Hall viscosity of quantum fluids

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Outline:

Definitions for Hall viscosity

1) Adiabatic transport

   BCS paired states

   Magnetic field

   Fractional quantum Hall states

   Quantization arguments

2) Kubo formulas --- stress-stress response

3) Relation with conductivity

Hall viscosity in quantum systems:

Avron, Seiler, and Zograf, 1995	-- general set-up, filled lowest LL
Levay, 1995	-- single ptcle, any LL
Tokatly and Vignale, 2007	-- “any LLL state”
Read, 2008/09	-- paired states and frac qu Hall states, relation with orbital spin and shift
Tokatly and Vignale, 2009	-- rederivation for Laughlin state
Haldane, 2009 (unpublished)	-- general discussion and rederivation
Read and Rezayi, 2010/11	-- general explication, numerics, and quantization arguments
Hughes, Leigh, and Fradkin, 2011	-- Dirac fermions/topological insulator
Dam Son and coworkers, 2011	-- effective field theory approaches, conductivity relation
Bradlyn, Goldstein, and NR, 2012	-- Kubo formulas, general conductivity relation

Viscosity is a fourth rank tensor

Landau and Lifshitz, "Elasticity"

Momentum density $\mathbf{g}(\mathbf{x}) = \frac{1}{2}\sum_j \{\mathbf{p}_j, \delta(\mathbf{x}_j - \mathbf{x})\}$ obeys continuity:

$$\partial g_a/\partial t + \partial_b \tau_{ba} = 0$$

---defines stress tensor $\tau_{ab}(\mathbf{x}, \mathbf{t})$.
In a solid, we have for expectation of stress

$$\tau_{ab} = -\lambda_{abef} u_{ef} - \eta_{abef} \partial u_{ef}/\partial t + \ldots,$$

where the local strain is $u_{ab} = \frac{1}{2}\left(\partial_b u_a + \partial_a u_b\right)$,
u_a is displacement field, λ_{abef} are elastic coefficients (moduli),
and η_{abef} is the viscosity tensor. σ_{ab} and u_{ab} are both symmetric tensors if rotational invariance holds.

In a fluid with local velocity $\mathbf{v}$, elastic part becomes $p\delta_{ab}$ (pressure), we replace

$$\partial u_{ab}/\partial t = \frac{1}{2}\left(\partial_b v_a + \partial_a v_b\right)$$

and also add $m_p \overline{n} v_a v_b$ (momentum flux) to stress tensor.

Rate of loss of mechanical energy, or rate of entropy production, is

$$k_B T \left(\frac{\partial s}{\partial t} + \nabla \cdot \mathbf{j}_s \right) = \eta_{abef} \frac{\partial u_{ab}}{\partial t} \frac{\partial u_{ef}}{\partial t} \geq 0$$

Symmetric and antisymmetric parts:

Avron, Seiler, and Zograf (1995)

$$\eta_{abef} = \eta^{(S)}_{abef} + \eta^{(A)}_{abef},$$

$$\begin{aligned} \eta^{(S)}_{abef} &= +\eta^{(S)}_{efab}, \\ \eta^{(A)}_{abef} &= -\eta^{(A)}_{efab}. \end{aligned}$$

---at zero frequency, only symmetric part gives dissipation; if rotation invariant, it reduces to usual *bulk* and *shear* viscosities, ζ and $\eta^{\rm sh}$

---antisymmetric part vanishes if time reversal is a symmetry; if rotation invariant, it reduces to one number η^H in two dimensions (odd under reflections), none in higher dimensions: $\eta_{abef} = \eta^H (\delta_{be}\epsilon_{af} - \delta_{af}\epsilon_{eb})$

Hall viscosity η^H is analog of Hall conductivity

Avron *et al*.: framework of calculating $\eta^{(A)}_{abef}$ in a quantum fluid by adiabatic transport (Berry phase), varying aspect ratio of periodic boundary conditions.

N.R. (2009): extended to gapped paired superfluids and fractional quantum Hall fluids, discovered general form

$$\boxed{\eta^H = \tfrac{1}{2}\bar{n}\bar{s}\hbar,}$$

where $\bar{n}$ is number density, and $\bar{s}$ is (minus) mean “orbital spin” per particle, e.g. $\bar{s} = 1/2$ for p-ip superfluid (and also for filled LLL in Avron *et al*.).

Further, if the ground state for N particles on a sphere requires N_ϕ quanta of magnetic flux, and

$$N_\phi = \nu^{-1}N - \mathcal{S},$$

then the “shift” is

$$\boxed{\mathcal{S} = 2\bar{s}.}$$

Wen and Zee (1992)

Quantized if translation and rotation invariance are preserved.
Characteristic of a topological phase.

N.R. and Rezayi (2011)

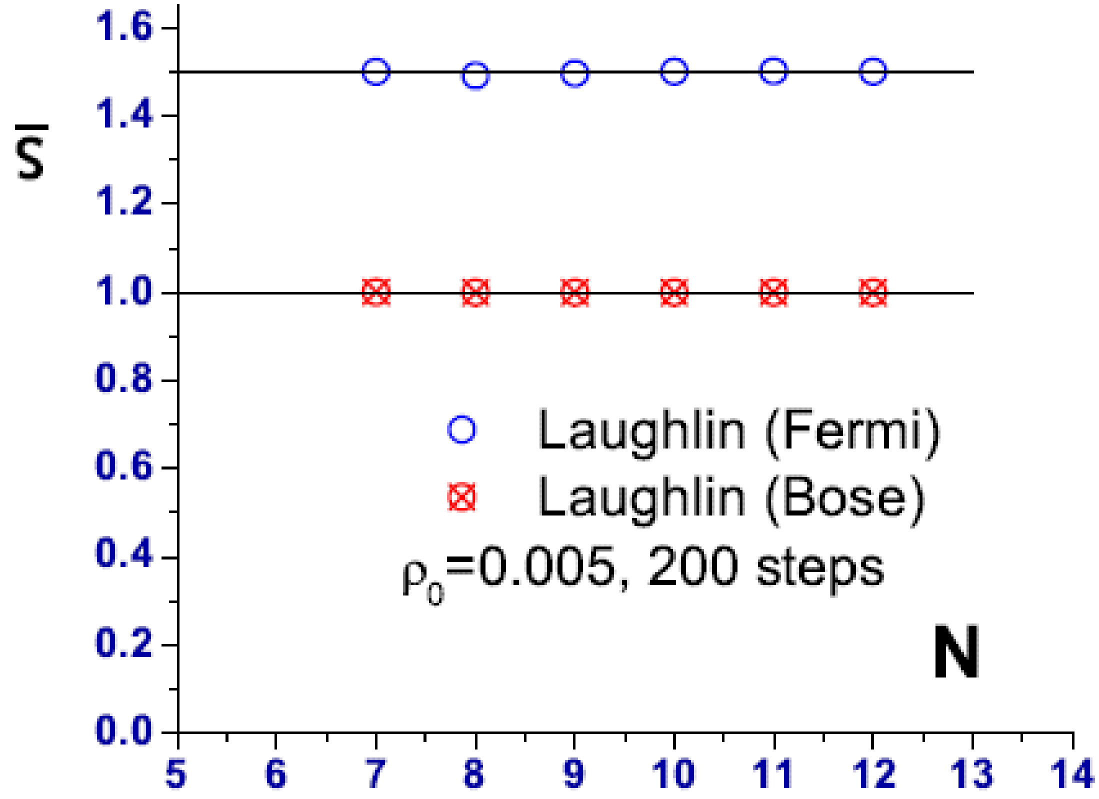


FIG. 1: (color online) $\bar{s}$ of Laughlin states for various sizes, showing rapid convergence with size. Both boson ($\nu = 1/2$) and fermion ($\nu = 1/3$) cases are shown. $\tau = e^{i\pi/3}$ at the center of the circular path, corresponding to hexagonal geometry. The data for each case lie very close to the horizontal line which is the corresponding expected result.

Ways to calculate viscosity in quantum fluids

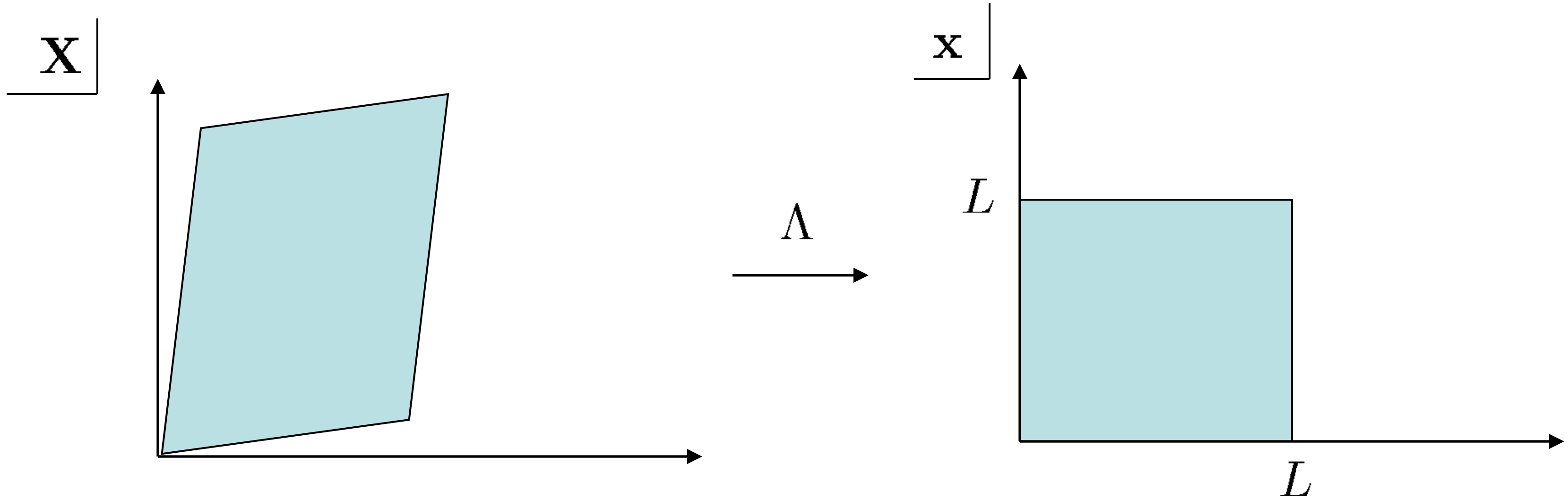
---adiabatic transport: antisymmetric $\omega = 0$ part in gapped systems only
---first part

---Kubo formula for stress-stress response: any kind of system or frequency
---second part

---momentum-momentum response: relation with conductivity
---third part

Strain is geometric

Avron, Seiler, and Zograf (1995)



Relate $\mathbf{X} = \Lambda^T \mathbf{x}$.
Then metric in two systems is $ds^2 = g_{ab} dx_a dx_b$ or usual $ds^2 = d\mathbf{X}^2$

$$G = I \qquad\qquad g = \Lambda \Lambda^T$$

In terms of $\mathbf{x}$, aspect ratio and boundary conditions are fixed. Area is L^2.
Λ or g describe strain: $\partial u_{ef}/\partial t = \frac{1}{2} \partial g_{ef}/\partial t$
Group of all invertible matrices Λ is $\mathrm{GL}(d, \mathbf{R})$.

Stress is variation with respect to metric

In $\mathbf{x}$ coordinates, stress $\tau_{ab} = -2\frac{\delta H}{\delta g_{ab}}$ (analog of electric current $J_a = \frac{\delta H}{\delta \mathcal{A}_a}$)

Time-varying $\dot{g}_{ab}$ is "gravitational field" ($\mathbf{E} = -\dot{\mathcal{A}}$ is electric field)

So

$$\tau_{ab} = -\tfrac{1}{2}\eta_{abef}\frac{\partial g_{ef}}{\partial t}$$

and we examine **adiabatic response** to slowly-varying g
(analog of slowly-varying $\mathbf{A}$ or boundary condition).

Use $L \times L$ square in $\mathbf{x}$ space.

Avron, Seiler, and Zograf (1995)
(w. periodic boundary conditions)

Adiabatic response and Berry phase

Avron and Seiler (1985)

Suppose Hamiltonian $H(\lambda)$ depends on parameters $\lambda = \{\lambda_\mu\} \quad (\mu = 1, \ldots, n)$

and that $\widehat{I}_\mu(\lambda) = -\dfrac{\partial H}{\partial \lambda_\mu}$ is some "current" operator.

Also $H(\lambda)|\varphi(\lambda)\rangle = 0$ for each value of λ (ignore "persistent currents").

Then as $\dot{\lambda} \to 0$, using quantum adiabatic theorem,

$$I_\mu(\lambda) = \langle \widehat{I}_\mu(\lambda) \rangle = \sum_\nu F_{\mu\nu}(\lambda) \dot{\lambda}_\nu$$

where $A_\mu = i\langle \varphi | \partial_\mu \varphi \rangle$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are Berry or adiabatic "connection" and "curvature". Or $\oint_C F$ is a Berry phase.

For us, $\mu = (a, b)$ and $\Lambda = e^\lambda$ as matrices, then

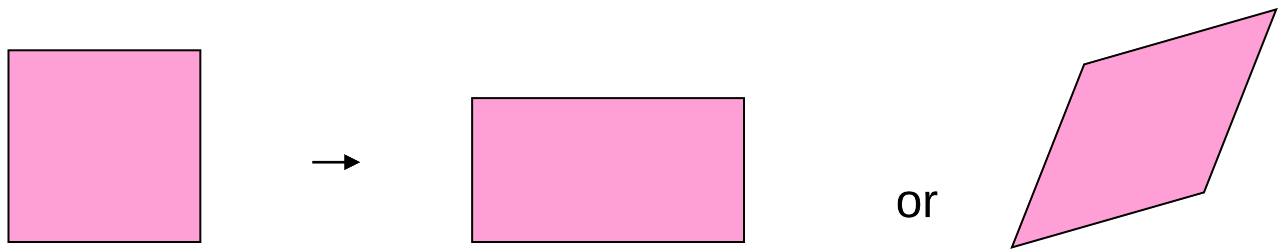
$$\boxed{\eta_{abcd} = -\frac{1}{L^d} g^{be} g^{df} F_{ae,cf}}$$

Avron, Seiler, and Zograf (1995)

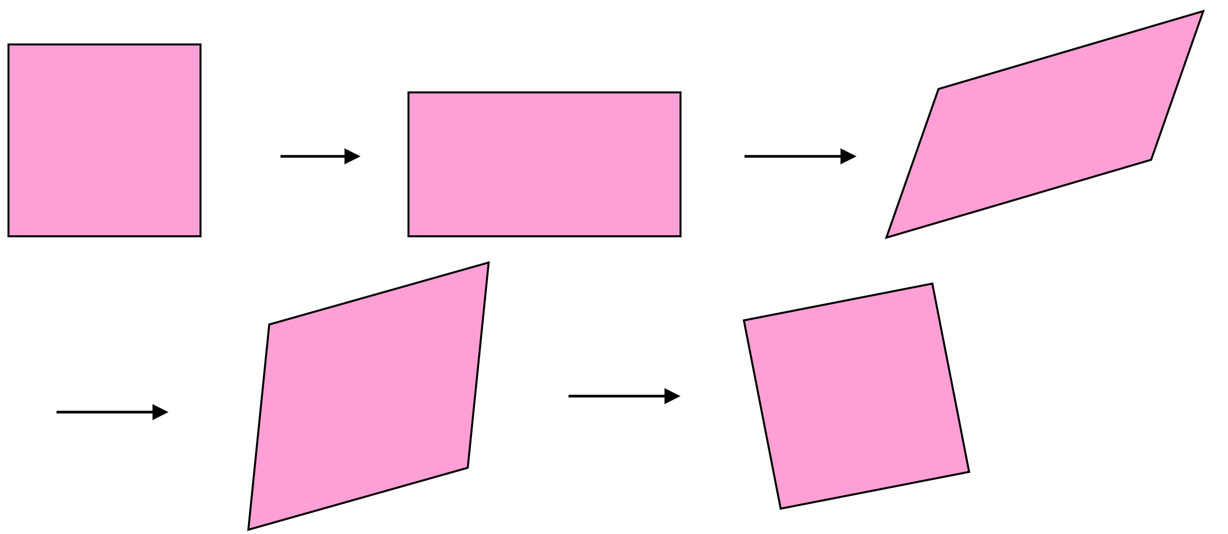
(Analog of "Chern number" approach to quantized Hall conductivity.)

Geometry of shear:

Two independent area-preserving shears in two dimensions:



These moves don't commute: if undo in reverse order, we get a net rotation:

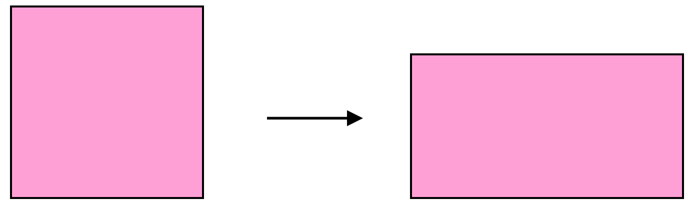


The transformations are described by $\mathrm{SL}(2,\mathbf{R})$ transformations of coordinates,

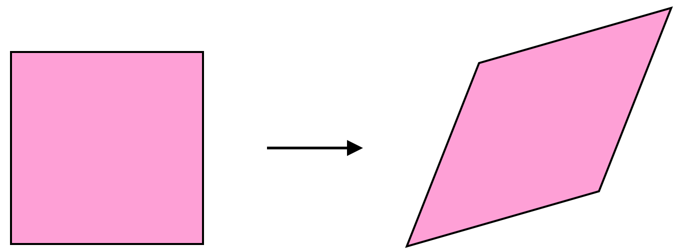
$$\mathbf{x} \to A\mathbf{x}, \qquad \det A = 1$$

Pure shears are symmetric matrices, e.g.:

$$\begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix} :$$



$$\begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix} :$$



and

$$\begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix} \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -2\varepsilon\varepsilon' \\ 2\varepsilon\varepsilon' & 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2, \varepsilon'^2).$$

is a small rotation. If the state has angular momentum (“spin”), then it picks up a phase related to the spin. **This is the Berry phase we will calculate.**

Single-particle toy model

N.R. and Rezayi (2011)

Consider Hamiltonian in infinite two dimensional space

$$H_\Lambda = -\frac{1}{2m_p}\nabla^2_{\mathbf{X}} + U(\mathbf{X})$$

and non-degenerate eigenstate $f(\mathbf{X})$. Rotational invariance of H_Λ (in $\mathbf{X}$) implies that $f(\mathbf{X}) = |f(\mathbf{X})|e^{-is\phi}$.

Again view in terms of $\mathbf{X} = \Lambda^T \mathbf{x}$ $(\det \Lambda = 1)$, states $\varphi_\Lambda(\mathbf{x}) = f(\Lambda^T \mathbf{x})$.
Now

$$|\varphi_\Lambda\rangle = e^{-i\mathrm{tr}\,\lambda^T J}|\varphi_I\rangle,$$ where in coord rep $J_{ab} = \frac{1}{2}i\{x_a, \frac{\partial}{\partial x_b}\}$

are generators of GL(2,**R**).
Adiabatic curvature:

$$\begin{aligned} F_{ab,cd}(0) &= i\langle\varphi_I|[J_{ab}, J_{cd}]|\varphi_I\rangle \\ &= \delta_{ad}\langle J_{cb}\rangle - \delta_{bc}\langle J_{ad}\rangle \\ &= \tfrac{1}{2}s(\delta_{ad}\epsilon_{cb} - \delta_{bc}\epsilon_{ad}) \end{aligned}$$

$\leftarrow$ Angular momentum!

(at any Λ, in terms of $\mathbf{X}$ coords).
---commutation of two pure shears gives rotation.
Similar to Berry phase when dragging orientation of a quantum spin—group SU(2)

BCS paired states

N.R. (2009)
N.R. and Rezayi (2011)

Think of $\mathbf{X}$ as relative coordinate in a pair of particles, bound by $U(\mathbf{X})$, zero center of mass momentum.

Generalize further to many such pairs (BCS wavefunction, ℓ-wave pairs, spinless). Use periodic boundary condition: unfortunately breaks the invariance properties.

But if pairs small compared with system size, this effect drops out.

Find as $N \to \infty$, with $\bar{n} = N/L^2$ fixed,

$$\eta_{abef} = \eta^H(\delta_{be}\epsilon_{af} - \delta_{af}\epsilon_{eb}), \text{ where } \eta^H = \tfrac{1}{2}\bar{n}\bar{s}\hbar, \ \bar{s} = -\ell/2$$

(Independent of shape, as should be for a fluid.)

Alternative derivation: $|\varphi\rangle = \prod_{\mathbf{k}}{}' (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$, depends on Λ through discrete $\mathbf{k}$ in finite size.
Calculate adiabatic curvature in thermo limit, result is same.

Note we see the full angular momentum ℓ of every pair, even at weak coupling.
C.f. resolution of “intrinsic angular momentum” controversy.

Magnetic field case --- non-interacting particles

Avron, Seiler, and Zograf (1995)
Levay (1995)
N.R. and Rezayi (2011)

Several ways to do it.

Hamiltonian, one particle in infinite plane:

$$H_\Lambda = -\frac{1}{2m_p}\sum_{ab} g^{ab} D_a D_b, \qquad \pi_a = -iD_a = -i(\partial/\partial x_a - i\mathcal{A}_a)$$

Define

$$b = -i\frac{1}{\sqrt{2}}(D_1 + iD_2), \quad b^\dagger = -i\frac{1}{\sqrt{2}}(D_1 - iD_2), \quad \mathcal{N} = b^\dagger b \quad (B=1)$$

Then for $\Lambda = I$,

$$H = \hbar\omega_c(\mathcal{N} + \frac{1}{2})$$

← Angular momentum of cyclotron motion!

Each Landau level $\mathcal{N} = 0, 1, 2, \ldots$ has "extensive" degeneracy; must transport the subspace, not single states.

Find by similar calculation $\quad F_{ab,cd}(0) = \frac{1}{2}(\mathcal{N} + 1/2)(\delta_{ad}\epsilon_{cb} - \delta_{bc}\epsilon_{ad})$

(times identity matrix in degeneracy space)

---angular momentum ("spin") is that of the cyclotron motion only

Non-interacting gas:

$$\eta^H = \tfrac{1}{2}\bar{n}\overline{(\mathcal{N}+1/2)}\hbar = \frac{\langle H \rangle}{2\omega_c L^2}$$

--- e.g. for filled LLL

At high T, we recover classical plasma result

$$\eta^H = \bar{n}\frac{k_B T}{2\omega_c}$$

Lifshitz and Pitaevskii, “Physical Kinetics”

Fractional quantum Hall states

NR and Rezayi (2011)

A droplet of QH fluid in plane for “special” (soluble) Hamiltonians has infinite multiplicity of degenerate edge states.

For transport of a degenerate subspace, the curvature evaluated in ground state is

$$\begin{aligned} F_{ab,cd}(0) &= i\langle\varphi_I|[J_{ab},J_{cd}]|\varphi_I\rangle \\ &\quad - i\langle\varphi_I|J_{ab}P_0J_{cd}|\varphi_I\rangle + i\langle\varphi_I|J_{cd}P_0J_{ab}|\varphi_I\rangle \end{aligned}$$

where P_0 is projector onto degenerate subspace.

First term is related to *total* angular momentum $\frac{1}{2}(\nu^{-1}N^2 - N\mathcal{S})$.

With some effort, for “nice” trial wavefunctions, the last two terms give $-\frac{1}{2}\nu^{-1}N^2 + \mathcal{O}(N^{1/2})$.

Remaining angular momentum is $-\bar{s} = -\frac{1}{2}\mathcal{S}$ per particle, and Hall viscosity is therefore

$$\eta^H = \frac{1}{2}\overline{ns}\hbar$$

as claimed.

The “shift”: For QH states and for paired states on a sphere, in ground state need N_ϕ magnetic flux quanta,

$$N_\phi = \nu^{-1} N - \mathcal{S}.$$

In “conformal block” states, $\mathcal{S} = 2(\frac{1}{2}\nu^{-1} + h_\psi)$, and $\mathcal{S} = 2\overline{s}$ holds even more generally (e.g. under particle-hole conjugation).

E.g. $\overline{s} = \frac{1}{2}\nu^{-1} + 1/2$ for Moore-Read state

p-ip paired state!

If $\nu = P/Q$ and P, Q have no common factors, then $P\mathcal{S}$ is an integer. Assuming $\mathcal{S} = 2\overline{s}$, it follows that $2P\overline{s}$ is integer, which was not obvious initially.

($P = 1$ for BCS paired states, in which $\nu = \infty$; $2\overline{s} = -\ell$ is an integer.)

Quantization/robustness arguments

N.R. and Rezayi (2011)

No “Chern number” arguments here.

Suppose we have a rotationally-invariant perturbation of a rotationally-invariant Hamiltonian. For sufficiently small pert, gap is still non-zero. Work at fixed N.

Return to $\lambda_1,\ \lambda_2$ for pure shear, $\lambda_3,\ \lambda_4,\ldots$ coefficients of perturbations. Consider adiabatic transport in **all** these. $\widehat{I}_\mu(\lambda)$ for $\mu = 1, 2$ is the (traceless) stress, and $\langle\widehat{I}_\mu(\lambda)\rangle$ vanishes by rotational invariance as system size $L^2 \to \infty$, even for slowly varying $\lambda_3,\ \lambda_4,\ldots$. But

$$I_\mu(\lambda) = \langle\widehat{I}_\mu(\lambda)\rangle = \sum_\nu F_{\mu\nu}(\lambda)\dot{\lambda}_\nu$$

so $F_{\mu\nu}/L^2 \to 0$ for $\mu = 1, 2,\ \nu = 3, 4, \ldots$. But

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0,$$

so $\partial_3 F_{12}/L^2 \to 0$, which says that $N\bar{s}$ is unchanged by the perturbation.

Translation and rotation invariance (no disorder) appears to be essential here.

Kubo formulas --- B=0

Bradlyn, Goldstein, NR (2012)

In $\mathbf{X}$ variables (but write as $\mathbf{x}$), a time-dependent strain yields modified Hamiltonian

$$H = H_0 \to H = H_0 + H_1$$

$$H_1 = -\frac{\partial \lambda_{ab}}{\partial t} J_{ab}$$

and again $J_{ab} = -\frac{1}{2}\sum_j \{x_{j,a}, p_{j,b}\}$ is the strain generator $(j = 1, \ldots, N)$.

Moreover, from continuity for momentum, $\mathbf{g}(\mathbf{x}) = \frac{1}{2}\sum_j \{\mathbf{p}_j, \delta(\mathbf{x}_j - \mathbf{x})\}$

$$\partial g_b/\partial t + \partial_a \tau_{ab} = 0$$

Fourier transform on $\mathbf{x}$ and expand in $\mathbf{q}$ to first order, find

$$\boxed{T_{ab} = -\partial J_{ab}/\partial t = -i[H_0, J_{ab}]} \qquad \left(T_{ab} = \int d^2x\, \tau_{ab}(\mathbf{x}) \right)$$

Now for viscosity, we want response of stress T_{ab} to applied rate of strain $\partial \lambda_{ab}(\mathbf{q} = 0)/\partial t$, as frequency goes to zero.

We should extract stress response to static strain, which gives elastic moduli, (not a pole in bulk viscosity at zero frequency).

For response: $\langle T_{ab}(t)\rangle - \langle T_{ab}(t)\rangle_0 = -\int_{-\infty}^{\infty} dt'\, X_{abcd}(t-t')\frac{\partial \lambda_{cd}}{\partial t}(t')$

Stress-strain form:

$$X_{abcd}(\omega) = -\lim_{\varepsilon\to 0} i\int_0^{\infty} dt\, e^{i\omega^+ t}\langle [T_{ab}(t), J_{cd}(0)]\rangle_0$$

$$(\omega^+ = \omega + i\varepsilon)$$

Stress-stress form:

$$X_{abcd}(\omega) = \lim_{\varepsilon\to 0}\frac{1}{\omega^+}\left(\langle [T_{ab}(0), J_{cd}(0)]\rangle_0 + \int_0^{\infty} dt\, e^{i\omega^+ t}\langle [T_{ab}(t), T_{cd}(0)]\rangle_0\right)$$

"contact term" as expected!

Strain-strain form:

$$X_{abcd}(\omega) = \lim_{\varepsilon\to 0}\left(-i\langle [J_{ab}(0), J_{cd}(0)]\rangle_0 + \omega^+\int_0^{\infty} dt\, e^{i\omega^+ t}\langle [J_{ab}(t), J_{cd}(0)]\rangle_0\right)$$

---reduces to adiabatic approach as freq -> 0

To get viscosity, we must extract the change in the expectation $\langle T_{ab}\rangle = PL^d\delta_{ab}$
so finally $[\kappa^{-1} = -L^d(\partial P/\partial L^d)_N]$

$$\eta_{abcd}(\omega) = X_{abcd}(\omega)/L^d - \frac{i}{\omega^+}(\kappa^{-1} - P)\delta_{ab}\delta_{cd}$$

Kubo formulas –- B > 0 (two dimensions)

Bradlyn, Goldstein, NR (2012)

Now
$$\mathbf{g}(\mathbf{x}) = \tfrac{1}{2}\sum_j \{\boldsymbol{\pi}, \delta(\mathbf{x}_j - \mathbf{x})\}$$

$$\partial g_b/\partial t + \partial_a \tau_{ab} = \frac{B}{m_p}\epsilon_{bc} g_c$$ (Lorentz force)

For
$$J_{ab} = -\tfrac{1}{2}\sum_j \{x_{j,a}, \pi_{j,b}\} + \tfrac{1}{2}B\epsilon_{bc}\sum_j x_{j,a}x_{j,c} + \tfrac{1}{2}\delta_{ab}\{B, \mathcal{P}\}$$

(symmetric gauge), which obey gl(2,**R**) commutation relations, we recover

$$T_{ab} = -\partial J_{ab}/\partial t = -i[H_0, J_{ab}]$$

and three forms of response function as before.

Example: non-interacting electrons $$T_{ab} = \frac{1}{2m_p}\sum_j \{\pi_{j,a}, \pi_{j,b}\}$$

$$\eta_{abcd}(\omega) = \frac{\langle H_0 \rangle}{L^2[(\omega^+)^2 - (2\omega_c)^2]}[i\omega^+(\delta_{ad}\delta_{bc} - \epsilon_{ad}\epsilon_{bc}) - 2\omega_c(\delta_{bc}\epsilon_{ad} - \delta_{ad}\epsilon_{cb})]$$

---Hall and (at non-zero frequency) shear viscosity; no bulk viscosity

Relation with conductivity

In Galilean-invariant systems, number current density is $\mathbf{j} = \mathbf{g}/m_p$.

Use $-\frac{\partial g_b}{\partial t} + \frac{B}{m_p}\epsilon_{bc} g_c = \partial_a \tau_{ab}$ to relate: $\bar{\eta}_{\mu\nu\alpha\beta}(\omega) = \frac{1}{2}[\eta_{\mu\nu\alpha\beta}(\omega) + \eta_{\alpha\nu\mu\beta}(\omega)]$

$$\bar{\eta}_{\mu\nu\alpha\beta}(\omega) = \frac{1}{2} m^2 \left(\omega\delta_{\nu\lambda} - i\omega_c \epsilon_{\nu\lambda}\right) \left.\frac{\partial^2 \sigma_{\lambda\rho}(\mathbf{q},\omega)}{\partial q_\mu \partial q_\alpha}\right|_{\mathbf{q}=\mathbf{0}} \left(\omega\delta_{\rho\beta} - i\omega_c \epsilon_{\rho\beta}\right)$$
$$- \frac{i\kappa_{\text{int}}^{-1}}{2\omega^+} \left(\delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}\right)$$

For B=0, relation is well-known (e.g. Taylor and Randeria 2010) usually without Hall viscosity.
For B>0, we obtain (σ^H is Hall conductivity)

$$\frac{1}{2} B^2 \frac{\partial^2}{\partial q_x^2} \sigma^H(\mathbf{q},\omega=0)\big|_{\mathbf{q}=\mathbf{0}} = \eta^H(\omega=0) - \frac{\kappa_{\text{int}}^{-1}}{\omega_c} + \frac{2i}{\omega_c} \lim_{\omega\to 0} \omega \eta^{\text{sh}}(\omega)$$

which generalizes the result of Hoyos and Son (2012).
Maybe this can be measured?

Use as numerical diagnostic tool

N.R. and Rezayi (2011)

In QH numerics, usually determine $\mathcal{S}$ for a state at given ν by looking for best $\mathbf{L} = 0$ ground state on sphere. Prey to “aliasing”.

Much better to numerically measure a parameter in periodic b.c. geometry ---unbiased, no aliasing.

Conclusion

- Hall viscosity: a non-dissipative transport coefficient

$$\eta^H = \frac{1}{2}\bar{n}\bar{s}\hbar,$$

- Orbital spin per particle, $\overline{s}$, is a true emergent property --- quantized

- Use as numerical diagnostic tool

- Experimental detection: quantum Hall systems, superfluids, cold atoms…?