $\mathbf{X}$
$(\mathbf{X})_{ij}$	–	the element in ith row and jth column of $\mathbf{X}$
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Nonlinear case

Estimation problem: fit the state to observations with no time dependence and no prior information about state

- Let us have observations $\mathbf{y}$.
- Let $\mathcal{H}$ be the observation function, so that $\mathcal{H}(\mathbf{x})$ gives observations associated with $\mathbf{x}$.
- Let us characterise the uncertainty of $\mathbf{y}$ by covariance $\mathbf{R}$: if $\mathbf{x}$ is the true state, then $\mathbf{y} = \mathcal{H}(\mathbf{x}) + \mathbf{v}$, and $(\mathbf{R})_{ij} = \mathrm{cov}[(\mathbf{v})_i, (\mathbf{v})_j]$.
- Estimation problem: find $\mathbf{x}^a$ such that

$$\mathbf{x}^a = \arg\min_{\mathbf{x}} \left\{ [\mathbf{y} - \mathcal{H}(\mathbf{x})]^{\mathrm{T}} (\mathbf{R})^{-1} [\mathbf{y} - \mathcal{H}(\mathbf{x})] \right\}. \tag{1}$$

- Statistical interpretation: if PDF of observation error is Gaussian,

$$\mathcal{P}[\mathbf{y} - \mathcal{H}(\mathbf{x})] = \exp\left\{ -\frac{1}{2} [\mathbf{y} - \mathcal{H}(\mathbf{x})]^{\mathrm{T}} (\mathbf{R})^{-1} [\mathbf{y} - \mathcal{H}(\mathbf{x})] \right\},$$

then (1) gives the maximal likelihood solution.

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+ prior information about the state

- Let us have a prior estimate of the state $\mathbf{x}^f$ ("forecast"), with the uncertainty characterised by covariance $\mathbf{P}^f$.
- Estimation problem: find $\mathbf{x}^a$ such that

$$\mathbf{x}^a = \arg\min_{\mathbf{x}} \Big\{ (\mathbf{x} - \mathbf{x}^f)^{\mathrm{T}} (\mathbf{P}^f)^{-1} (\mathbf{x} - \mathbf{x}^f) + [\mathbf{y} - \mathcal{H}(\mathbf{x})]^{\mathrm{T}} (\mathbf{R})^{-1} [\mathbf{y} - \mathcal{H}(\mathbf{x})] \Big\} . \tag{2}$$

+ time dimension

- Let us have a sequence of observations $\mathbf{y}_1, \ldots, \mathbf{y}_k$ obtained at time $t_1, \ldots, t_k$, with error covariances $\mathbf{R}_1, \ldots, \mathbf{R}_k$.
- Estimation problem: find $\{\mathbf{x}_i^a\}$, $i = 1, \ldots, k$ such that

$$\{\mathbf{x}_i^a\}_{i=1}^k = \arg \min_{\{\mathbf{x}_i\}_{i=1}^k} \Big\{ (\mathbf{x}_1 - \mathbf{x}_1^f)^{\mathrm{T}} (\mathbf{P}_1^f)^{-1} (\mathbf{x}_1 - \mathbf{x}_1^f) + \sum_{i=1}^k [\mathbf{y}_i - \mathcal{H}_i(\mathbf{x}_i)]^{\mathrm{T}} (\mathbf{R}_i)^{-1} [\mathbf{y}_i - \mathcal{H}_i(\mathbf{x}_i)] \Big\} . \tag{3}$$

+ model error

- Let $\mathbf{x}_{i+1} = \mathcal{M}_i(\mathbf{x}_i) + \mathbf{w}_i$, with $\mathbf{w}_i$ characterised by covariance $\mathbf{Q}_i$: $(\mathbf{Q}_i)_{lm} = \mathrm{cov}[(\mathbf{w}_i)_l, (\mathbf{w}_i)_m]$.
- The estimation problem becomes:

$$\{\mathbf{x}_i^a\}_{i=1}^k = \arg \min_{\{\mathbf{x}_i\}_{i=1}^k} \mathcal{L}(\mathbf{x}_1, \ldots, \mathbf{x}_k),$$

$$\begin{aligned}\mathcal{L}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \Big\{ & (\mathbf{x}_1 - \mathbf{x}_1^f)^{\mathrm{T}} (\mathbf{P}_1^f)^{-1} (\mathbf{x}_1 - \mathbf{x}_1^f) \\ & + \sum_{i=1}^k [\mathbf{y}_i - \mathcal{H}_i(\mathbf{x}_i)]^{\mathrm{T}} (\mathbf{R}_i)^{-1} [\mathbf{y}_i - \mathcal{H}_i(\mathbf{x}_i)] \\ & + \sum_{i=2}^k [\mathbf{x}_i - \mathcal{M}_{i-1}(\mathbf{x}_{i-1})]^{\mathrm{T}} (\mathbf{Q}_{i-1})^{-1} [\mathbf{x}_i - \mathcal{M}_{i-1}(\mathbf{x}_{i-1})] \Big\}. \qquad (4)\end{aligned}$$

Linear case

Standard assumptions of linearity:

$$\mathbf{x}_{i+1} \equiv \mathcal{M}_i(\mathbf{x}_i) = \mathbf{M}_i \mathbf{x}_i,$$
$$\mathcal{H}_i(x_i) = \mathbf{H}_i \mathbf{x}_i.$$

Sufficient assumptions – affine $\mathcal{M}$ and $\mathcal{H}$:

$$\mathcal{M}_i(\mathbf{x}_i^2) = \mathcal{M}_i(\mathbf{x}_i^1) + \mathbf{M}_i\,(\mathbf{x}_i^2 - \mathbf{x}_i^1), \qquad \mathbf{M}_i = \nabla_x \mathcal{M}_i(\mathbf{x}_i),$$
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- Consider the minimisation problem for the linear case.

- Let us assume that

$$\min_{\{\mathbf{x}_i\}_{i=1}^{k-1}} \mathcal{L}_k(\mathbf{x}_1, \ldots, \mathbf{x}_k) = (\mathbf{x}_k - \mathbf{x}_k^f)^{\mathrm{T}} (\mathbf{P}_k^f)^{-1} (\mathbf{x}_k - \mathbf{x}_k^f) + \mathrm{Const.}$$

- Then

$$\min_{\{\mathbf{x}_i\}_{i=1}^{k}} \mathcal{L}_{k+1}(\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}) = (\mathbf{x}_{k+1} - \mathbf{x}_{k+1}^f)^{\mathrm{T}} (\mathbf{P}_{k+1}^f)^{-1} (\mathbf{x}_{k+1} - \mathbf{x}_{k+1}^f) + \mathrm{Const},$$

where

$$\begin{aligned}
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\mathbf{x}_k^a &= \mathbf{x}_k^f + \mathbf{K}_k [\mathbf{y}_k - \mathcal{H}(\mathbf{x}_k^f)], \\
\mathbf{K}_k &= \mathbf{P}_k^f (\mathbf{H}_k)^{\mathrm{T}} [\mathbf{H}_k \mathbf{P}_k^f (\mathbf{H}_k)^{\mathrm{T}} + \mathbf{R}_k]^{-1} \\
\mathbf{P}_k^a &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^f.
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where

$$\left.\begin{aligned} \mathbf{x}_{k+1}^f &= \mathcal{M}_k(\mathbf{x}_k^a), \\ \mathbf{P}_{k+1}^f &= \mathbf{M}_k \mathbf{P}_k^a (\mathbf{M}_k)^{\mathrm{T}} + \mathbf{Q}_k, \end{aligned}\right\} \quad \text{"propagation"}$$

$$\left.\begin{aligned} \mathbf{x}_k^a &= \mathbf{x}_k^f + \mathbf{K}_k[\mathbf{y}_k - \mathcal{H}(\mathbf{x}_k^f)], \\ \mathbf{K}_k &= \mathbf{P}_k^f (\mathbf{H}_k)^{\mathrm{T}} [\mathbf{H}_k \mathbf{P}_k^f (\mathbf{H}_k)^{\mathrm{T}} + \mathbf{R}_k]^{-1} \\ \mathbf{P}_k^a &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^f. \end{aligned}\right\} \quad \text{"analysis"}$$

Kalman filter, proof (analysis part)

Regroup the terms:

$$(\mathbf{x}_k - \mathbf{x}_k^f)^{\mathrm{T}}(\mathbf{P}_k^f)^{-1}(\mathbf{x}_k - \mathbf{x}_k^f) + [\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{R}_k)^{-1}[\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k)] \tag{5}$$

$$\Rightarrow (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\mathbf{x}_k - \mathbf{x}_k^a) + C, \tag{6}$$

and find $\mathbf{x}_k^a$ and $\mathbf{P}_k^a$.

- $(\mathbf{x}_k)^2$:

$$\boxed{(\mathbf{P}_k^a)^{-1} = (\mathbf{P}_k^f)^{-1} + (\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\mathbf{H}_k}. \tag{7}$$

- $(\mathbf{x}_k)^1$:

$$\boxed{\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{P}_k^a(\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\left[\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k^f)\right]}. \tag{8}$$

Kalman filter, proof (analysis part)

Regroup the terms:

$$(\mathbf{x}_k - \mathbf{x}_k^f)^{\mathrm{T}}(\mathbf{P}_k^f)^{-1}(\mathbf{x}_k - \mathbf{x}_k^f) + [\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{R}_k)^{-1}[\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k)] \tag{5}$$

$$\Rightarrow (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\mathbf{x}_k - \mathbf{x}_k^a) + C, \tag{6}$$

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$$= \cancel{(\mathbf{x}_k)^{\mathrm{T}}}(\mathbf{P}_k^a)^{-1}(-\mathbf{x}_k^a)$$

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$$\left. + (\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\left[\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k^f) + \cancel{\mathbf{H}_k\mathbf{x}_k^f}\right]\right\}$$

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Kalman filter, proof (analysis part, 2)

- Rearranging (7):

$$(\mathbf{P}_k^a)^{-1} = (\mathbf{P}_k^f)^{-1} + (\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\mathbf{H}_k \qquad \Rightarrow$$

$$\mathbf{P}_k^f = \mathbf{P}_k^a + \mathbf{P}^f(\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\mathbf{H}\mathbf{P}_k^a \qquad \Rightarrow$$

$$\mathbf{P}_k^a = \left[\mathbf{I} + \mathbf{P}^f(\mathbf{H}_k)^{\mathrm{T}}(\mathbf{R}_k)^{-1}\mathbf{H}\right]^{-1}\mathbf{P}_k^f.$$

Matrix inversion lemma:

$$(\mathbf{A} + \mathbf{X}\mathbf{B}\mathbf{Y})^{-1} = (\mathbf{A})^{-1} - (\mathbf{A})^{-1}\mathbf{X}[(\mathbf{B})^{-1} + \mathbf{Y}(\mathbf{A})^{-1}\mathbf{X}]^{-1}\mathbf{Y}(\mathbf{A})^{-1}$$

set $\quad \mathbf{A} = \mathbf{I}, \quad \mathbf{X} = \mathbf{P}_k^f(\mathbf{H}_k)^{\mathrm{T}}, \quad \mathbf{B} = (\mathbf{R}_k)^{-1}, \quad \mathbf{Y} = \mathbf{H}_k \quad \Rightarrow$

$$\mathbf{P}_k^a = \left\{\mathbf{I} - \mathbf{P}_k^f(\mathbf{H}_k)^{\mathrm{T}}\left[\mathbf{H}_k\mathbf{P}_k^f(\mathbf{H}_k)^{\mathrm{T}} + \mathbf{R}_k\right]^{-1}\mathbf{H}_k\right\}\mathbf{P}_k^f,$$

or

$$\boxed{\begin{aligned} \mathbf{P}_k^a &= (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_k^f, && (9)\\ \mathbf{K}_k &\equiv \mathbf{P}_k^f(\mathbf{H}_k)^{\mathrm{T}}\left[\mathbf{H}_k\mathbf{P}_k^f(\mathbf{H}_k)^{\mathrm{T}} + \mathbf{R}_k\right]^{-1}. && (10)\end{aligned}}$$

(8) $\Rightarrow$

$$\boxed{\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k\left[\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k^f)\right].} \qquad (11)$$

Kalman filter, proof (propagation part)

- We are aiming to show that

$$\min_{\mathbf{x}_k} \left\{ (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}} (\mathbf{P}_k^a)^{-1} (\mathbf{x}_k - \mathbf{x}_k^a) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}} (\mathbf{Q}_k)^{-1} [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)] \right\}$$
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  and find $\mathbf{x}_{k+1}^f$ and $\mathbf{P}_{k+1}^f$.

- This problem is solved in two stages:
  1. Find expression for the cost function at minimum over $\mathbf{x}_k$
  2. Factorise the expression and transform to the final form

- Solution:

$$\boxed{\begin{aligned} \mathbf{x}_{k+1}^f &= \mathcal{M}_k(\mathbf{x}_k^a), && (12) \\ \mathbf{P}_{k+1}^f &= \mathbf{M}_k \mathbf{P}_k^a (\mathbf{M}_k)^{\mathrm{T}} + \mathbf{Q}_k. && (13) \end{aligned}}$$

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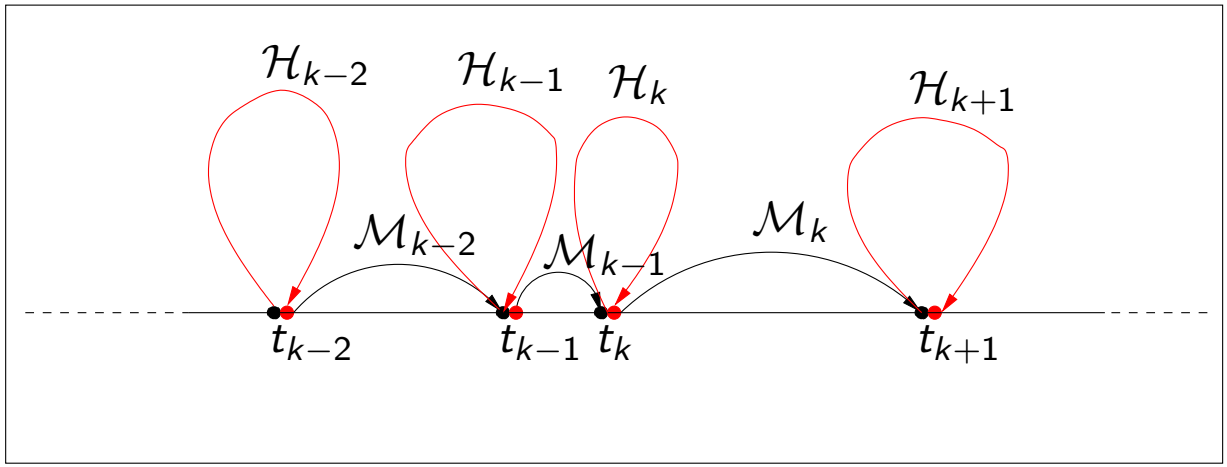

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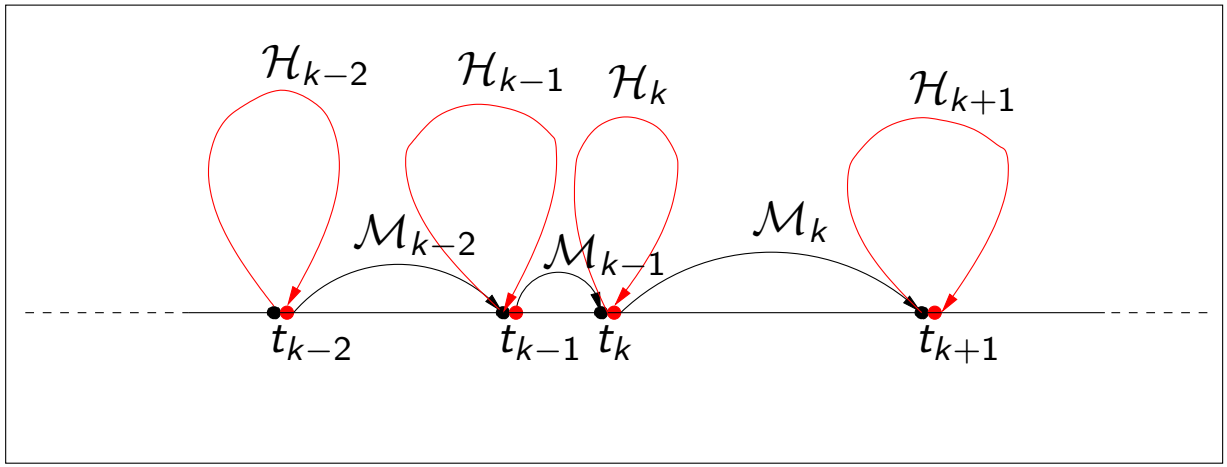

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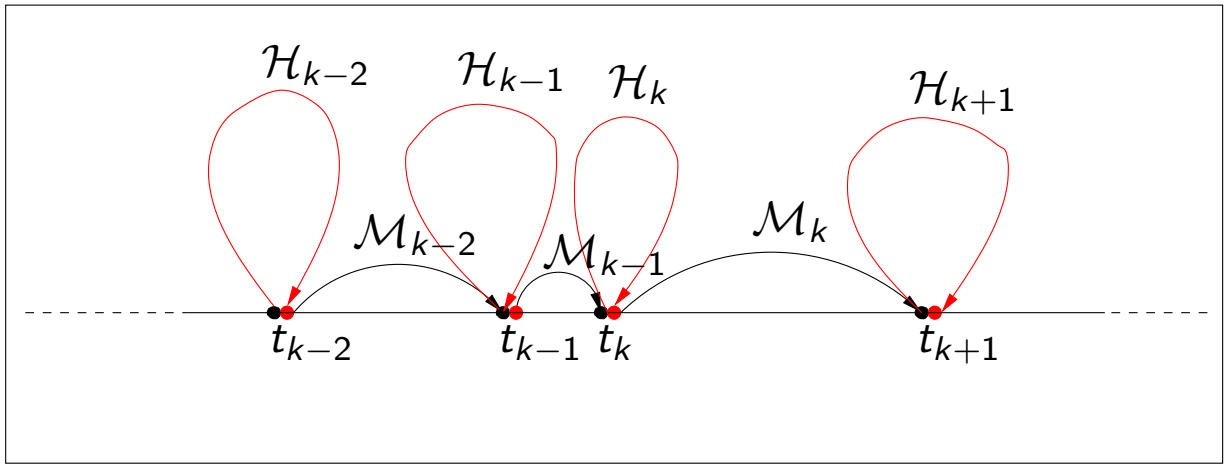

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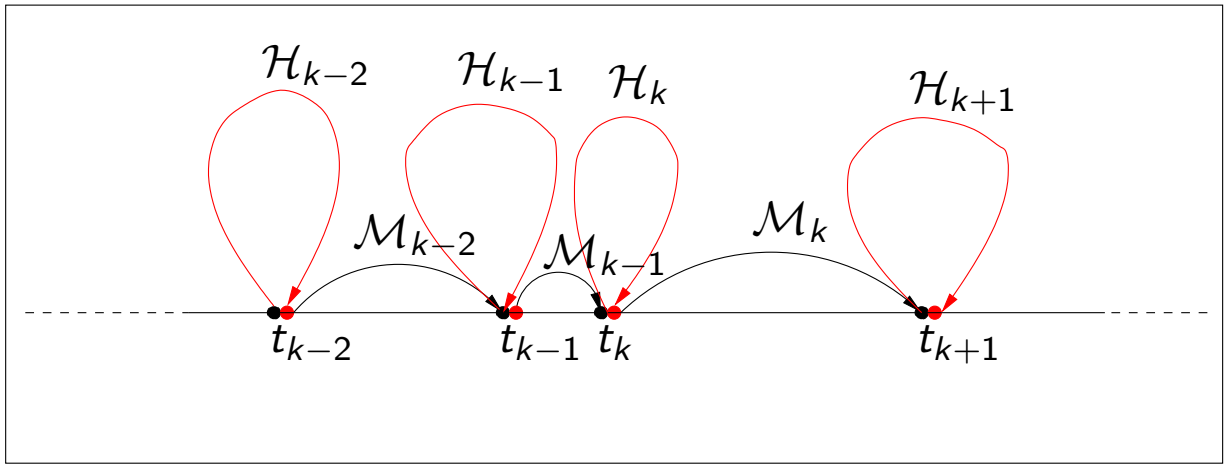

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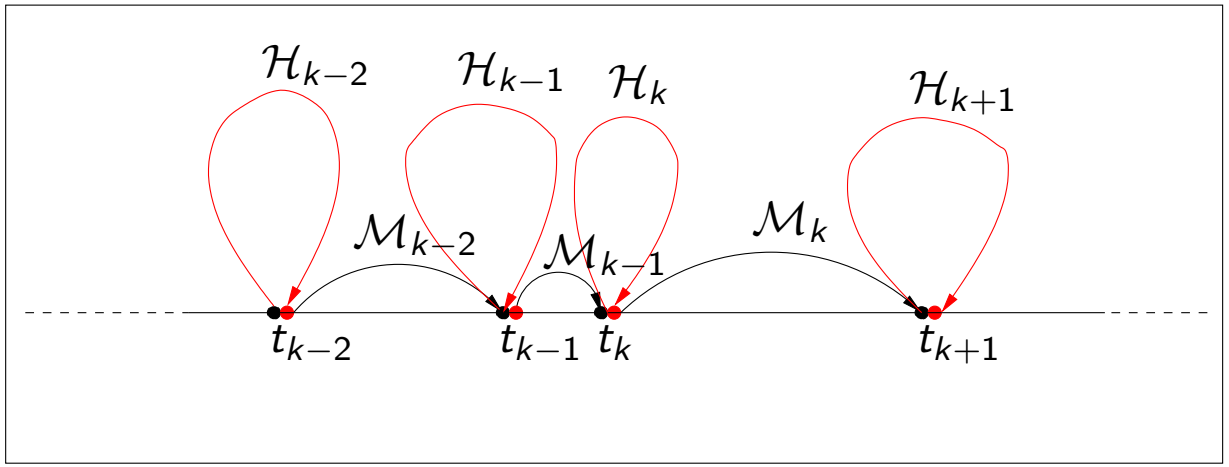

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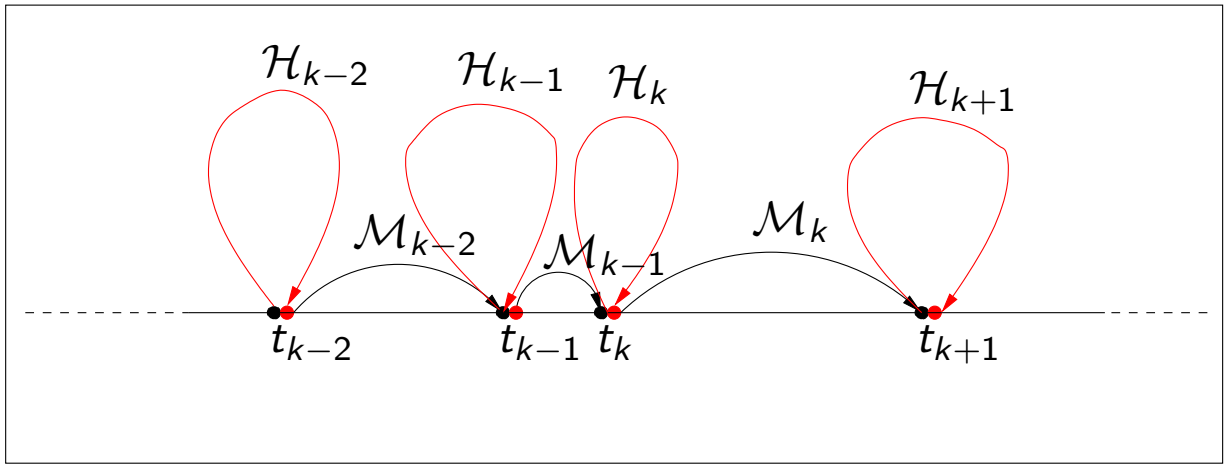

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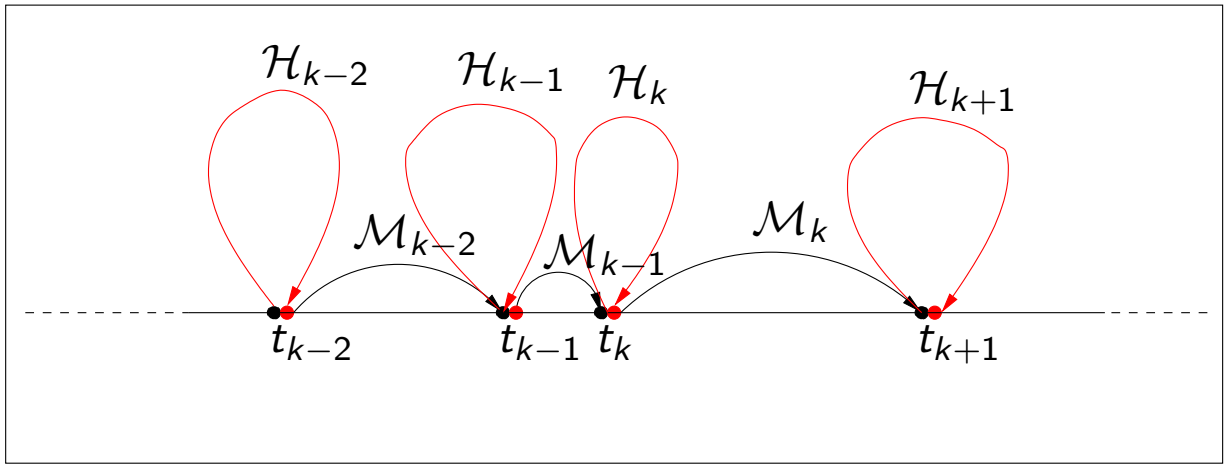

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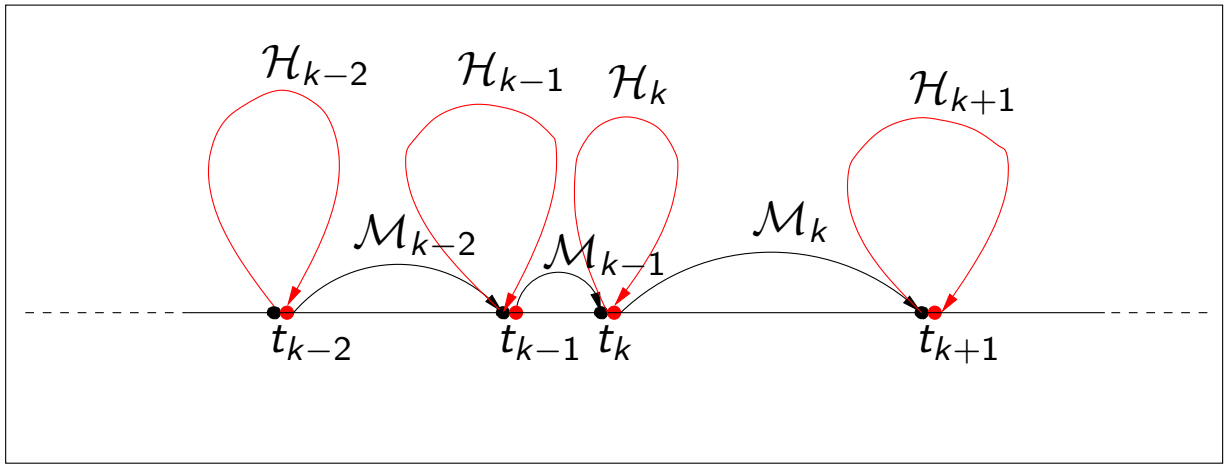

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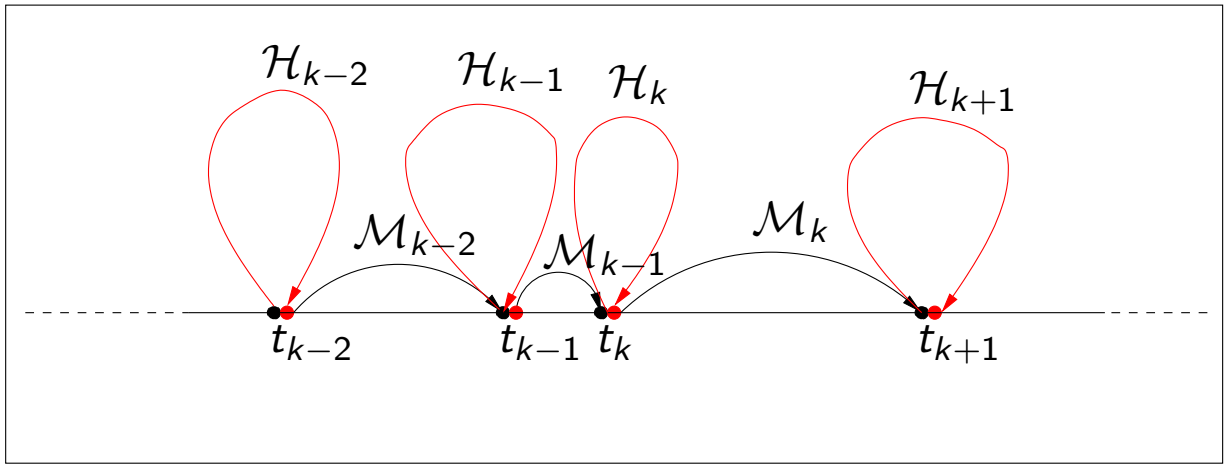

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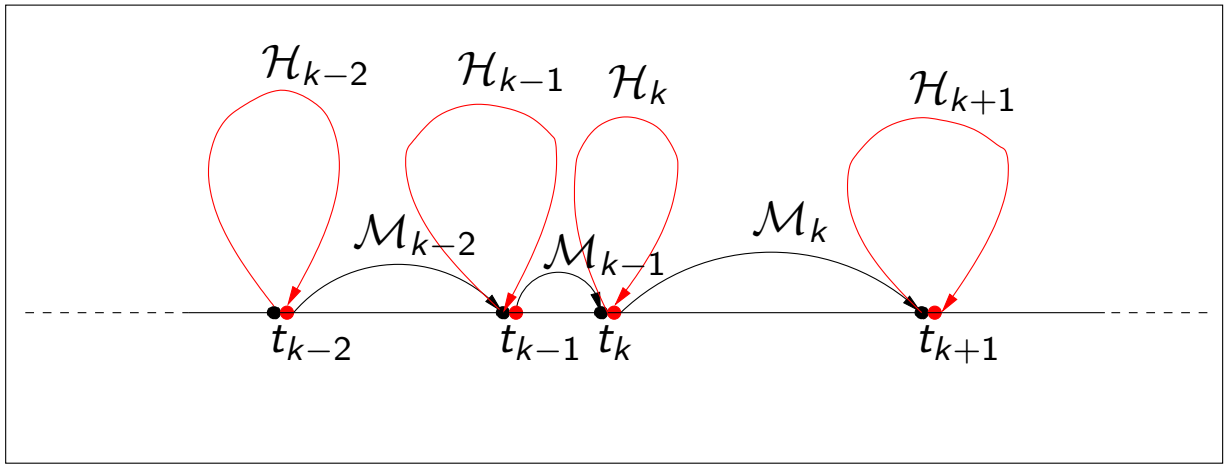

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- The KF provides optimal solution for the current state of the system given past observations. It does not provide optimal solution for all past states.
- The state of the DA system at any stage is given by (i) state estimate $\mathbf{x}$ and (ii) state error covariance estimate $\mathbf{P}$.
- The assimilation cycle breaks into two stages: propagation and analysis.
- The observation and propagation operators can be affine, so that

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- In the Gaussian case the KF solution coincides with the maximal likelihood solution.
- The analysed state estimate results from matching first-order terms, while the analysed covariance estimate is a result of matching second-order terms.
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- The state of the DA system at any stage is given by (i) state estimate $\mathbf{x}$ and (ii) state error covariance estimate $\mathbf{P}$.
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- In the Gaussian case the KF solution coincides with the maximal likelihood solution.
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Extended Kalman filter

- $\mathcal{H}(\mathbf{x}), \mathcal{M}(\mathbf{x})$ are nonlinear
- EKF: uses the estimate

$$\mathbf{M}_i = \nabla_x \mathcal{M}_i(\mathbf{x}_i^a),$$
$$\mathbf{H}_i = \nabla_x \mathcal{H}_i(\mathbf{x}_i^f),$$

- Requires

$$|\nabla_x \mathcal{M}_i(\mathbf{x} + \delta\mathbf{x}) - \nabla_x \mathcal{M}_i(\mathbf{x})| \ll |\nabla_x \mathcal{M}_i(\mathbf{x})|,$$
$$|\nabla_x \mathcal{H}_i(\mathbf{x} + \delta\mathbf{x}) - \nabla_x \mathcal{H}_i(\mathbf{x})| \ll |\nabla_x \mathcal{H}_i(\mathbf{x})|.$$

- Therefore, for EKF to work the state must be "linearly" constrained - that is, constrained to a degree when linearised operators can be applied within the characteristic uncertainty range.

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Kalman smoother

- Let us assume that

$$\min_{\{\mathbf{x}_i\}_{i\neq k+1}} \mathcal{L}_K(\mathbf{x}_1,\ldots,\mathbf{x}_k,\mathbf{x}_{k+1},\ldots,\mathbf{x}_K)$$
$$= (\mathbf{x}_{k+1}-\mathbf{x}^s_{k+1})^{\mathrm{T}}(\mathbf{P}^s_{k+1})^{-1}(\mathbf{x}_{k+1}-\mathbf{x}^s_{k+1}) + \mathrm{Const.}$$

- Then

$$\min_{\{\mathbf{x}_i\}_{i\neq k}} \mathcal{L}_K(\mathbf{x}_1,\ldots,\mathbf{x}_k,\mathbf{x}_{k+1},\ldots,\mathbf{x}_K) = (\mathbf{x}_k-\mathbf{x}^s_k)^{\mathrm{T}}(\mathbf{P}^s_k)^{-1}(\mathbf{x}_k-\mathbf{x}^s_k) + \mathrm{Const},$$

where

$$\mathbf{x}^s_k = \mathbf{x}^a_k + \mathbf{P}^s_k(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\left[\mathbf{x}^s_{k+1} - \mathcal{M}_k(\mathbf{x}^a_k)\right],$$
$$\mathbf{P}^s_k = \left[(\mathbf{P}^a_k)^{-1} + (\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\mathbf{M}_k\right]^{-1}.$$

- Compare this with the KF analysis solution:

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Proof

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\begin{aligned}
&\mathcal{L}_K(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_K) \\
&\quad = \mathcal{L}_{k-1}(\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) + [\mathcal{M}_{k-1}(\mathbf{x}_{k-1}) - \mathbf{x}_k]^{\mathrm{T}} (\mathbf{Q}_{k-1})^{-1} [\mathcal{M}_{k-1}(\mathbf{x}_{k-1}) - \mathbf{x}_k] \\
&\qquad + [\mathbf{y}_k - \mathcal{H}_k(\mathbf{x}_k)]^{\mathrm{T}} (\mathbf{R}_k)^{-1} [\mathbf{y}_k - \mathcal{H}_i(\mathbf{x}_k)] \\
&\qquad + [\mathcal{M}_k(\mathbf{x}_k) - \mathbf{x}_{k+1}]^{\mathrm{T}} (\mathbf{Q}_k)^{-1} [\mathcal{M}_k(\mathbf{x}_k) - \mathbf{x}_{k+1}] + \mathrm{Terms}(\mathbf{x}_{k+1}, \ldots, \mathbf{x}_K).
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\end{aligned}
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Compare (14) with (5) and (6). ■.

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Conclusions

- The KS provides a recursive solution of the least squares mininisation problem in the linear case for the states of the DA system at all times.
- These states can be obtained by recursion back in time, starting from the final state given by the KF.
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Appendix 1: To the KF derivation

- Rearrange the cost function

$$\mathcal{L}(\mathbf{x}_k) \equiv (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\cdot) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

$$\left(\quad \text{use} \quad \mathcal{M}_k(\mathbf{x}_k) = \mathcal{M}_k(\mathbf{x}_k^a) + \mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a) \quad \right)$$

$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Observation:

$$\min_{\mathbf{w}} (\mathbf{w}^{\mathrm{T}} A\mathbf{w} - 2\mathbf{b}^{\mathrm{T}}\mathbf{w}) = -\mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{b}$$

- Apply with

$$\mathbf{A} = (\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k$$

$$\mathbf{b} = (\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]$$

- Obtain:

$$\min_{\mathbf{x}_k} \mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

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$$\left(\quad \text{use} \quad \mathcal{M}_k(\mathbf{x}_k) = \mathcal{M}_k(\mathbf{x}_k^a) + \mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a) \quad \right)$$

$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Observation:

$$\min_{\mathbf{w}}\,(\mathbf{w}^{\mathrm{T}}A\mathbf{w} - 2\mathbf{b}^{\mathrm{T}}\mathbf{w}) = -\mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{b}$$

- Apply with

$$\mathbf{A} = (\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k$$

$$\mathbf{b} = (\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]$$

- Obtain:

$$\min_{\mathbf{x}_k}\,\mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

Appendix 1: To the KF derivation

- Rearrange the cost function

$$\mathcal{L}(\mathbf{x}_k) \equiv (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\cdot) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

$$\left(\quad \text{use} \quad \mathcal{M}_k(\mathbf{x}_k) = \mathcal{M}_k(\mathbf{x}_k^a) + \mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a) \quad \right)$$

$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Observation:

$$\min_{\mathbf{w}}(\mathbf{w}^{\mathrm{T}}A\mathbf{w} - 2\mathbf{b}^{\mathrm{T}}\mathbf{w}) = -\mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{b}$$

- Apply with

$$\mathbf{A} = (\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k$$

$$\mathbf{b} = (\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]$$

- Obtain:

$$\min_{\mathbf{x}_k} \mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

Appendix 1: To the KF derivation

- Rearrange the cost function

$$\mathcal{L}(\mathbf{x}_k) \equiv (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\cdot) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

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$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Observation:

$$\min_{\mathbf{w}} (\mathbf{w}^{\mathrm{T}} A\mathbf{w} - 2\mathbf{b}^{\mathrm{T}}\mathbf{w}) = -\mathbf{b}^{\mathrm{T}}\mathbf{A}\mathbf{b}$$

- Apply with

$$\mathbf{A} = (\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k$$

$$\mathbf{b} = (\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]$$

- Obtain:

$$\min_{\mathbf{x}_k} \mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

Appendix 1: To the KF derivation

- Rearrange the cost function

$$\mathcal{L}(\mathbf{x}_k) \equiv (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\cdot) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

$$\left(\quad \text{use} \quad \mathcal{M}_k(\mathbf{x}_k) = \mathcal{M}_k(\mathbf{x}_k^a) + \mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a) \quad \right)$$

$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Obtain:

$$\min_{\mathbf{x}_k} \mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

$$(\mathbf{A} + \mathbf{XBY})^{-1} = (\mathbf{A})^{-1} - (\mathbf{A})^{-1}\mathbf{X}\left[(\mathbf{B})^{-1} + \mathbf{Y}(\mathbf{A})^{-1}\mathbf{X}\right]^{-1}\mathbf{Y}(\mathbf{A})^{-1}$$

with

$$\mathbf{A} = \mathbf{Q}_k, \quad \mathbf{X} = \mathbf{M}_k, \quad \mathbf{Y} = \mathbf{P}_k^a, \quad \mathbf{B} = \mathbf{P}_k^a$$

- Obtain:

$$\mathbf{P}_{k+1}^f = \mathbf{M}_k\mathbf{P}_k^a(\mathbf{M}_k)^{\mathrm{T}} + \mathbf{Q}_k, \quad \mathbf{x}_{k+1}^f = \mathcal{M}_k(\mathbf{x}_k^a)$$

Appendix 1: To the KF derivation

- Rearrange the cost function

$$\mathcal{L}(\mathbf{x}_k) \equiv (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}(\mathbf{P}_k^a)^{-1}(\cdot) + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

$$\left(\quad \text{use} \quad \mathcal{M}_k(\mathbf{x}_k) = \mathcal{M}_k(\mathbf{x}_k^a) + \mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a) \quad \right)$$

$$= (\mathbf{x}_k - \mathbf{x}_k^a)^{\mathrm{T}}\left[(\mathbf{P}_k^a)^{-1} + (Q_k)^{-1}\mathbf{M}_k\right](\cdot)$$

$$- 2\,[\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k^{-1})\,[\mathbf{M}_k(\mathbf{x}_k - \mathbf{x}_k^a)] + [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}(\mathbf{Q}_k)^{-1}[\cdot]$$

- Obtain:

$$\min_{\mathbf{x}_k} \mathcal{L}(\mathbf{x}_k) = [\mathbf{x}_{k+1} - \mathcal{M}_k(\mathbf{x}_k^a)]^{\mathrm{T}}\left[(\mathbf{Q}_k)^{-1} - (\mathbf{Q}_k)^{-1}\mathbf{M}_k(\mathbf{A}_k)^{-1}(\mathbf{M}_k)^{\mathrm{T}}(\mathbf{Q}_k)^{-1}\right][\cdot]$$

- Apply the Matrix Inversion Lemma

$$(\mathbf{A} + \mathbf{XBY})^{-1} = (\mathbf{A})^{-1} - (\mathbf{A})^{-1}\mathbf{X}\left[(\mathbf{B})^{-1} + \mathbf{Y}(\mathbf{A})^{-1}\mathbf{X}\right]^{-1}\mathbf{Y}(\mathbf{A})^{-1}$$

with

$$\mathbf{A} = \mathbf{Q}_k, \quad \mathbf{X} = \mathbf{M}_k, \quad \mathbf{Y} = \mathbf{P}_k^a, \quad \mathbf{B} = \mathbf{P}_k^a$$

- Obtain:

$$\mathbf{P}_{k+1}^f = \mathbf{M}_k\mathbf{P}_k^a(\mathbf{M}_k)^{\mathrm{T}} + \mathbf{Q}_k, \quad \mathbf{x}_{k+1}^f = \mathcal{M}_k(\mathbf{x}_k^a)$$

Matrix inversion lemma

or Woodbury matrix identity, or Sherman-Morrison-Woodbury formula, or inverse of a small-rank adjustment ...

$$(\mathbf{A} + \mathbf{X}\mathbf{B}\mathbf{Y})^{-1} = (\mathbf{A})^{-1} - (\mathbf{A})^{-1}\mathbf{X}\left[(\mathbf{B})^{-1} + \mathbf{Y}/, (\mathbf{A})^{-1}\mathbf{X}\right]^{-1}\mathbf{Y}\,(\mathbf{A})^{-1}$$