Reflection K-matrices of Temperley-Lieb spin chains and algebraic structures

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Abstract

The general solutions of the reflection equation associated with Temperley-Lieb R-matrices are constructed. Their parametrization is defined and the Hamiltonians of corresponding integrable open spin systems are given.

Introduction

Heisenberg (*XXX*-)spin 1/2 chain (1928) ($su(2)$-invariant),

Anisotropic *XXZ*-spin 1/2 chain ($su_q(2)$-invariant),

Integrable higher spin $s = 1, 3/2, 2, ...$ chains (1980)

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$$tr({}^t b\, b^{-1}) = -(q + \frac{1}{q}). \tag{1.1}$$

$$q\, K^2 + \ c_1\, K + (q + \frac{1}{q})^{-1}(c_1^2 + q\, c_2)\mathrm{I} = 0 \tag{1.2}$$

Hecke and Temperley-Lieb Algebras, $\mathcal{B}_N$, $H_N(q)$, $TL_N(q)$

Both Hecke algebra $H_N(q)$ and TL algebra $TL_N(q)$ are quotients of the group algebra of the braid group $\mathcal{B}_N$ generated by $(N-1)$ generators $\check{R}_j$, $j = 1, 2, \dots, N-1$, their inverses $\check{R}_j^{-1}$ and the relations:

$$\check{R}_j \check{R}_k \check{R}_j = \check{R}_k \check{R}_j \check{R}_k, \text{ for } |j-k| = 1 \quad \text{and} \quad \check{R}_j \check{R}_k = \check{R}_k \check{R}_j, \text{ for } |j-k| > 1. \tag{2.1}$$

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$$X_j X_k X_j - X_j = X_k X_j X_k - X_k, \quad |j-k| = 1. \tag{2.3b}$$

Finally the TL algebra $TL_N(q)$ is obtained as the quotient algebra of the Hecke algebra $H_N(q)$ by the set of equations requiring that each side of (2.3b) be zero. To sum up, $TL_N(q)$ is defined by the generators X_j, $j = 1, 2, \ldots, N-1$ and their relations:

$$\begin{aligned} X_j^2 &= -\nu(q) X_j, \\ X_j X_k X_j &= X_j, \quad |j-k| = 1, \\ X_j X_k &= X_k X_j, \quad |j-k| > 1 \end{aligned} \tag{2.4}$$

with $\nu(q) = q + 1/q$.

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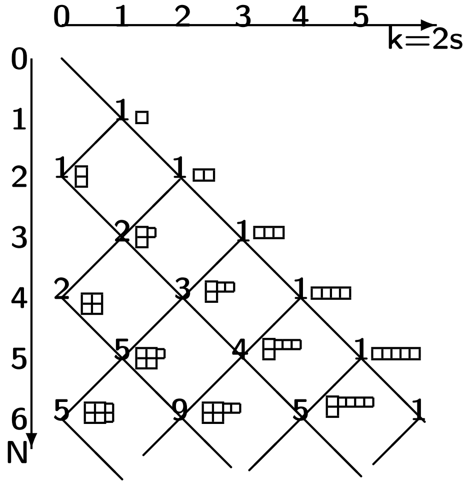
with $\nu(q) = q + 1/q$. The dimension of the Hecke algebra, $N!$, is the same as the dimension of the symmetric group, whereas the dimension of $TL_N(q)$ is equal to the Catalan number $C_N = (2N)!/N!(N+1)!$. Implementation of the TL constraint thus considerably reduces the dimension of the algebra.

Schur - Weyl duality $sl(2)$-case



Tensor category of $sl(2)$ finite dim. reps

$$sl(2), \quad \left\{S^z, X^+, X^-\right\}, \quad s = 0, \frac{1}{2}, 1, \dots;$$

$$\text{IrReps } V_{2s} \quad \dim V_k = k+1 \quad 2s = k \in \mathbb{Z}_{\geq 0}$$

$$V_1 \otimes V_1 = V_0 \oplus V_2;$$

$$V_1 \otimes V_k = V_{k-1} \oplus V_{k+1};$$

$$V_1^{\otimes N} = \bigoplus_{k=0,1}^{N} V_k \otimes \mathbb{C}^{\nu_k};$$

$$V_l \otimes V_k = \bigoplus_{|l-k|}^{l+k} V_m$$

Multiplicity free "C-G"-decomposition.

This "ring of representations" corresponds to some quantum algebra $\mathfrak{b}_n$ if one starts from $V_0 \simeq \mathbb{C}^1, \quad V_1 \simeq \mathbb{C}^n$ (e.g. $\dim V_2 = n^2 - 1$)

$$n p_k = p_{k+1} + p_{k-1}, \quad p_{-1} = 0,\ p_0 = 1 \qquad \dim V_k = p_k(n).$$

$p_k(x)$ — Chebyshev polynomial of the 2-nd kind.

V_k as corepresentations of dual Hopf algebra and FRT-formalism

$$R_{1\,2} T_1 T_2 = T_2 T_1 R_{1\,2}$$

(Semi)ring of irreps: sl(3) case

Highest weight finite dimensional irreducible representations of $sl(3)$ are parametrized by fundamental weights $m\omega_1 + n\omega_2$, $V_{m\,n}$.
According to the Clebsch-Gordan decomposition

$$V_{1,0} \otimes V_{m,n} = V_{m+1,n} \oplus V_{m-1,n+1} \oplus V_{m,n-1}$$

$$V_{01} \otimes V_{m,n} = V_{m,n+1} \oplus V_{m+1,n-1} \oplus V_{m-1,n}$$

$$xp_{m,n} = p_{m+1,n} + p_{m-1,n+1} + p_{m,n-1}$$

$$yp_{m,n} = p_{m,n+1} + p_{m+1,n-1} + p_{m-1,n}$$

$$sl(3):\ x = y = 3, \quad p_{m,n} = \frac{1}{2}(m+1)(n+1)(m+n+2), \quad p_{0,0} = 1,$$

where $p_{m,n}(x,y)$ — generalized Chebyshev polynomials.

Recurrent relations for generalized Chebyshev polynomials

$$xp_{m,0}(x,y) = p_{m+1,0}(x,y) + yp_{m-1,0}(x,y) - p_{m-2,0}(x,y).$$

$$p_{m,0}(x,y) = x^m - (m-1)x^{m-2}y + \frac{(m-3)(m-2)}{2}y^2 - b_m x^{m-6}y^3 + \dots$$

The generating function for polynomials $p_{m,0}(x,y)$ looks as

$$\varphi(t;x,y) = \frac{1}{1 - tx + yt^2 - t^3}.$$

For more general polynomials we have the following recurrent relations

$$xp_{m,n}(x,y) - yp_{m-1,n}(x,y) = p_{m+1,n}(x,y) - p_{m-2,n}(x,y).$$

The matrix realization on $\mathcal{H}$ of the idempotent generator X_j now reads in terms of b:

$$X_j = \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{j-1} \otimes \left(\sum_{\substack{c,d,c',d \\ \in \{1 \ldots n\}}} b_{cd} \bar{b}_{c'd'} E_{cc'} \otimes E_{dd'} \right) \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{N-j-1} \tag{2.5}$$

where we have now denoted by $\mathbb{I}$ the identity matrix in $End(\mathbb{C}^n)$ and we have used the canonical basis of $n \times n$ matrices, $E_{cc'}$ denoting the $n \times n$ matrix with entries $(E_{cc'})_{xx'} = \delta_{cx} \delta_{c'x'}$.
Direct computation shows that the set of relations (2.4) are satisfied and they fix the value of the parameter q up to a duality $q \to 1/q$:

$$-\nu(q) = \mathrm{tr}\, {}^t b \bar{b} = -\left(q + \frac{1}{q} \right). \tag{2.6}$$

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$$R_{12}(u)R_{13}(uw)R_{23}(w) = R_{23}(w)R_{13}(uw)R_{12}(u). \tag{2.11}$$

Classification of the solutions of the constant RE

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This equation reads in term of the b matrix:

$$qb\otimes({}^tK^2\bar{b}) + tr({}^tb^tK\bar{b})b\otimes({}^tK\bar{b}) = q({}^tK^2b)\otimes\bar{b} + tr({}^tb^tK\bar{b})({}^tKb)\otimes\bar{b}. \tag{3.4}$$

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Since matrices b and $\bar{b}$ are invertible

$$\mathbb{I}\otimes(q\ K^2+\ tr({}^tb\,{}^tK\bar{b})\ K) = (q\ K^2+\ tr({}^tb\,{}^tK\bar{b})\ K)\otimes\mathbb{I}. \tag{3.5}$$

This establishes that $q\ K^2+\ tr({}^t\bar{b}Kb)\ K$ is proportional to identity.

$$K^2+\frac{1}{q}tr({}^t\bar{b}Kb)K=k_2\mathbb{I}, \quad k_2=\frac{1}{qtr({}^t\bar{b}b)}((tr{}^t\bar{b}Kb)^2+qtr({}^t\bar{b}K^2b)). \tag{3.6}$$

the complete resolution of the reflection equation for these constant TL R-matrices will be realized in two steps:

1. Parametrize all matrices K with a minimal polynomial of degree 2 (or less).
2. Fix the value of the coefficient of the linear term to its expression in (3.6).

Step 1 is separated into three obvious subcases:
$1a$: Minimal polynomial of degree 1. The matrix K is then proportional to the Identity and automatically solves the reflection equation without further conditions.
$1b$: Minimal polynomial of degree 2 with two distinct roots. The matrix K is then diagonalizable with the same two zeroes as eigenvalues.
$1c$: Minimal polynomial of degree 2 with a double root. The matrix K is then only trigonalizable (i.e. is written with Jordanian cells)

Diagonalizable K-matrices

Diagonalizable $n \times n$ matrix with two distinct eigenvalues denoted λ and μ

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The projector P is then constructed from two sets of data encapsulating all the information on V_μ and V_λ albeit with redundancies:
a: a set of m independent vectors building a basis of V_μ, defining in this way an $n \times m$ rectangular matrix B of maximal rank m. The redundancy in this parametrization correspond to the arbitrariness in the choice of the basis in V_μ , described by the transformation $B \to Bg$ for any g in $Gl(m)$.

b: a set of m independent vectors building a basis of $\bar{V}_\lambda$ defined as the m-dimensional vector space of solutions to the rank $n-m$ homogeneous linear system:

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Requiring that the square $m \times m$ matrix ${}^t AB$ be invertible. P is then built as:

$$ P = B({}^t AB)^{-1} A^t \tag{3.9} $$

as is immediately checked by operating P on B (vectors of V_μ), yielding again B, and C (vectors of V_λ), yielding 0.

Hence, any diagonalizable K-matrix with 2 eigenvalues can be written as:

$$K = \lambda \mathbb{I} + (\mu - \lambda) B ({}^t A B)^{-1} A^t. \tag{3.10}$$

The number of relevant parameters is thus $2(n-m)m+1$.

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We now realize Step 2 by imposing that the value of the coefficient of the linear term in the minimal polynomial, which is identified with the sum of the two zeroes $\lambda + \mu$, be identified to its expression in (3.6), i.e.:

$$\lambda + \mu = -\frac{1}{q} tr(b^{t}\bar{b}K) \tag{3.11}$$

that is:

$$\lambda(1+\frac{1}{q}tr(b^t\bar{b})-\frac{1}{q}tr(b^t\bar{b}B({}^tAB)^{-1}A^t))+\mu(1+\frac{1}{q}tr(b^t\bar{b}B({}^tAB)^{-1}A^t)=0. \tag{3.12}$$

This fixes univocally the ratio $\frac{\lambda}{\mu}$

$$K=\lambda\{\mathbb{I}+\{\frac{-q+1/q}{q+tr(b^t\bar{b}B^tA)}\}B^tA\}; \quad ({}^tAB)=\mathbb{I}. \tag{3.12}$$

Non-diagonalizable K-matrices

This time the m-dimensional image of the cokernel of N yields a rectangular matrix B up to rhs multiplication by g in $Gl(m)$. The $n-m$-dimensional kernel of N, can again be characterized by another $n \times m$ rectangular matrix A. However this time one must impose a complete inclusion condition of the image vectors defining B in the kernel, in other words ${}^tAB = 0$. N is then immediately obtained as $N = B^tA$. Because of the condition ${}^tAB = 0$ the scale of N is not fixed; this scale fixing is here obtained by the implementation of Step 2 to impose:

$$\lambda(q - q^{-1}) = tr(b^t \bar{b} B^t A). \tag{3.13}$$

The components of diagonal global $Gl(m)$ gauge transformation $A \to A(g^{-1})^t \;\; ; \;\; B \to Bg$ are equal to $= 2nm - 2m^2$.

Complete parametrization

Both situations can now be summarized into a single representation:
Proposition
Any solution to the Temperley-Lieb constant reflection equation (3.1) takes the form:

$$K = \lambda \mathbb{I} + B^t A \tag{3.14}$$

where A and B are rectangular $n \times m$ matrices of rank m, $m \leq [\frac{n}{2}]$ defined up to a diagonal $Gl(m)$ gauge transformation g:

$$A \to A(g^{-1})^t \;\; ; \;\; B \to Bg \tag{3.15}$$

Matrices A and B are submitted to the condition:

$$ {}^{t}AB = (\mu - \lambda)\mathbb{I} \tag{3.16} $$

and

$$ -\frac{1}{q}\lambda + q\mu = -\frac{1}{q} tr(b^{t}\bar{b}B^{t}A). \tag{3.16} $$

If $\mu = \lambda$ one recovers the non-diagonalizable case

If $\mu \neq \lambda$ one recovers the diagonalizable case.

Spectral parameter dependent K matrices. Yang-baxterization for the associated K-matrices

the affine Hecke algebra $\hat{H}_N(q)$. It has one more generator K with relations:

$$\check{R}_1 K \check{R}_1 K = K \check{R}_1 K \check{R}_1 \qquad K \check{R}_j = \check{R}_j K, j > 1. \tag{4.1}$$

As in the Hecke case will define a quotient of $\hat{H}_N(q)$. There exists then consistent realizations of the Yang-Baxterized K-matrix by Laurent polynomials $K(u)$ in u, u^{-1}, depending on the coefficients of p_n and K^m, $m = 0, 1, \ldots n-1$. They are solutions to the algebraic reflection equation:

$$\check{R}_1(u/w) K(u) \check{R}_1(uw) K(w) = K(w) \check{R}_1(uw) K(u) \check{R}_1(u/w) \tag{4.2}$$

where $\check{R}_1(u/w)$ is the Yang-Baxterized $\check{R}$ matrix.

Suitable matrix representations of both R and K are considered, respectively in $End(\mathbb{C}^n \otimes \mathbb{C}^n)$ and $End(\mathbb{C}^n)$ becomes the well-known Sklyanin reflection equation

$$\check{R}_{12}(u/w)K_1(u)\check{R}_{12}(uw)K_1(w) = K_1(w)\check{R}_{12}(uw)K_1(u)\check{R}_{12}(u/w). \tag{4.3}$$

$K(u)$ is given by the expression:

$$K(u) = u^2 K - \frac{1}{u^2}K^{-1} + c\mathbb{I} \tag{4.4}$$

with an arbitrary central element c. After a suitable normalization of K, one gets the regularity property of $K(u)$: $K(u)|(u=1) = \mathbb{I}$.
A boundary interaction on theleft and right boundary sites described by matrices $K^-(u)$ and $K^+(u)$ respectively.

Spectral parameter dependent K matrices.// Spin chains

Taking the R-matrix $R_{0j}(u) = \mathcal{P}_{0j}\check{R}_{0j}$ as an L-operator at each site j with auxiliary space labeled by 0 index, one constructs the monodromy matrix:

$$T(u) = L_{0N}(u)L_{0N-1}(u)\cdots L_{01}(u) \tag{4.5}$$

and the two-row monodromy matrix:

$$\mathcal{T}(u) = T(u)K_0^-(u)T^{-1}(1/u), \tag{4.6}$$

where $K_0^-(u)$ is a solution of the reflection equation. The generating functional of integrals of motions (including the Hamiltonian) is:

$$\tau(u) = trK_0^+(u)T(u)K_0^-(u)T^{-1}(1/u). \tag{4.7}$$

where in addition $K_0^+(u)$ is a solution of the suitably defined dual reflection equation.

The spin chain Hamiltonian becomes then proportional to the local expression:

$$H = \Sigma_{k=1}^{N-1} \frac{d}{du} \check{R}_{kk+1}(u=1) + \frac{1}{2}\frac{d}{du} K_1^-(u=1) +$$

$$+ (trK_0^+(1))^{-1} trK_0^+(1) \frac{d}{du} \check{R}_{N0}(u=1) \qquad (4.8)$$

where the contribution of the boundary conditions is explicit.
To illustrate what could be done, when the suitable algebraic tools available, let us finally concentrate on the simplest particular case which indeed can be treated very extensively using general algebraic arguments.

Restricting oneself to the free ends case:

$$K_1^-(u) = \mathbb{I} \qquad K_1^+(u) = M = b^t \bar{b} \tag{4.9}$$

where M is the matrix entering into the crossing-unitarity relation for R.

The spin chain Hamiltonian and the higher conserved quantities then lose altogether their boundary contributions and become elements of the TL algebra, for instance:

$$H = \Sigma_{k=1}^{N-1} \frac{d}{du} \check{R}_{kk+1}(u=1) \in TL_N(q) \tag{4.10}$$

This Hamiltonian is now symmetric w.r.t. the quantum algebra $\mathcal{U}_q(n)$ and can be restricted to the irreducible representation subspaces of $TL_N(q)$ in a decomposition of the phase space:

$$\mathcal{H} = \bigotimes_{1}^{N} \mathbb{C}^n = \bigoplus_{k=0}^{[N/2]} W_k \otimes \mathbb{C}^{\nu(k)} \tag{4.11}$$

where W_k denotes the irrep of $TL_N(q)$ corresponding to the two-row Young diagramme with partition $\{(\lambda_1, \lambda_2)|\lambda_1 + \lambda_2 = N, \lambda_2 = k\}$ and $\nu(k)$ is the multiplicity of this irrep in the decomposition. Hence the spectrum of H consists here of multiplets of subspaces $\{E_k^{(j)}\}, j = 1, 2, \cdots dim W_k$. associated with the irreps W_k, each with multiplicity $\nu(k)$.

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