Adiabatic continuation of Fractional Chern Insulators to Fractional Quantum Hall States

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August 16, 2012, Nordita, Stockholm

TOPOLOGICAL STATES OF MATTER: INSULATORS, SCs, AND QH LIQUIDS

TRINITY HALL CAMBRIDGE

The Leverhulme Trust

Cavendish Laboratory

UNIVERSITY OF CAMBRIDGE

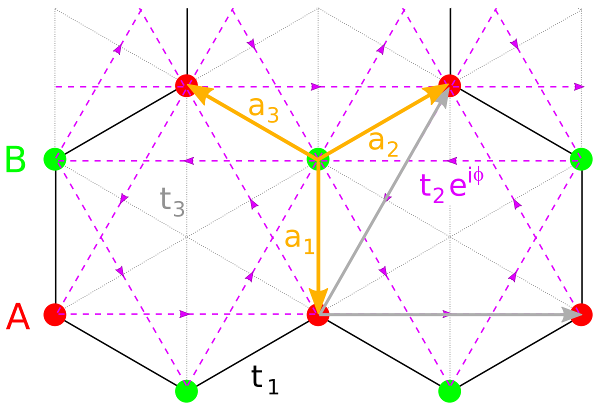
Outline

- Introduction: Strongly correlated phases in topological flat bands.

- Searching wavefunctions for Fractional Chern Insulators by mapping single particle wavefunctions

- Numerical study of the relationship between Fractional Chern Insulators and the Fractional Quantum Hall Effect

Topological bands in two-dimensions

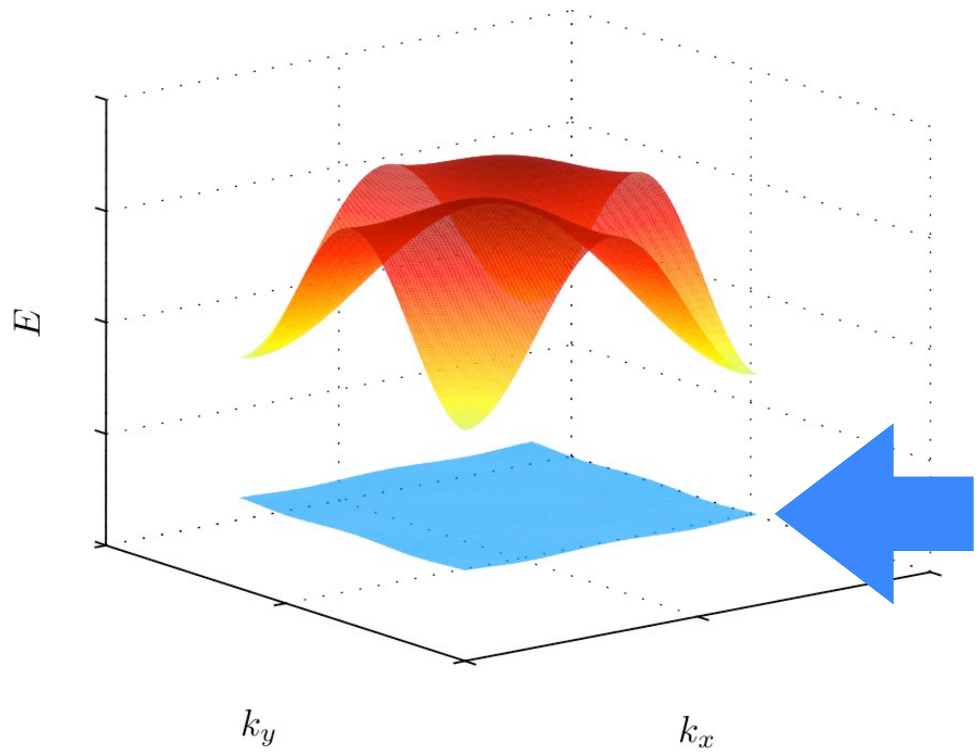


- diagonalize Hamiltonian by Fourier transform

$$\mathcal{H} = \sum_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\alpha} h_{\alpha\beta}(\mathbf{k}) \hat{a}_{\mathbf{k},\beta}$$

$$\mathcal{H}|n,\mathbf{k}\rangle = \epsilon_n(\mathbf{k})|n,\mathbf{k}\rangle$$

- study Berry curvature in n^{th} band:



Berry connection: $\boldsymbol{\mathcal{A}}(n,\mathbf{k}) = -i\langle n,\mathbf{k}|\nabla_{\mathbf{k}}|n,\mathbf{k}\rangle$

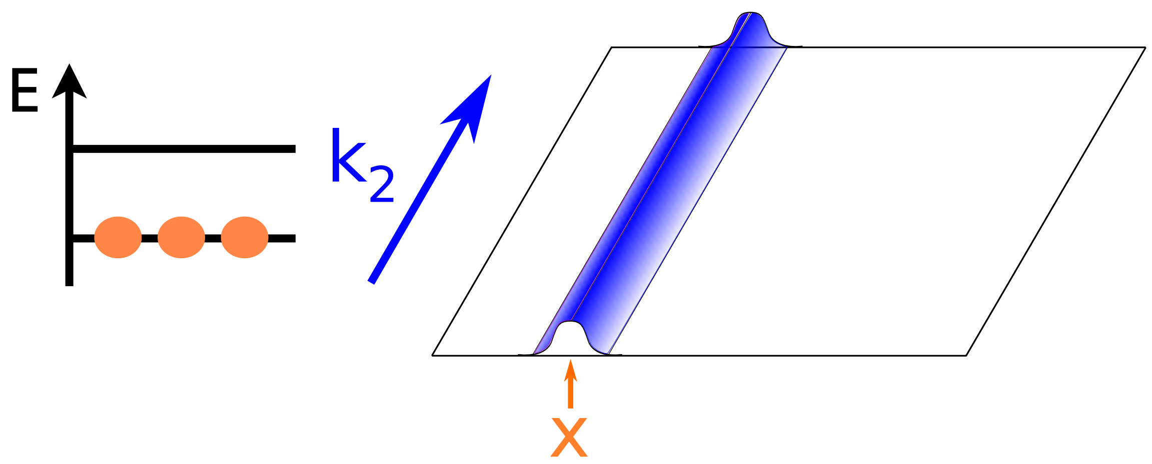
Berry curvature: $\mathcal{B}(k) = \nabla_{\mathbf{k}} \wedge \boldsymbol{\mathcal{A}}(k)$

Chern number: $C = \frac{1}{2\pi}\int_{BZ} d^2\mathbf{k}\, \mathcal{B}(\mathbf{k})$

C integer (focus on C=1 in this talk!)

FQHE and strong correlations in flat bands

Fractional Quantum Hall Effect



- Perfectly degenerate Landau level (C=1)

$$\mathcal{H} = \sum_{i<j} V(r_i - r_j)$$

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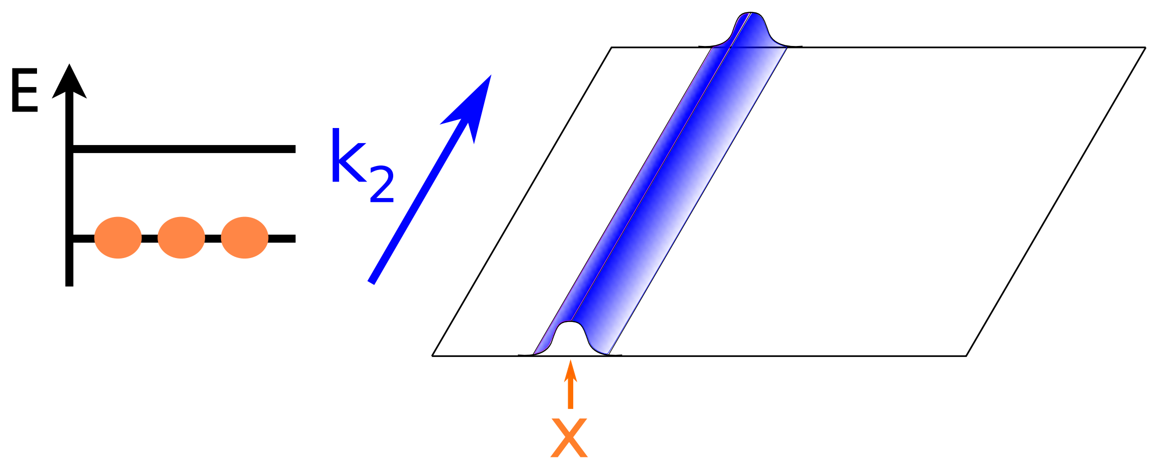

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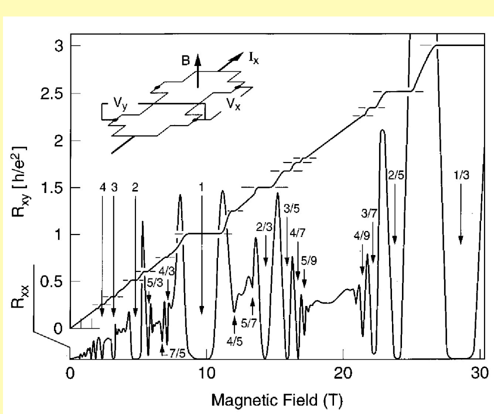
- Variety of gapped quantum liquids:

FQH states



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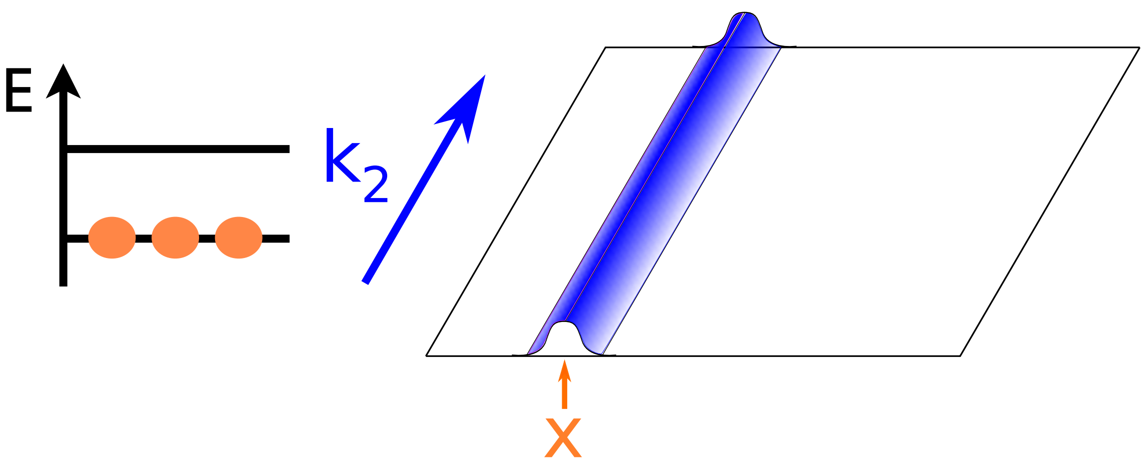

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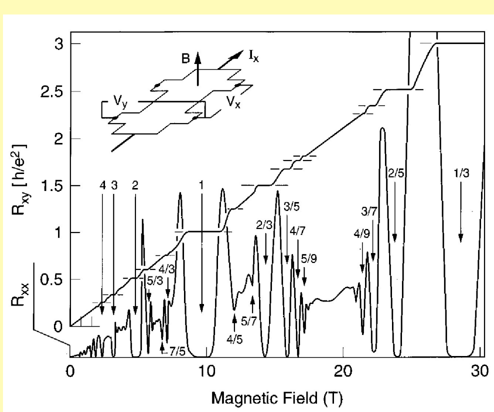
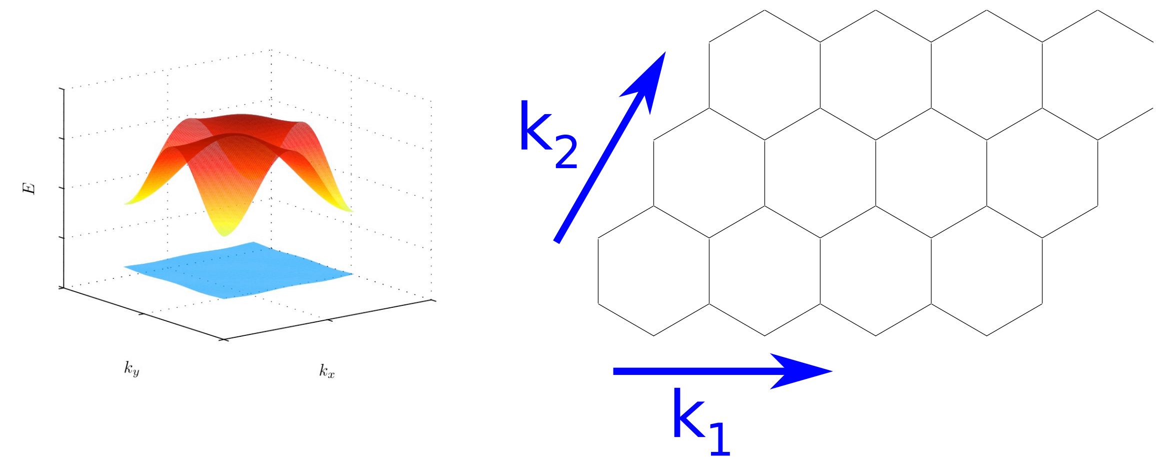


Fractional Chern Insulator



- Topological Flat band (C=1) with small dispersion

$$\mathcal{H} = \sum_{i<j} V(r_i - r_j) + \sum_{k} \epsilon_0(k)\hat{n}_k$$

FQHE and strong correlations in flat bands

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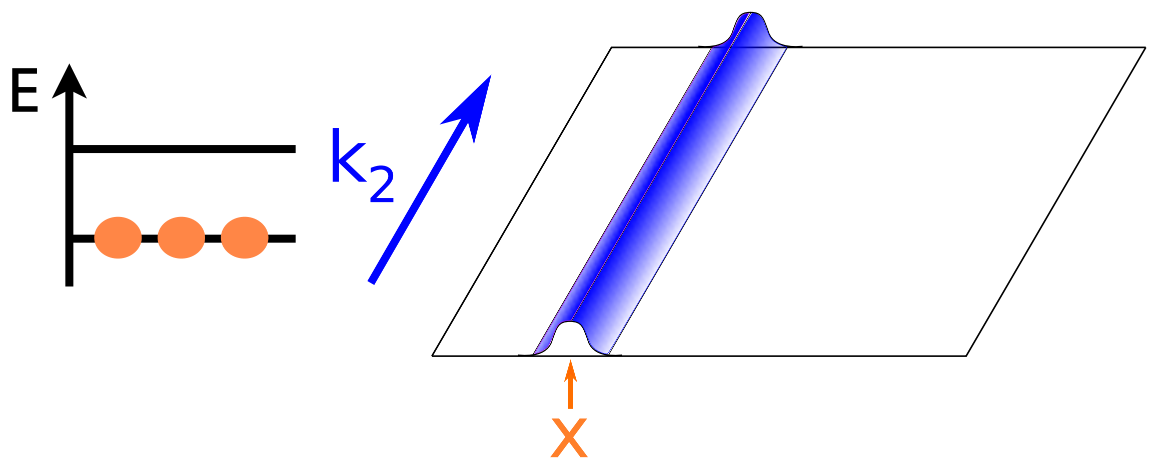

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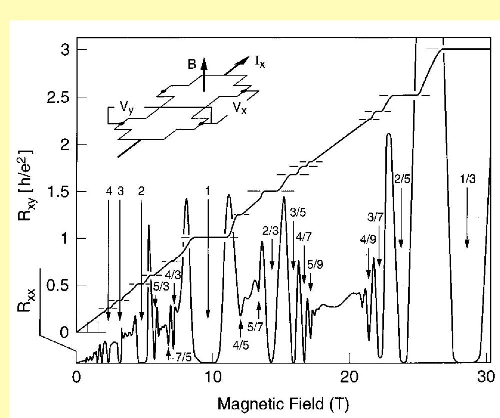

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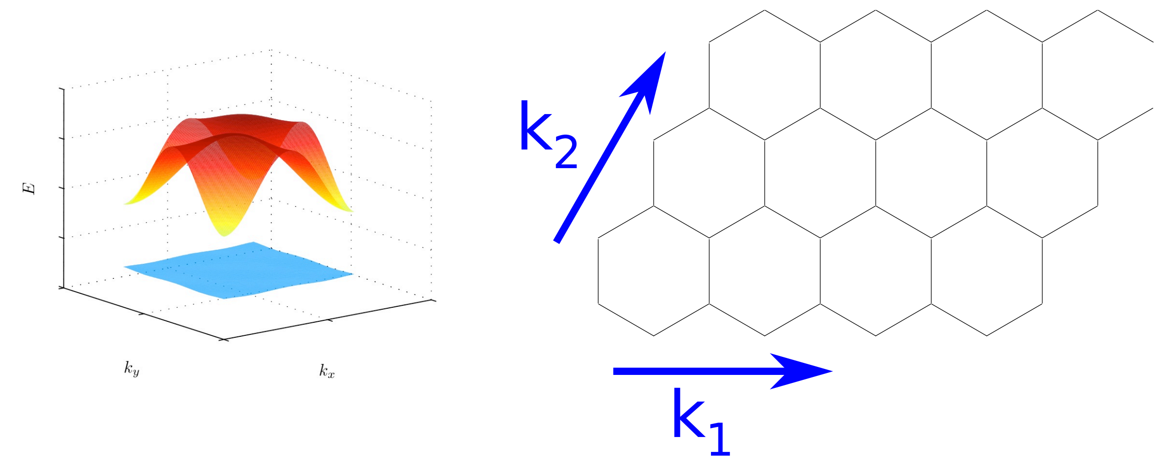

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Interactions in Chern #1 Bands = FQHE ?

Proposition: correlated states reproduce the physics of FQHE

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PRL **106,** 236804 (2011) PHYSICAL REVIEW LETTERS week ending 10 JUNE 2011

Fractional Quantum Hall States at Zero Magnetic Field

Titus Neupert,[1] Luiz Santos,[2] Claudio Chamon,[3] and Christopher Mudry[1]
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Nearly Flatbands with Nontrivial Topology

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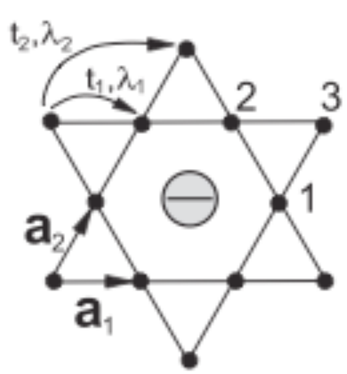

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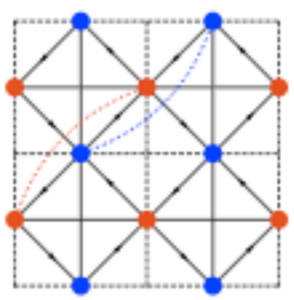

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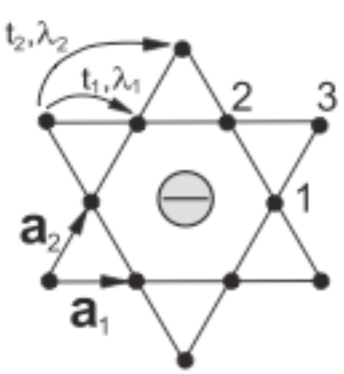

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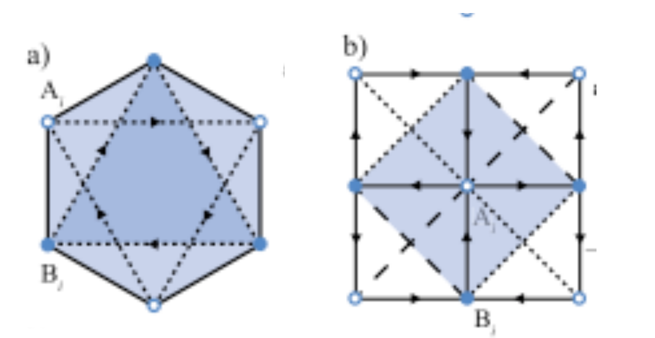

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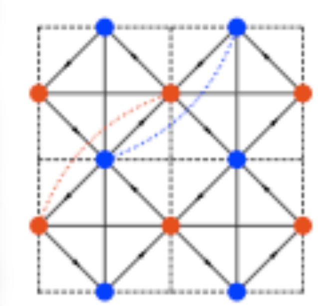
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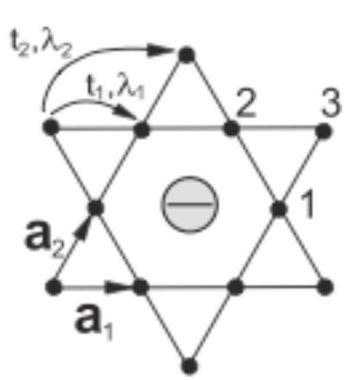

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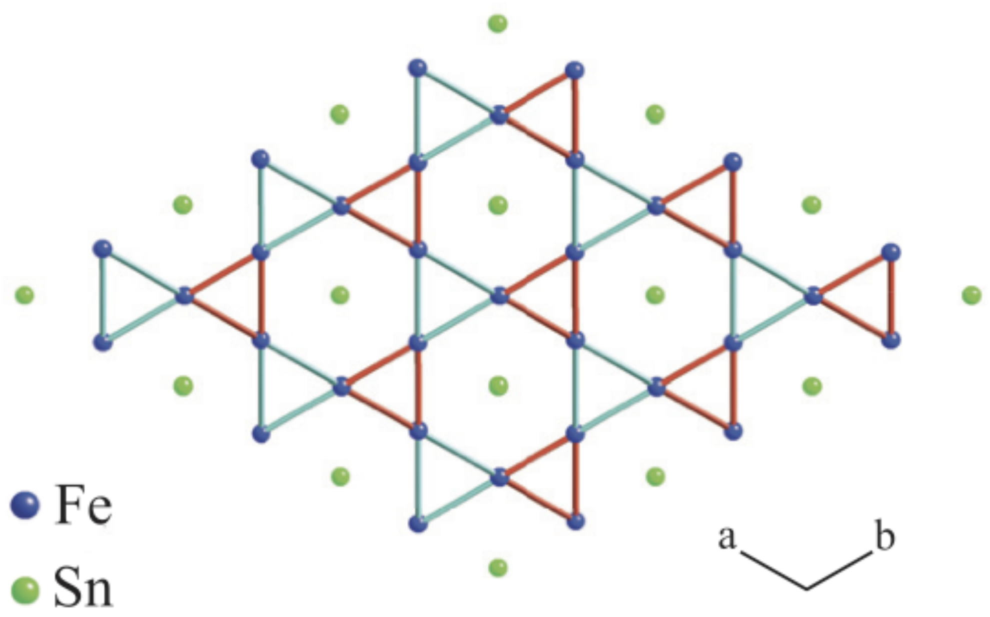
Where to realize interacting topological flat-band models?

- Thin films of kagomé lattice compound Fe_3Sn_2



- Sn relatively heavy and charged

Induces spin orbit coupling, possibly able to realize a TFB in thin films [E. Tang et al., PRL 2011]

Need fine tuned parameters and charge density, so difficult to find a suitable compound

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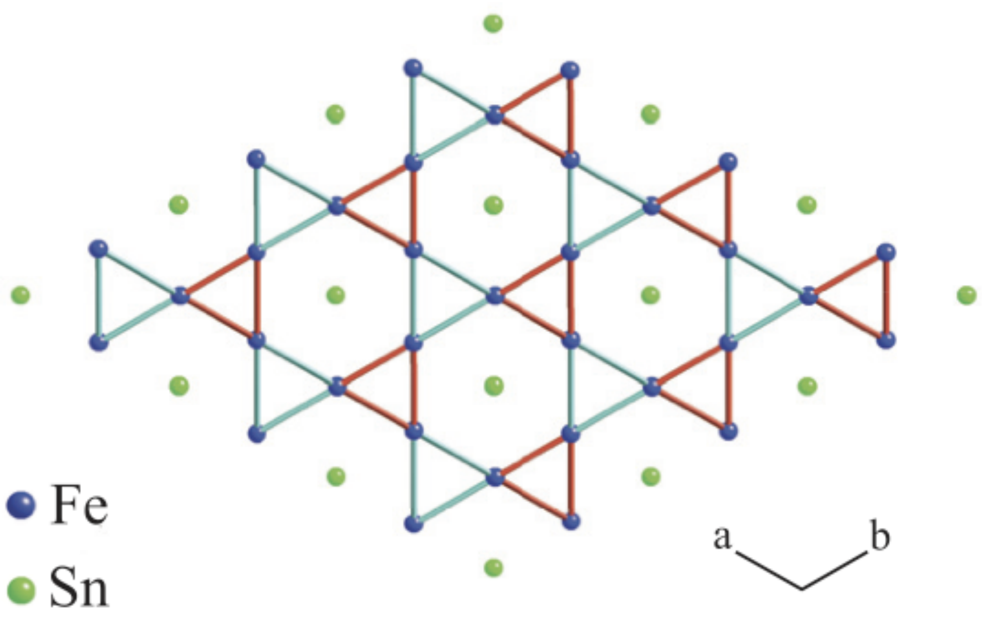

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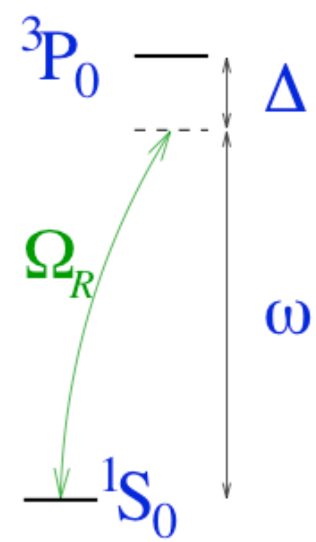
- Cold atomic gases in optical lattices

see, e.g. Cooper & Dalibard 2011

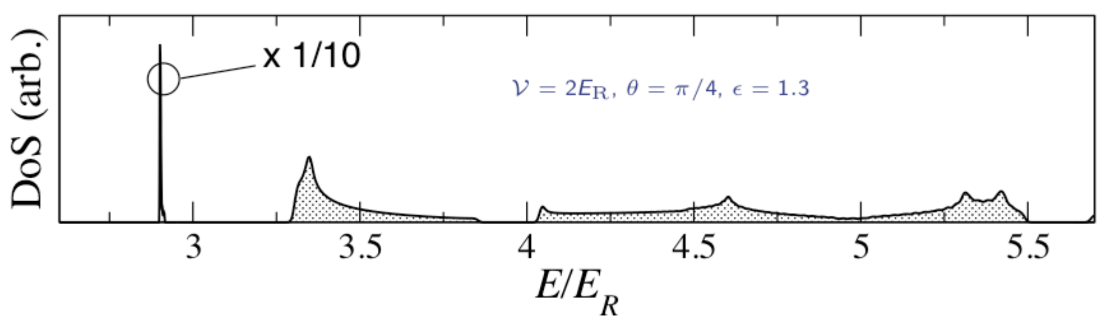
$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \hat{V}(\mathbf{r})$$

spatially varying coupling between N internal states



can be implemented routinely using coupling to Raman lasers

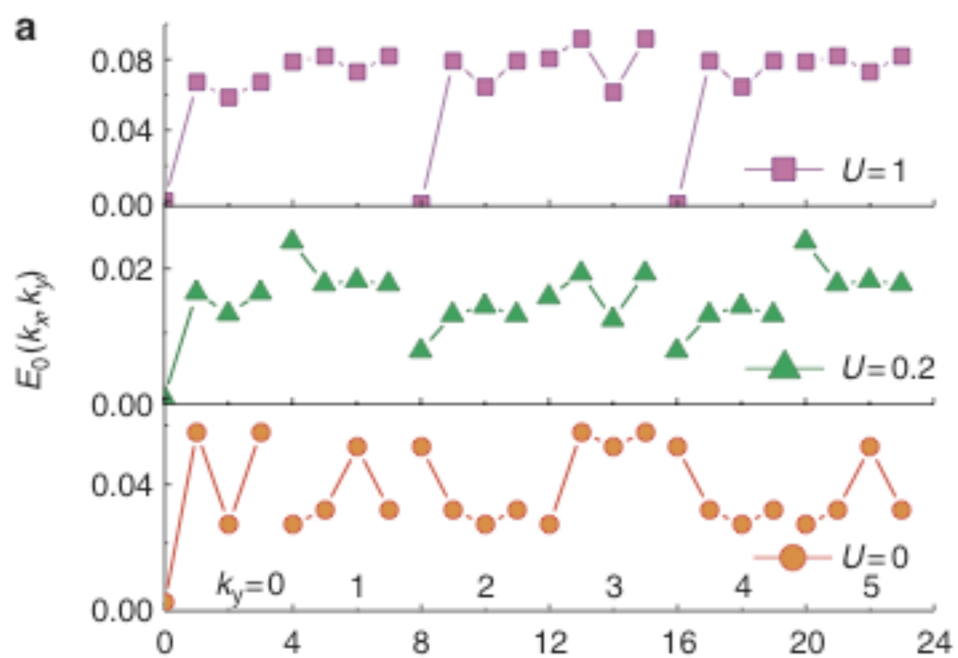
- example spectrum with flat C=1 band



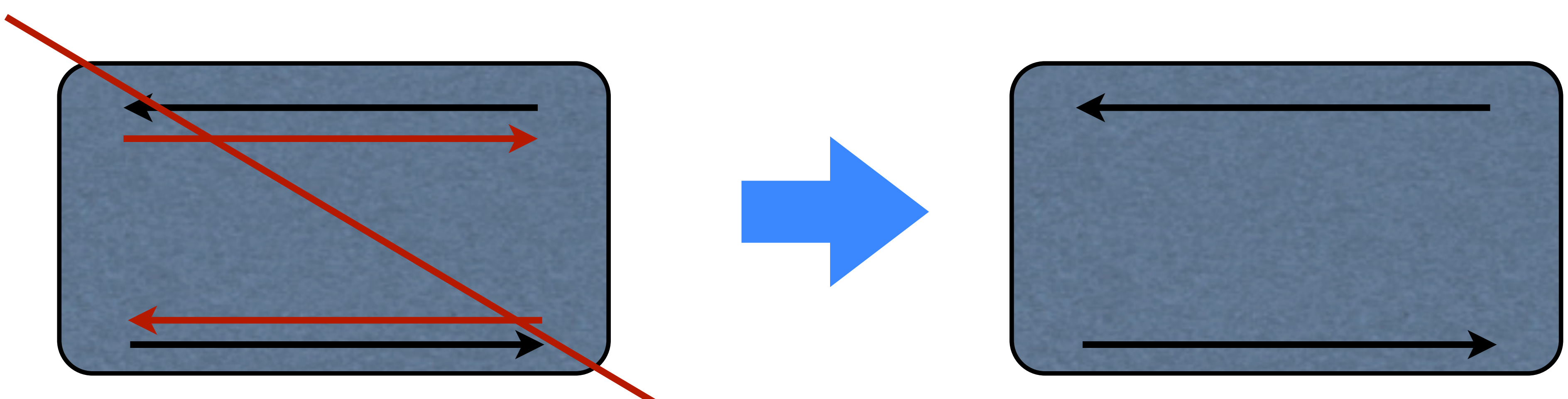
Ease to fine - tune, but also small energy scale for interactions

Numerical evidence for "Fractional Chern Insulators"

- existence of a gap & groundstate degeneracy [checkerboard lattice]
- chern number of groundstate manifold



[D. Sheng]



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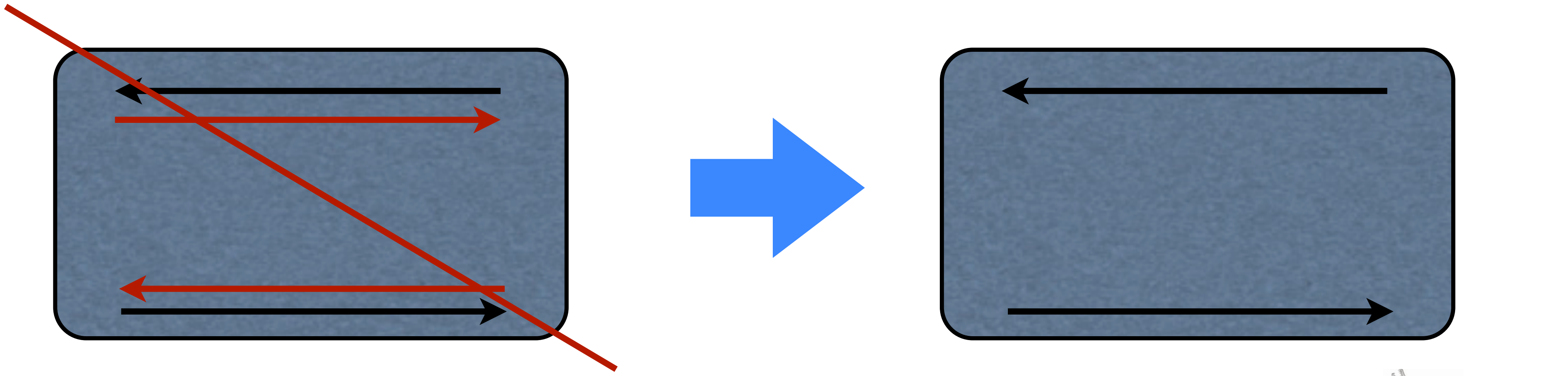

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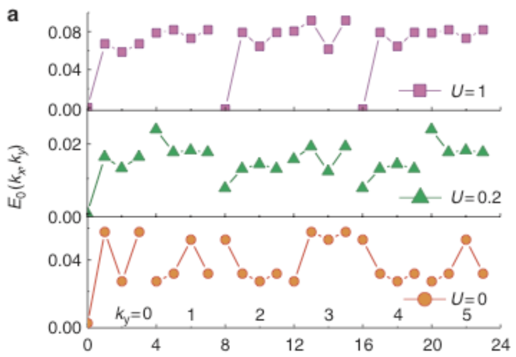
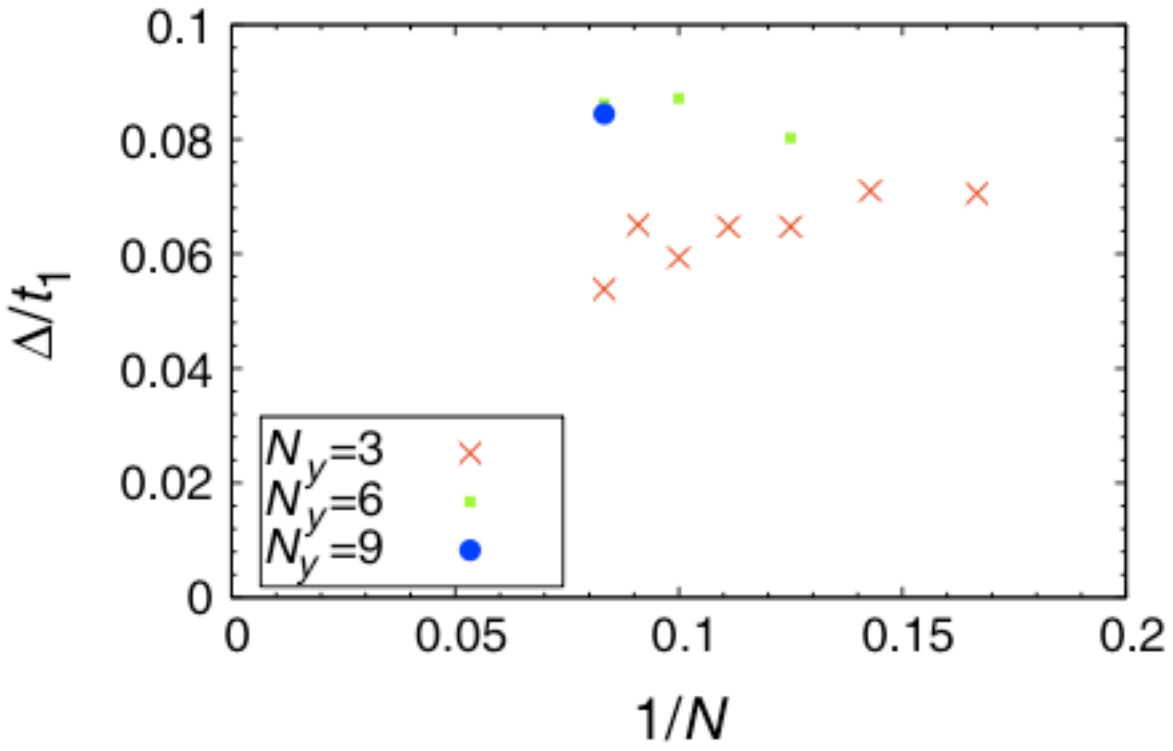
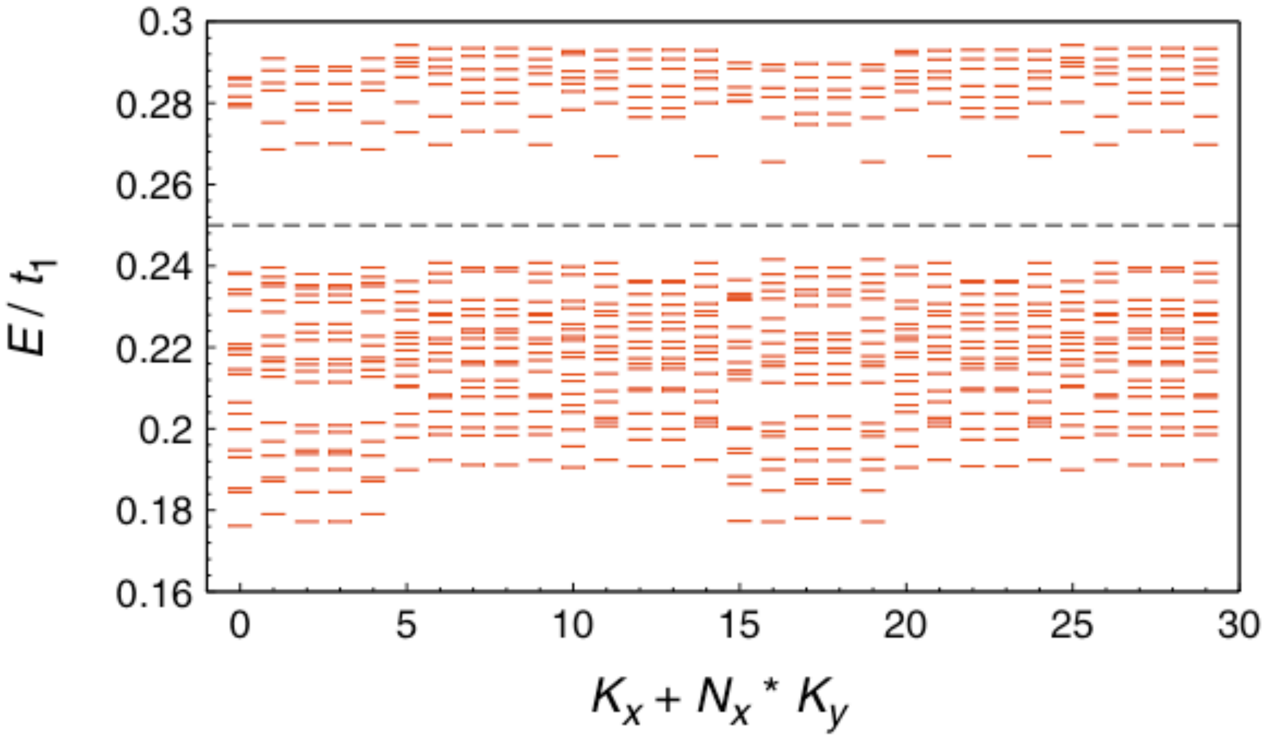


- Finite size scaling of gap



- Particle Entanglement Spectra : count of excitations matches FQHE (here - Laughlin state)



"Fractional Chern Insulators (FCI)" [N. Regnault & A. Bernevig, PRX '11]

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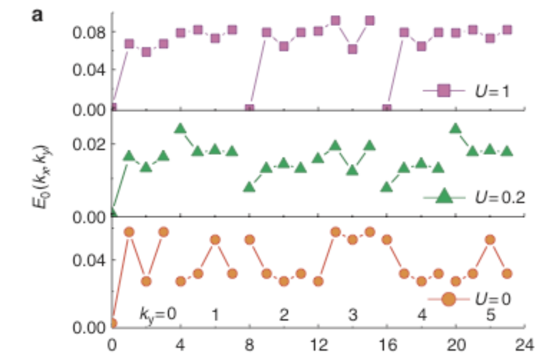

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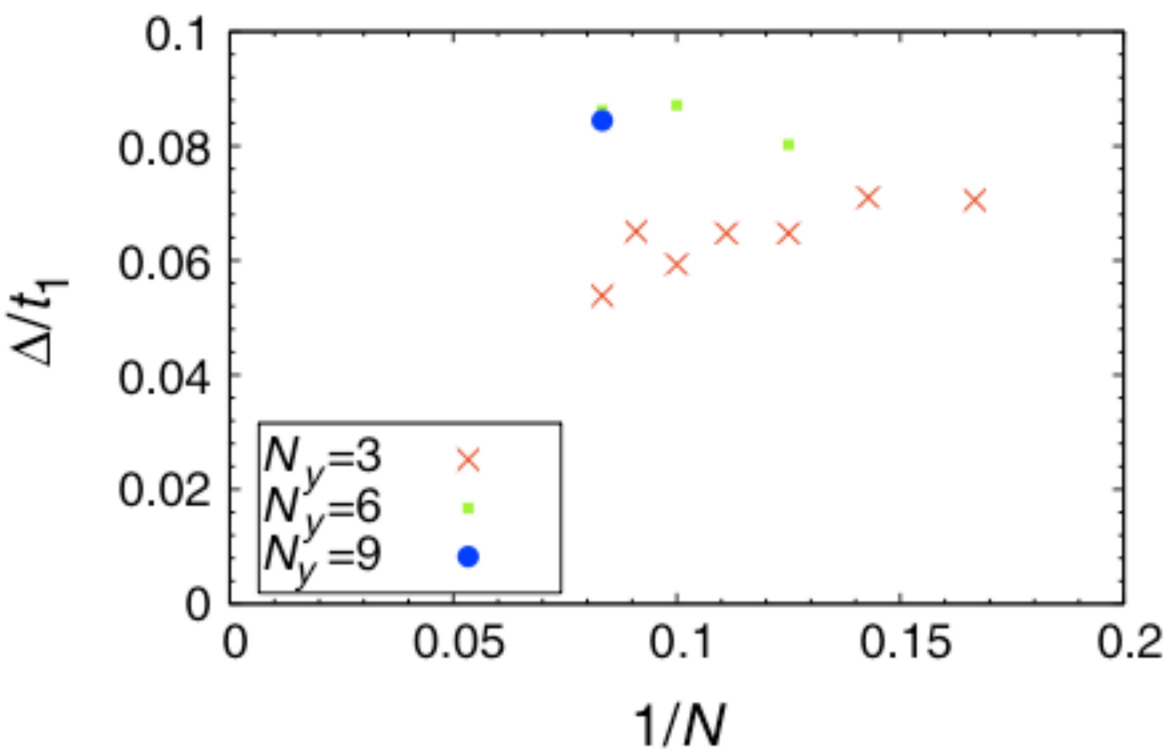

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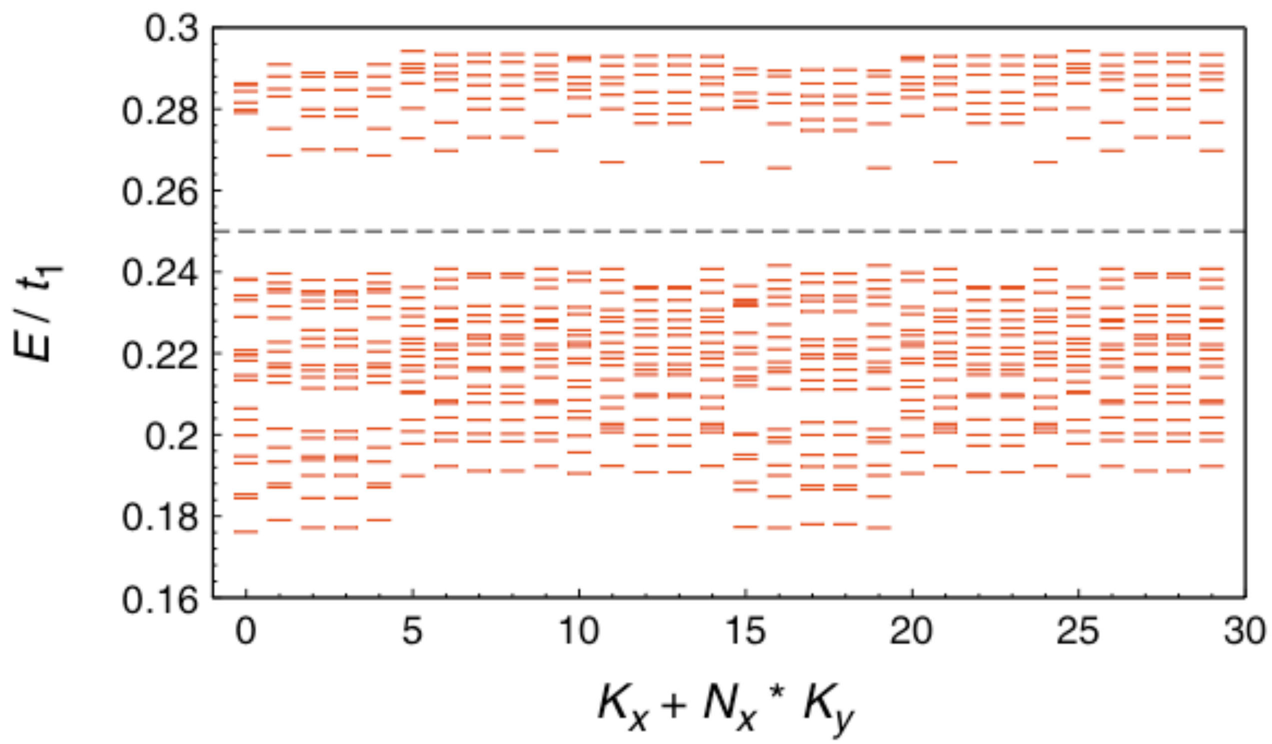

“Fractional Chern Insulators (FCI)” [N. Regnault & A. Bernevig, PRX ’11]

- Strong numerical evidence for QHE physics, but no clear organising principle for different lattice models

Understanding Fractional Quantum Hall states

- Single particle states are analytic functions in symmetric gauge

$$\vec{A} = \frac{1}{2}\vec{r} \wedge \vec{B}$$

$$\phi_m \propto z^m e^{-|z|^2/4\ell_0}$$

$$z_j = x_j + iy_j$$

- Many particle states are still analytic functions - can write explicitly!

e.g. Laughlin: $$\Psi_{\nu=\frac{1}{m}} = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4\ell_0}$$

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✦ Incompressibility ✦ Quasiparticle excitations: charge / statistics

✦ Correlations / You name the observable...

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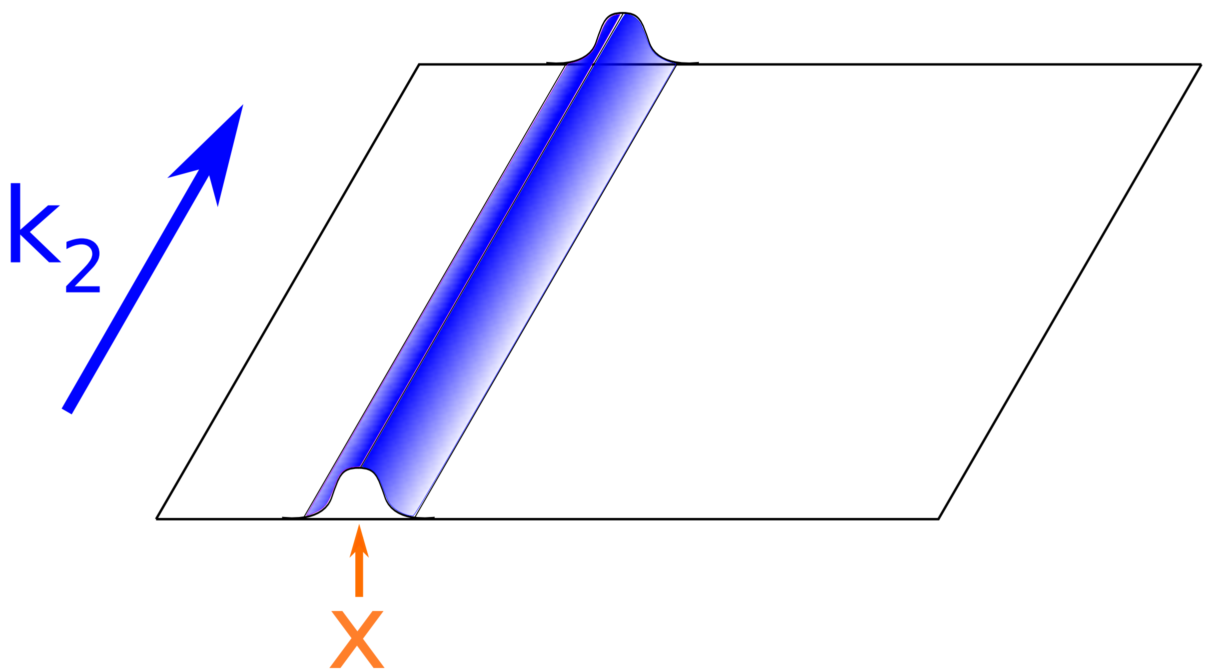
Can we construct wavefunctions for Chern Insulators?

From FQHE to Fractional Chern Insulator

Fractional Quantum Hall States (FQHE)



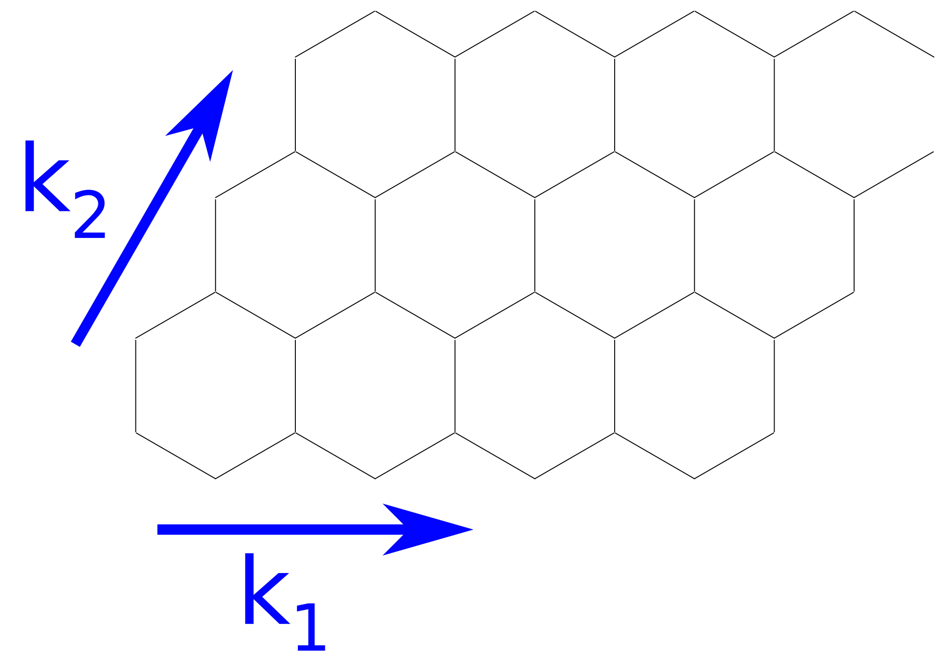
- Perfectly degenerate Landau level (C=1)

$$\mathcal{H} = \sum_{i<j} V(r_i - r_j)$$

- states indexed by linear momentum ($\vec{A} = Bx\vec{e}_y$)

$$\phi_{k_y}(x,y) = e^{ik_y y} e^{\frac{1}{2\ell_0^2}(x-k_y\ell_0^2)}$$

Fractional Chern Insulator (FCI)



- Topological Flat band (C=1) with small dispersion

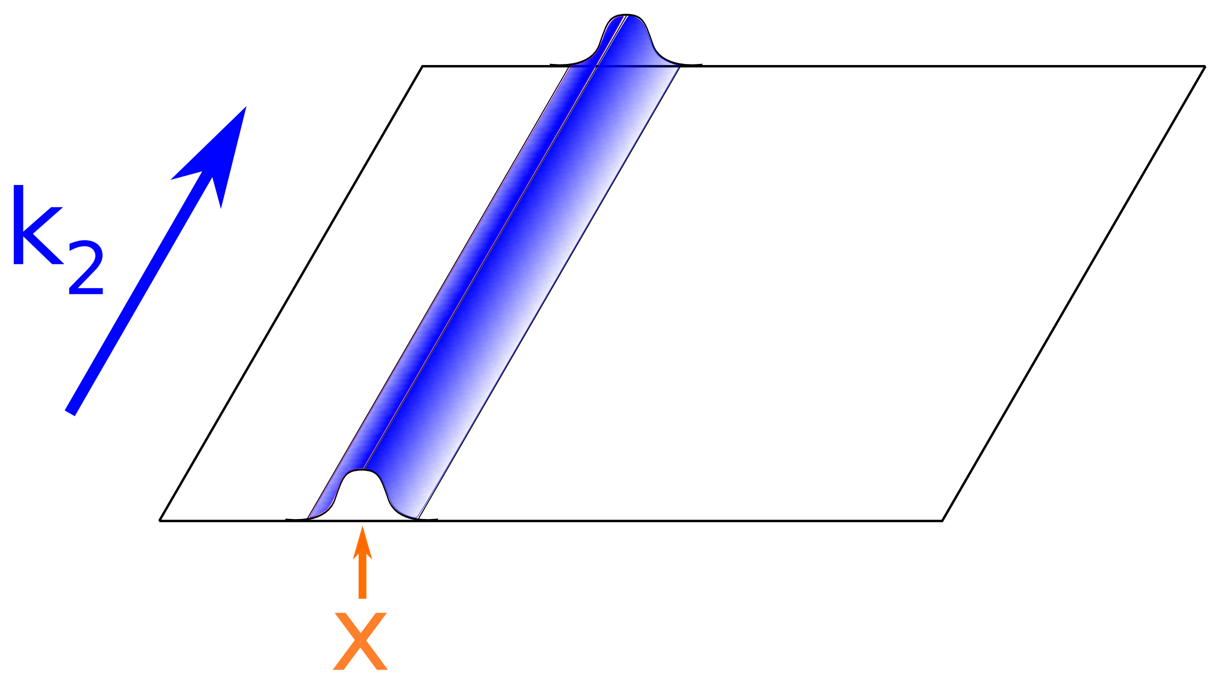
$$\mathcal{H} = \sum_{i<j} V(r_i - r_j) + \sum_k \epsilon_0(k)\hat{n}_k$$

- Lattice momentum conservation

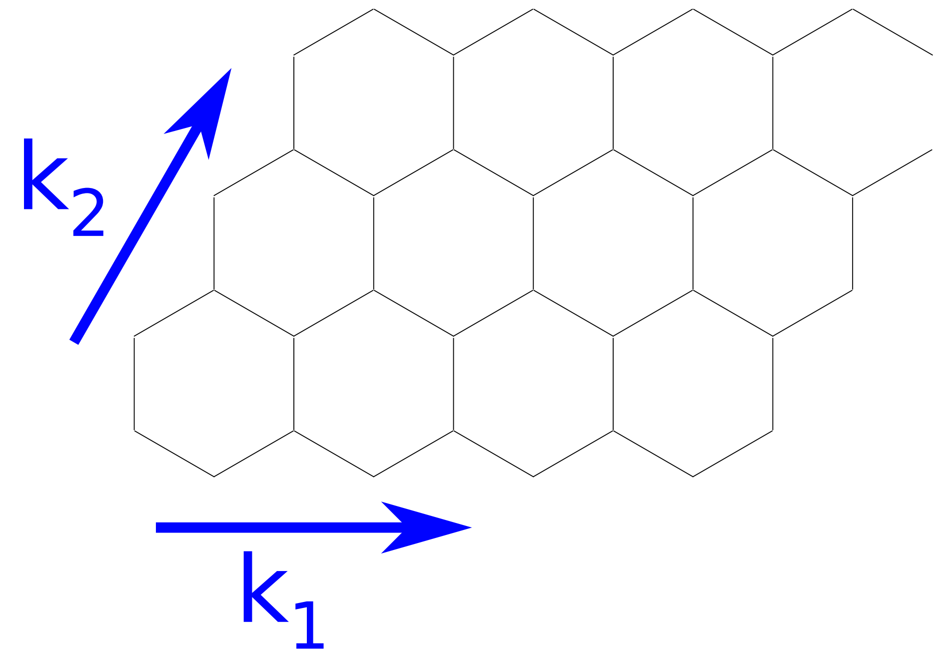
$$\langle \vec{r}_\alpha | n, k_x, k_y \rangle = \sum_{\vec{k}} e^{-i\vec{k}\vec{r}} u^n_\alpha(k)$$

Idea: Mapping Single Particle Orbitals

FQHE

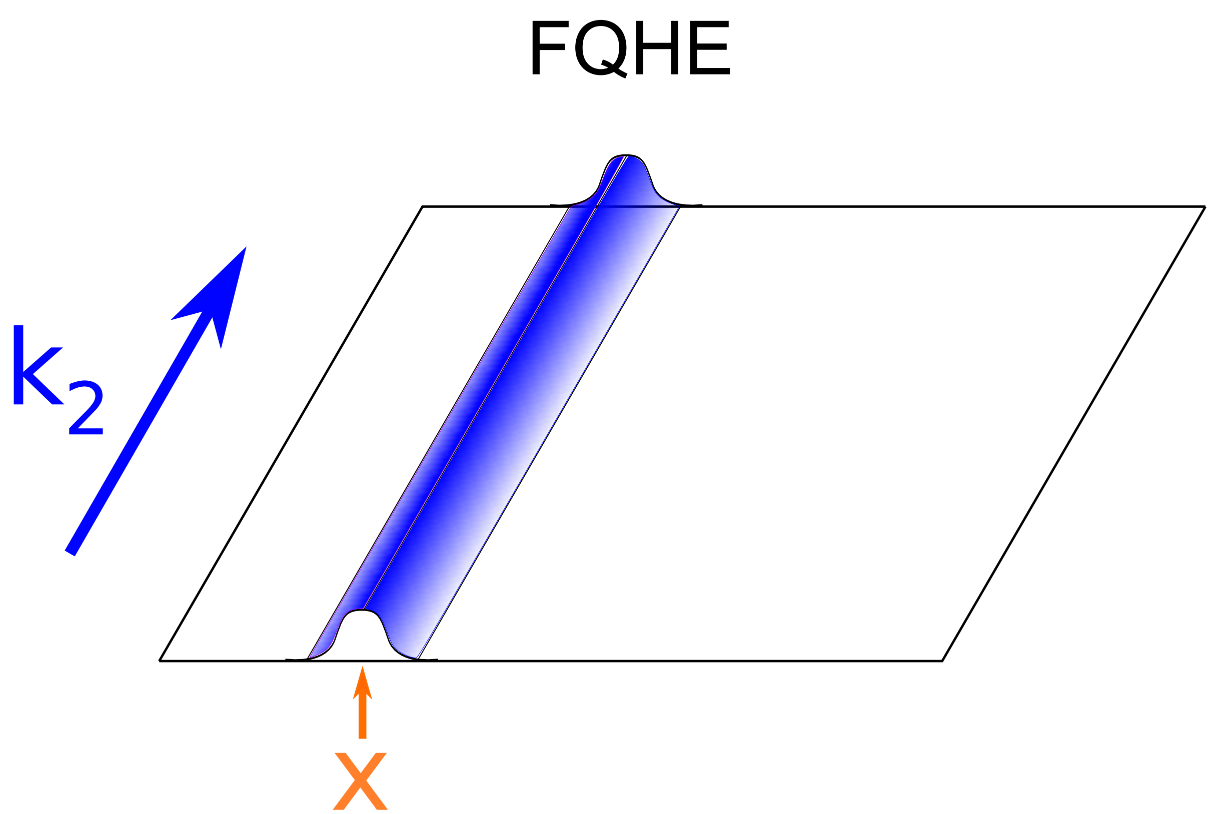


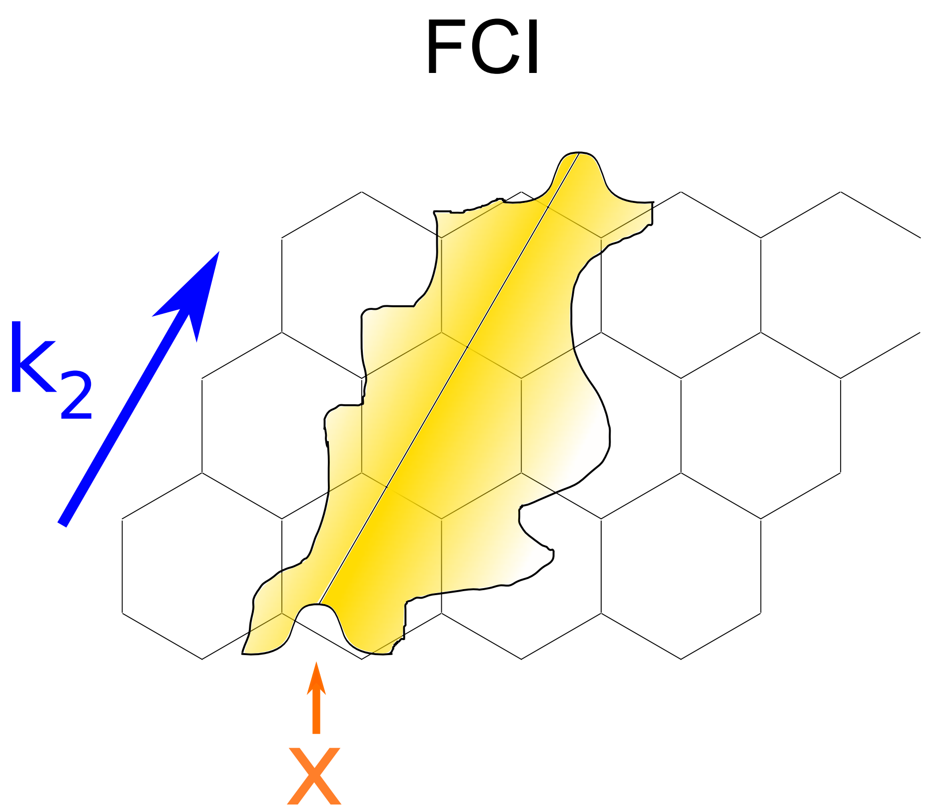
FCI



- Proposal by X.-L. Qi [PRL '11]: Get FCI Wavefunctions by mapping single particle orbitals

Idea: Mapping Single Particle Orbitals





- Proposal by X.-L. Qi [PRL '11]: Get FCI Wavefunctions by mapping single particle orbitals
- Idea: use Wannier states which are localized in the x-direction
- keep translational invariance in y (cannot create fully localized Wannier state if C>0!)

$$|W(x,k_y)\rangle = \sum_{k_x} f_{k_x}^{(x,k_y)} |k_x, k_y\rangle$$

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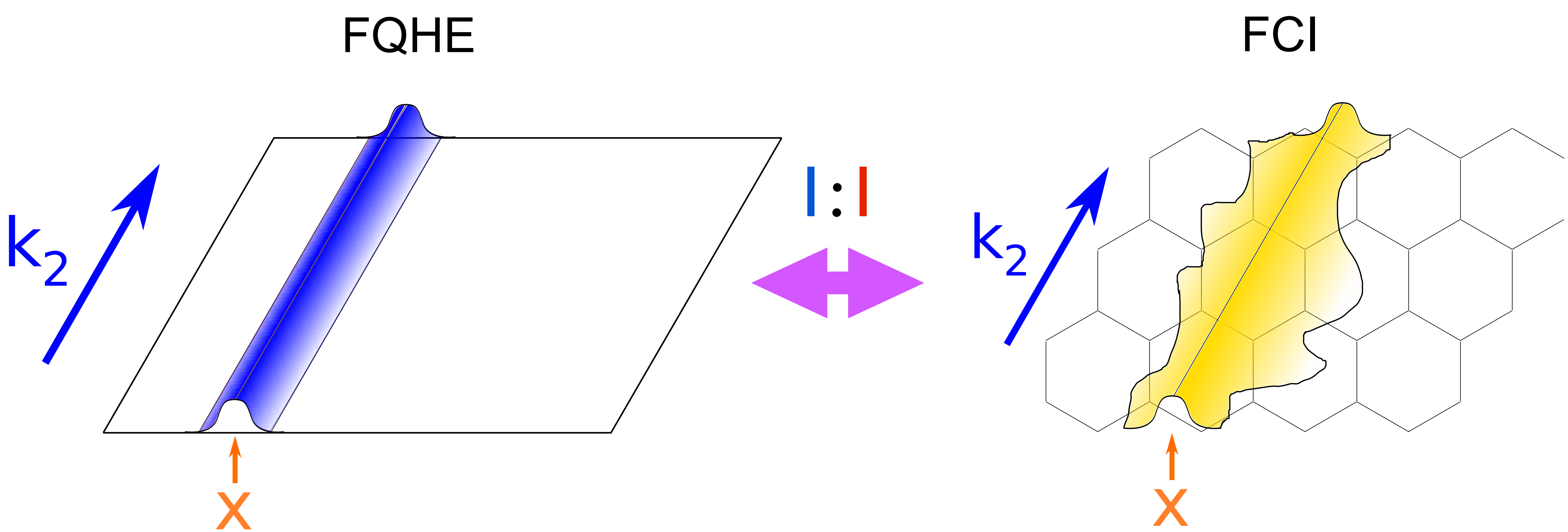

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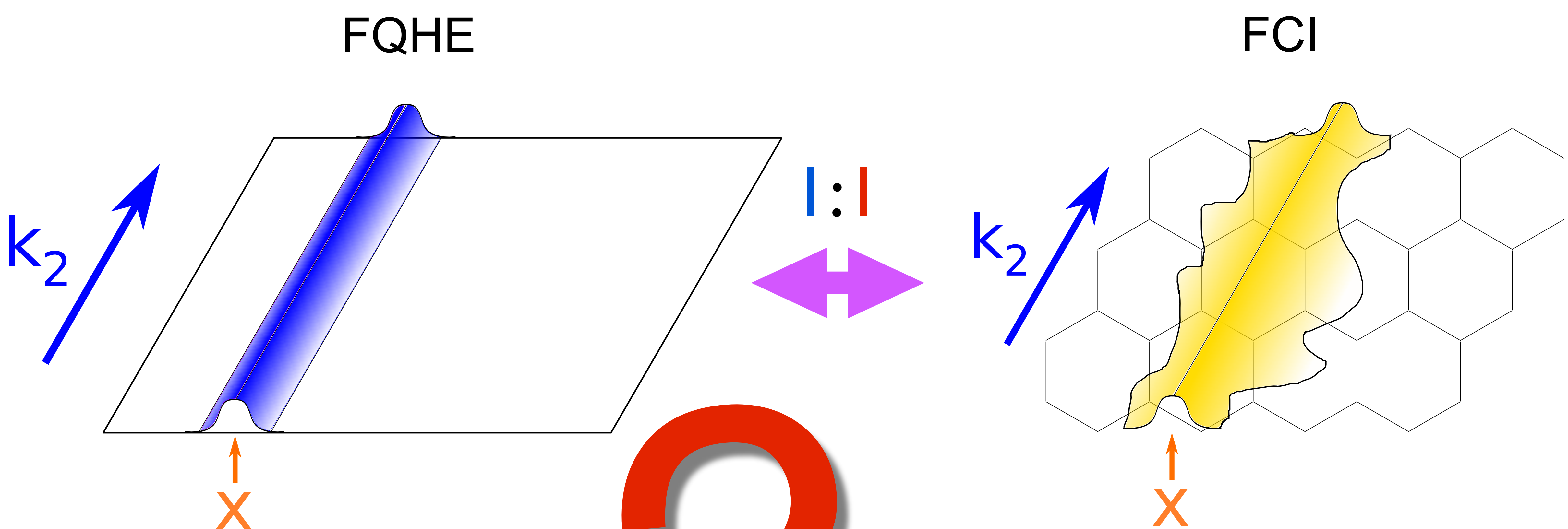

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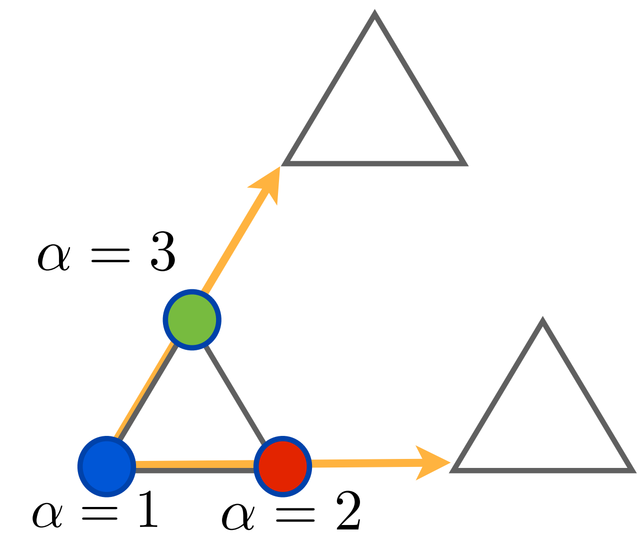
Wannier states in Chern bands



- Some formalism

$$\mathcal{H} = \sum_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\alpha} h_{\alpha\beta}(\mathbf{k}) \hat{a}_{\mathbf{k},\beta}$$ Hamiltonian

$$h_{\alpha\beta}(\mathbf{k}) u^n_\beta(\mathbf{k}) = \epsilon_n(\mathbf{k}) u^n_\alpha(\mathbf{k})$$ Eigenstates

$$\mathcal{A}(n,\mathbf{k}) = -i \sum_\alpha u^{n*}_\alpha(\mathbf{k}) \nabla_{\mathbf{k}} u^n_\alpha(\mathbf{k})$$ Berry connection

$$c^\dagger_{\mathbf{k},\alpha} = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot(\mathbf{R}+\boldsymbol{\delta}_\alpha)} c^\dagger_{\mathbf{R},\alpha}$$

- construction of a Wannier state at fixed k_y

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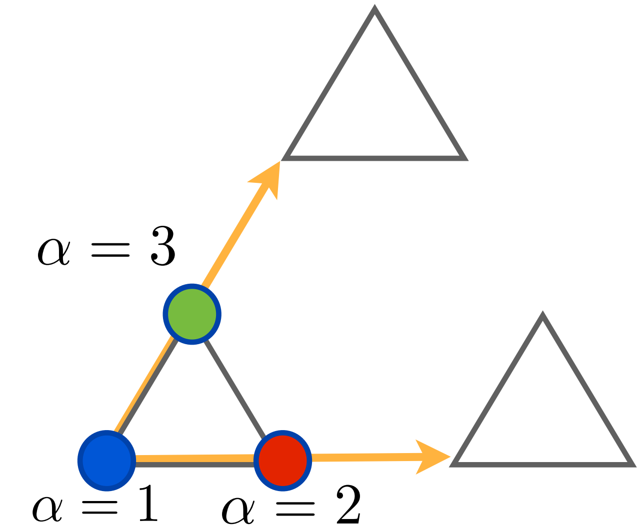
Fourier transform

Wannier states in Chern bands



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'Parallel transport' of phase

Fourier transform

Berry connection indicates change of phase due to displacement in BZ

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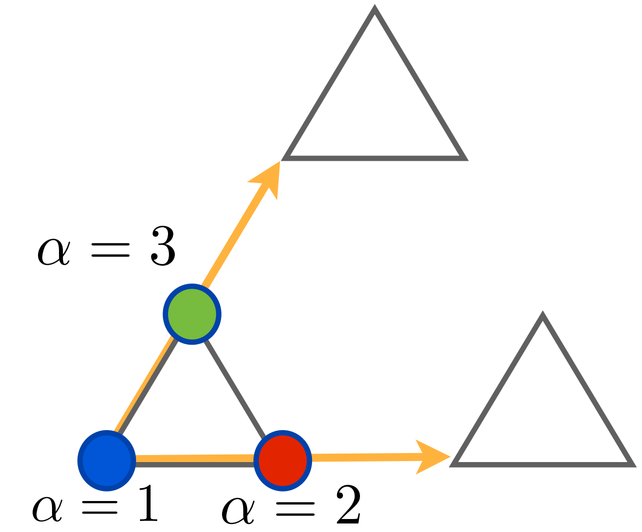

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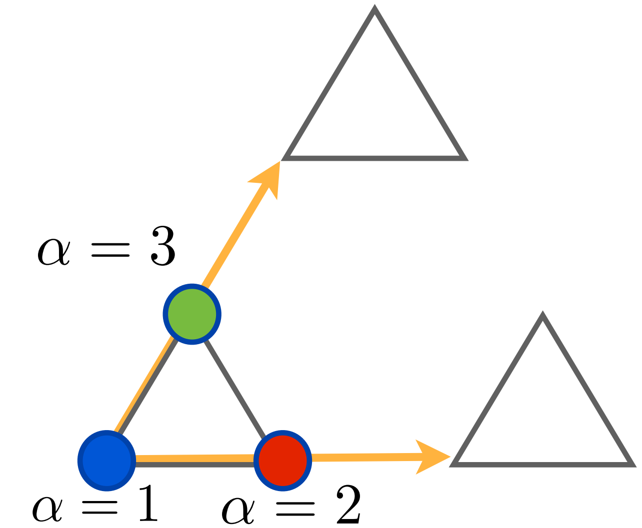
'Parallel transport' of phase — 'Polarization' — Fourier transform

Berry connection indicates change of phase due to displacement in BZ

ensures periodicity of WF in $k_y \rightarrow k_y + 2\pi$

Wannier states in Chern bands



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ky-dependent phase factor, or `gauge'

'Parallel transport' of phase

Berry connection indicates change of phase due to displacement in BZ

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ensures periodicity of WF in $k_y \rightarrow k_y + 2\pi$

Fourier transform

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'Parallel transport' of phase — Berry connection indicates change of phase due to displacement in BZ

'Polarization' — ensures periodicity of WF in $k_y \to k_y + 2\pi$

Fourier transform

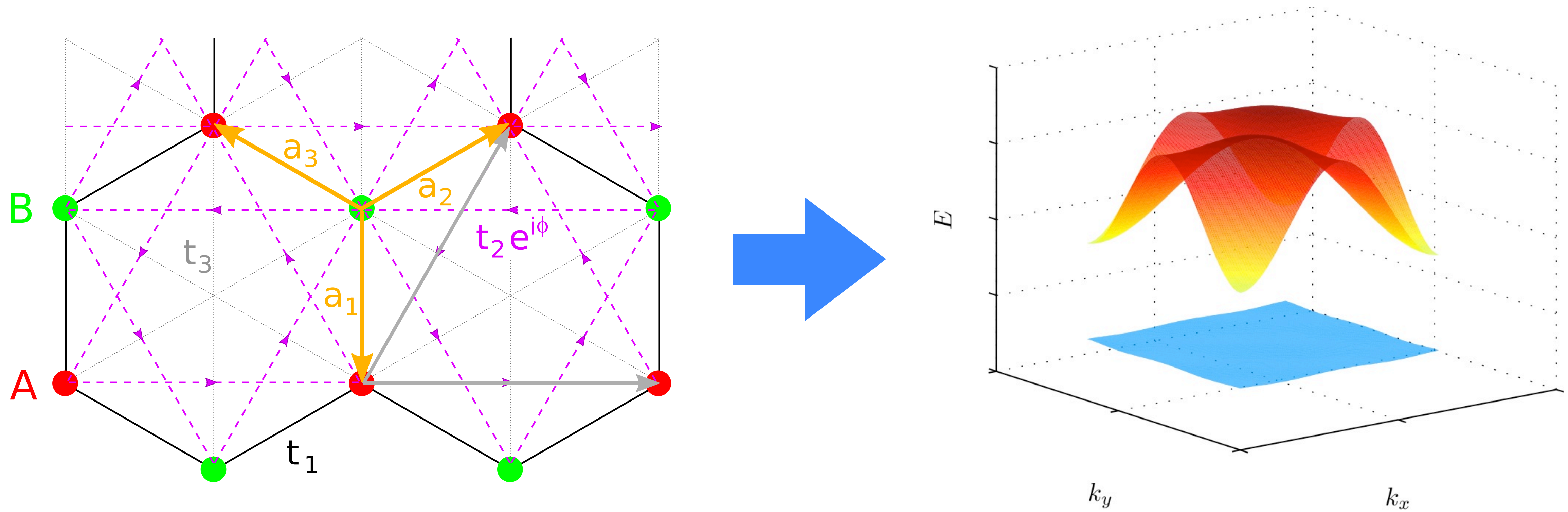
- or, more simply we can think of the Wannier states as the eigenstates of the position operator

$$\hat{X}^{cg} = \lim_{q_x \to 0} \frac{1}{i} \frac{\partial}{\partial q_x} \bar{\rho}_{q_x} \quad \Rightarrow \quad \hat{X}^{cg}|W(x,k_y)\rangle = [x - \theta(k_y)/2\pi]|W(x,k_y)\rangle$$

- role of polarization: displacement of centre of mass of the Wannier state

$$\theta(k_y) = \int_0^{2\pi} \mathcal{A}_x(p_x,k_y)dp_x$$

An example: The Haldane Model



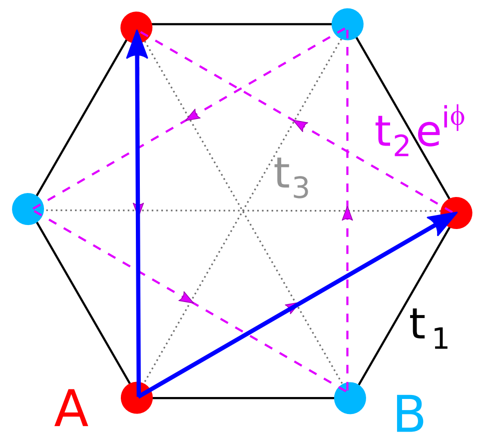
$$\mathcal{H} = -t_1 \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.) - t_2 \sum_{\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} e^{i\phi_{\mathbf{r}\mathbf{r}'}} + h.c.)$$
$$- t_3 \sum_{\langle\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle\rangle} (\hat{a}^\dagger_{\mathbf{r}} \hat{a}_{\mathbf{r}} + h.c.) + \frac{U}{2} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}} (\hat{n}_{\mathbf{r}} - 1)$$

- tight binding model on hexagonal lattice
- with fine-tuned hopping parameters: obtain flat lower band

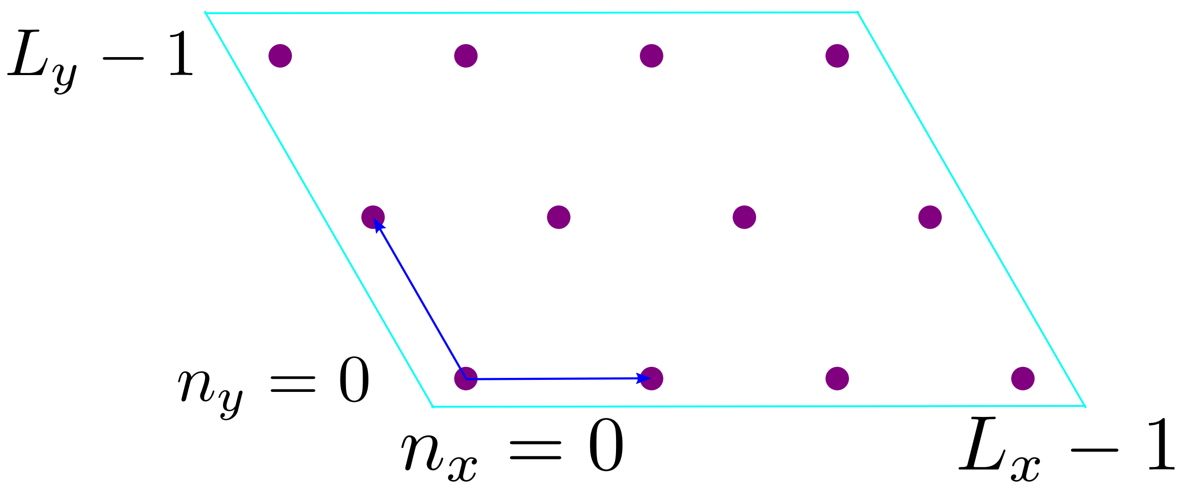
$$t_1 = 1,\ t_2 = 0.60,\ t_2 = -0.58 \text{ and } \phi = 0.4\pi$$

Conventions for numerical evaluation

Real Space



Reciprocal Space



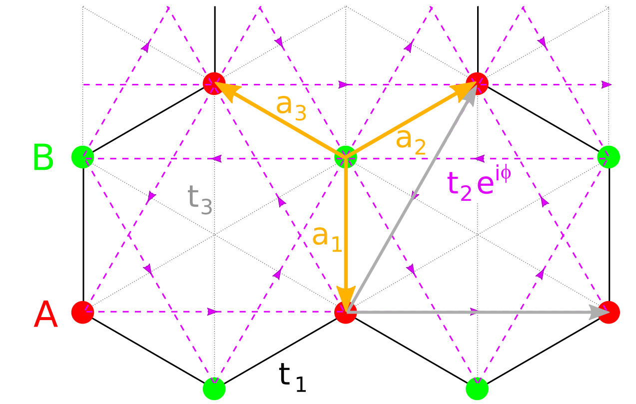
$$\mathbf{v}_1 = \sin(\gamma)\mathbf{e}_x + \cos(\gamma)\mathbf{e}_y$$
$$\mathbf{v}_2 = \mathbf{e}_y$$

$$\mathbf{G}_1 = 2\pi\mathbf{e}_x / L_1 \sin(\gamma)$$
$$\mathbf{G}_2 = 2\pi[-\cot(\gamma)\mathbf{e}_x + \mathbf{e}_y]/L_2$$

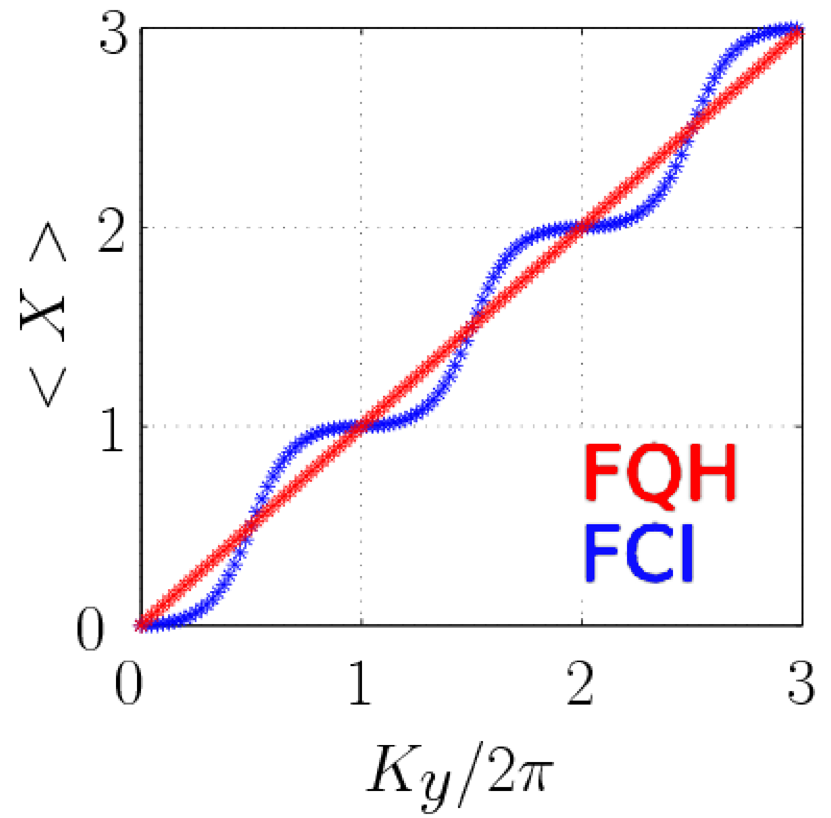
A few remarks:

- choose gauge such that Bloch functions satisfy: $u^n_\beta(\mathbf{k} + L_i\mathbf{G_i}) = u^n_\beta(\mathbf{k})$
- use discretized Berry connection $A^n_x(q_1, q_2) = \Im \log\left[u^{n*}_\alpha(q_1, q_2) u^n_\alpha(q_1 + 1, q_2)\right]$
- discretize its integrals by the rectangle rule $\displaystyle\int_0^{k_x} \mathcal{A}_x(p_x, k_y)dp_x \to \sum_{\tilde{q}_1=0}^{q_1(k_x)} A^n_x(\tilde{q}_1, q_2)$

Qi's Mapping



- Can introduce a canonical order of states with monotonously increasing position:



$$k_y = 2\pi n_y / L_y$$

$$\boxed{K_y} = k_y + 2\pi x = \boxed{2\pi j / L_y}$$

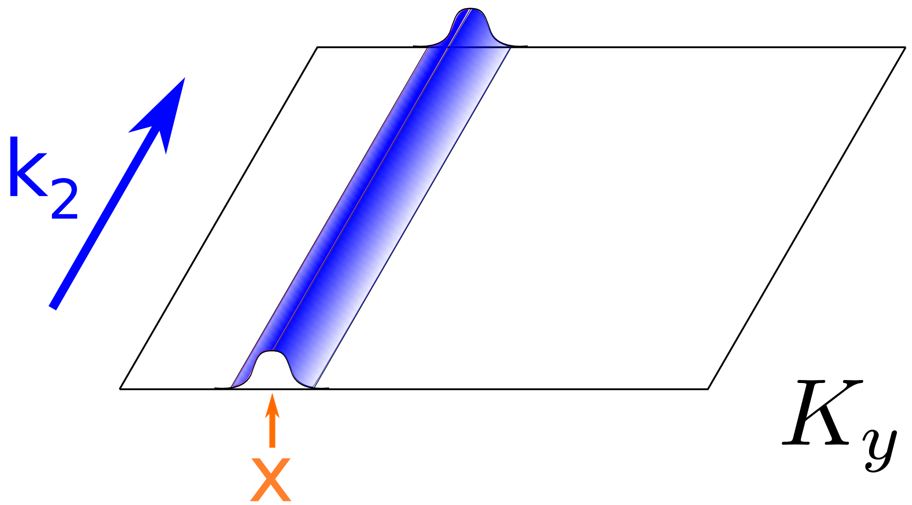
$$j = n_y + L_y x = 0, 1, \ldots, N_\phi - 1$$

- Increase in position for $k_y \rightarrow k_y + 2\pi$ = Chern-number C, as

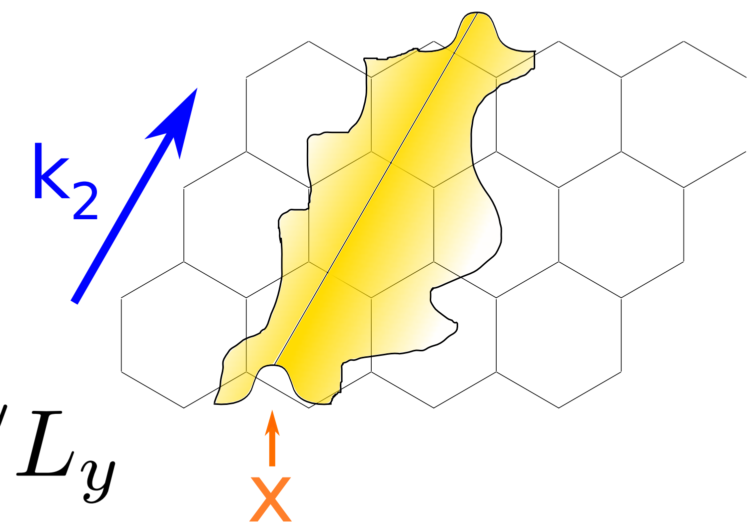
$$\frac{\partial}{\partial k_y} \langle \hat{X}^{cg} \rangle |_x = -\frac{1}{2\pi} \frac{\partial \theta(k_y)}{\partial k_y} = \int_0^{2\pi} \mathcal{B}(p_x, k_y) dp_x$$

Case study: Bosons with contact interactions

$$\mathcal{H}_{\rm int} \propto \sum_{i<j} \delta(r_i - r_j)$$



$\Leftrightarrow$



$$K_y = k_y + 2\pi x = 2\pi j / L_y$$

$k_y^{\max} = N_\phi - 1$

expand Hamiltonian in single-particle orbitals
(finite size, periodic boundary conditions)

$K_y^{\max} = L_x \times L_y - 1$

$$\mathcal{H} = \sum_{j_1, j_2, j_3, j_4} V_{j_1 j_2; j_3 j_4} \hat{c}^\dagger_{j_1} \hat{c}^\dagger_{j_2} \hat{c}_{j_3} \hat{c}_{j_4}$$

- Landau level momentum conserved:

$$V_{j_1 j_2; j_3 j_4} \propto \delta_{j_1+j_2,\, j_3+j_4}$$

- Linearized momentum

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T. Scaffidi, GM, arxiv:1207.3539

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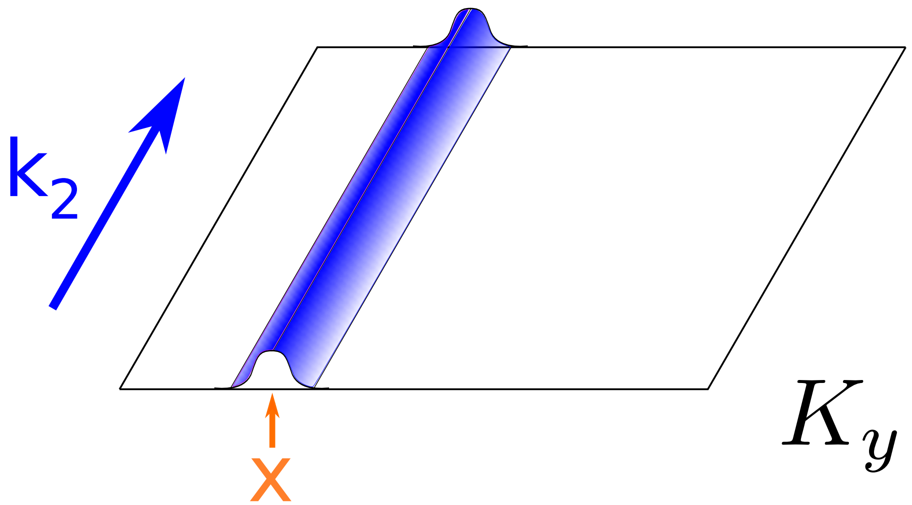

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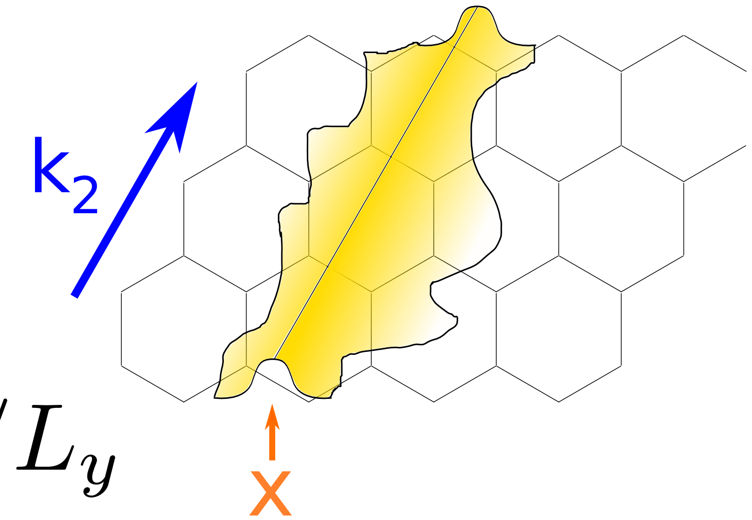

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Different conservation laws
→ Problems live in different Hilbert spaces

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Matrix elements in the Wannier basis

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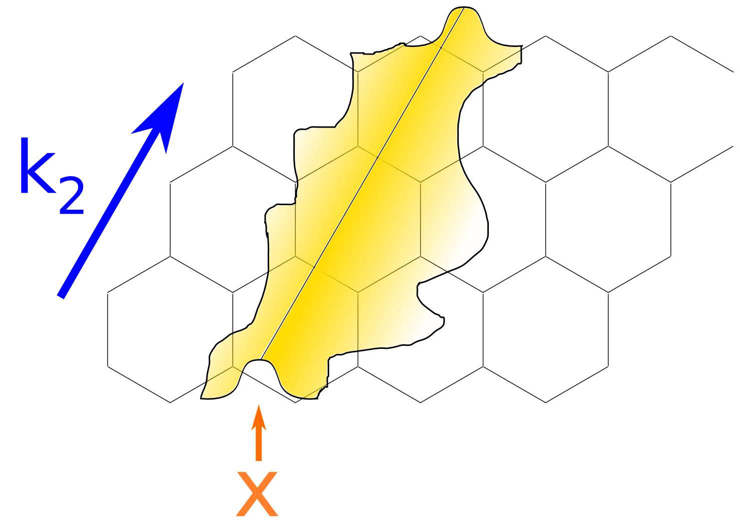
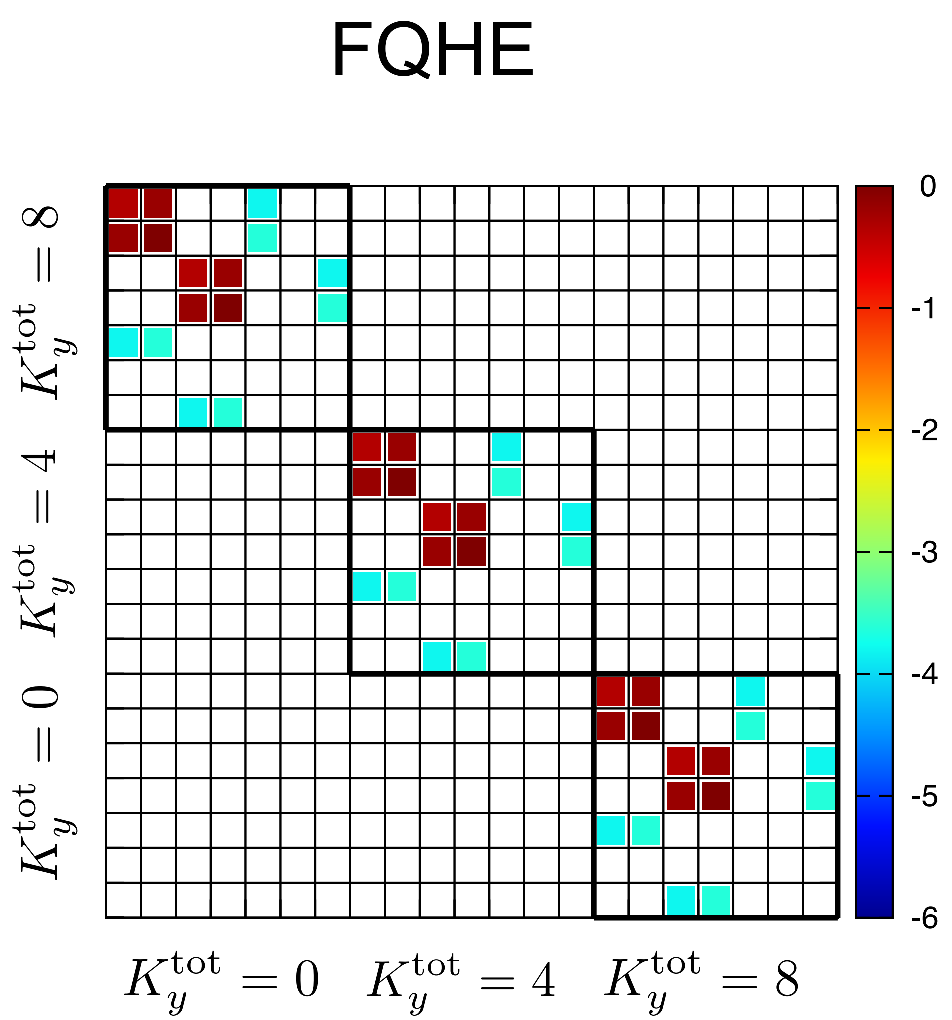


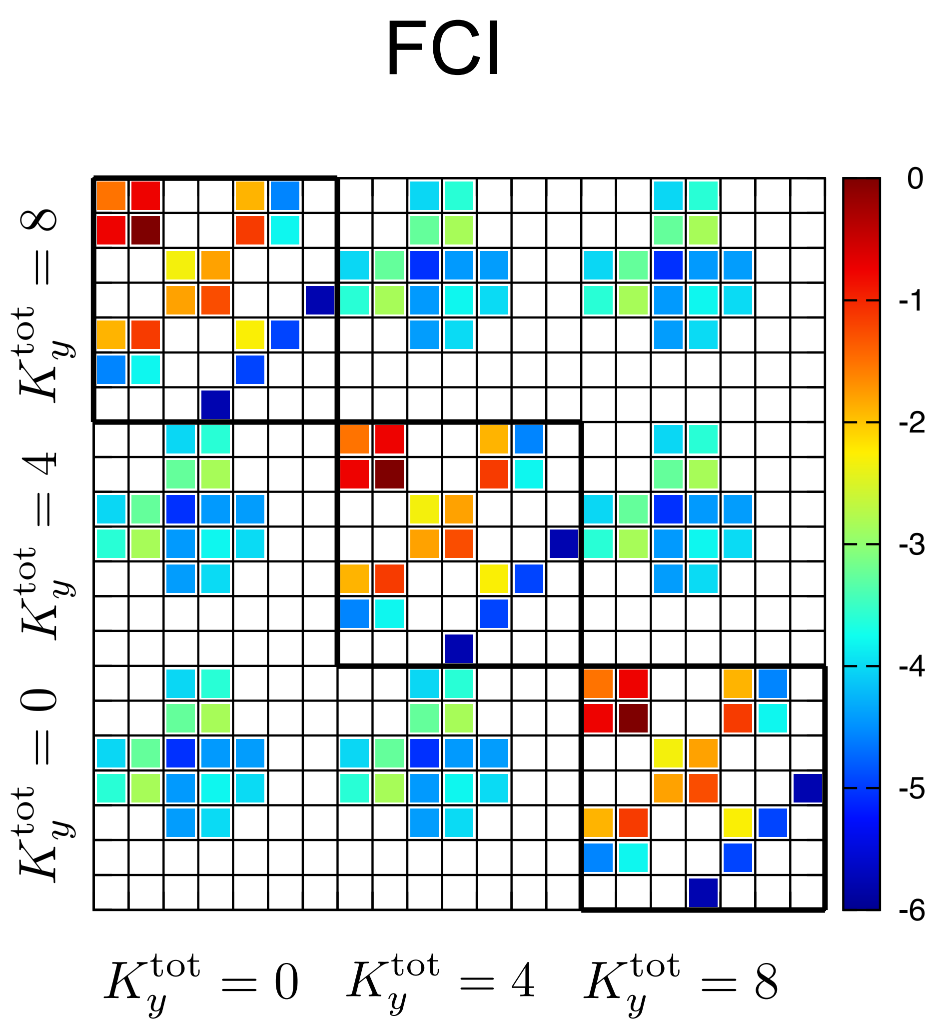
$$\mathcal{H}^{FCI} = \sum_{\substack{k_{y1},k_{y2},k_{y3},k_{y4} \\ x_1,x_2,x_3,x_4 \\ k_{y1}+k_{y2}=k_{y3}+k_{y4}}} \hat{c}^\dagger_{W(k_{y1},x_1)} \hat{c}^\dagger_{W(k_{y2},x_2)} \hat{c}_{W(k_{y3},x_3)} \hat{c}_{W(k_{y4},x_4)}$$

$$V_{j_1 j_2; j_3 j_4} \left\{ \begin{array}{l} \displaystyle\sum_{\substack{k_{x1},k_{x2},k_{x3},k_{x4} \\ k_{x1}+k_{x2}=k_{x3}+k_{x4}}} f^{*(x_1,k_{y1})}_{k_{x1}} f^{*(x_2,k_{y2})}_{k_{x2}} f^{(x_3,k_{y3})}_{k_{x3}} f^{(x4,k_{y4})}_{k_{x4}} \\ \displaystyle\sum_{a=A,B} u^{*a}_{\alpha_0}(\mathbf{k_1}) u^{*a}_{\alpha_0}(\mathbf{k_2}) u^{a}_{\alpha_0}(\mathbf{k_3}) u^{a}_{\alpha_0}(\mathbf{k_4}) \end{array} \right.$$

Case study: Bosons with contact interactions

- Magnitude of matrix elements for delta interactions:



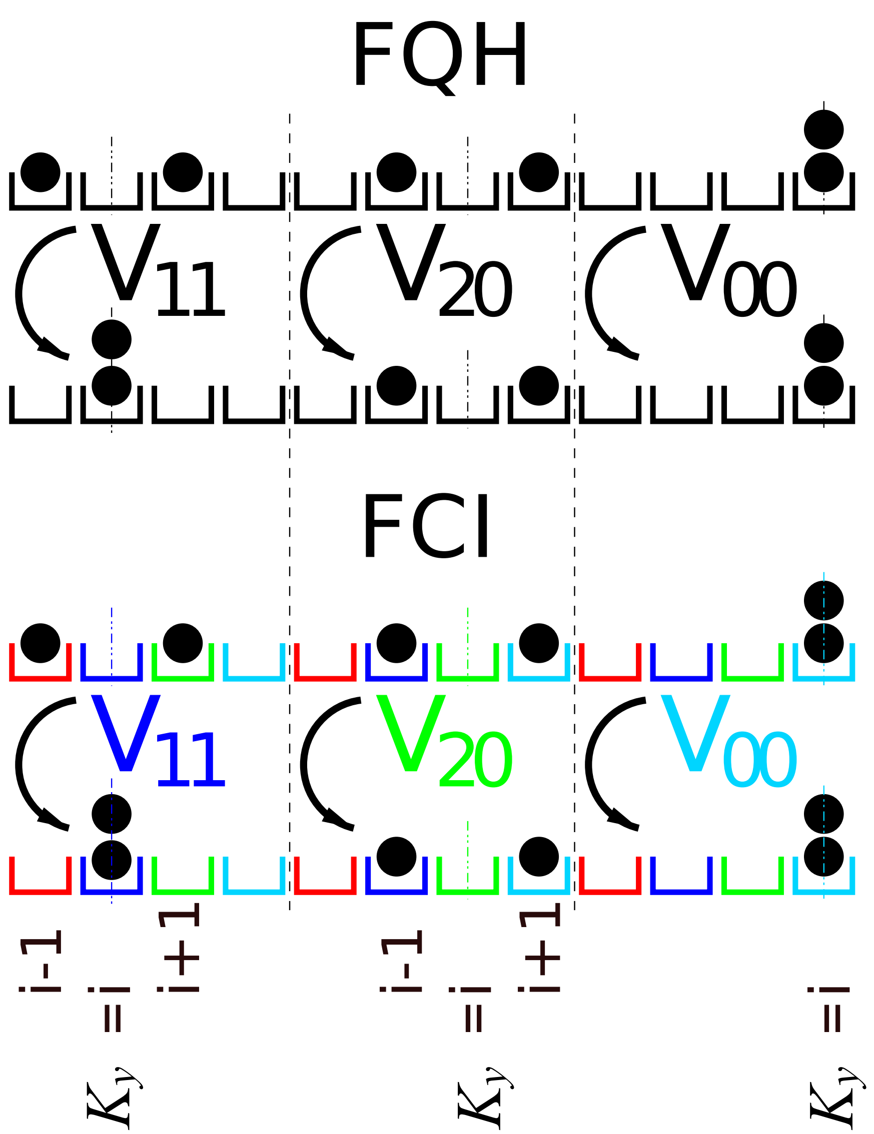


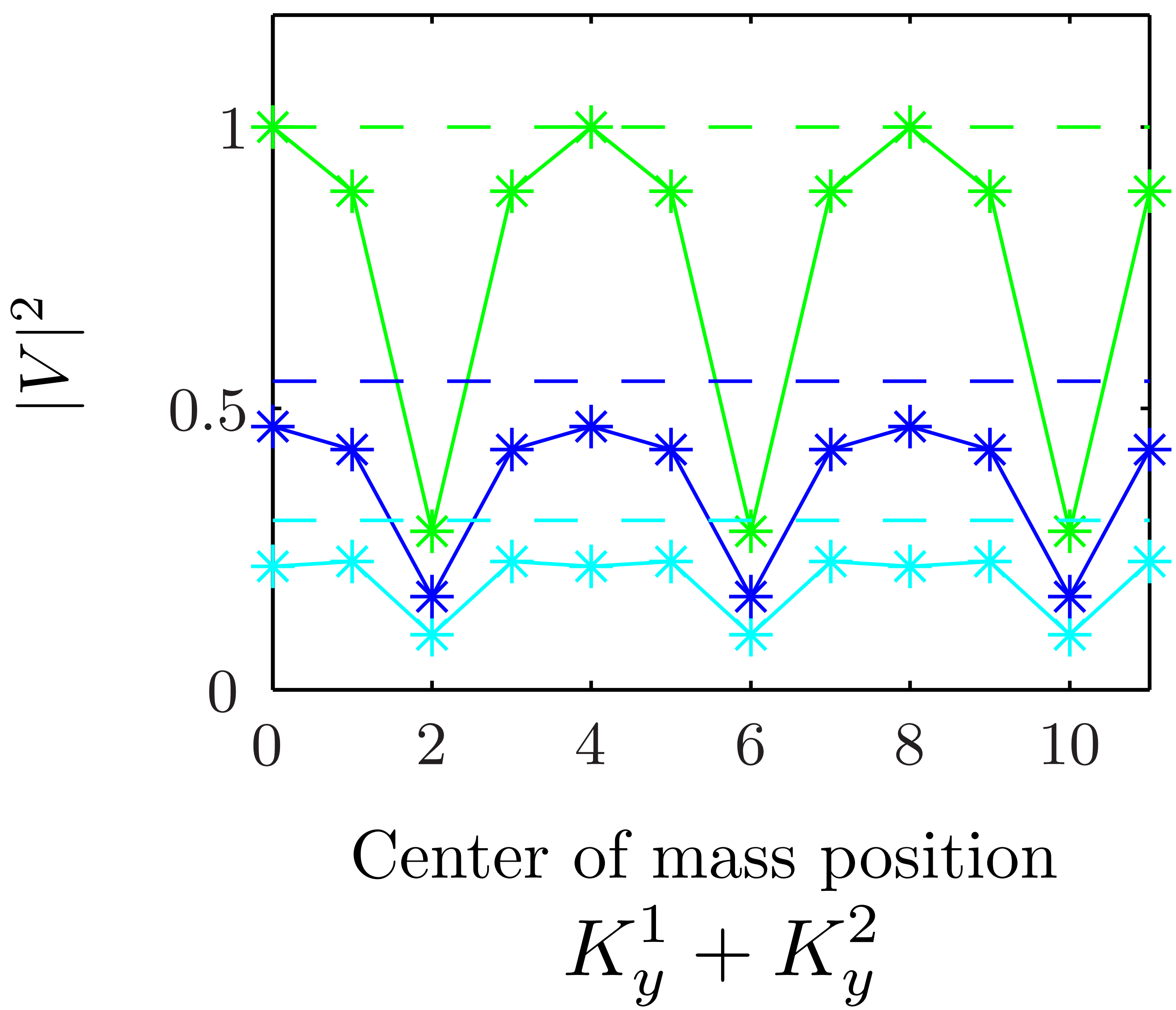
- System shown: $$N = 6,\ L_x \times L_y = 3 \times 4$$

- Matrix elements differ in magnitude, but overall similarities are present
- Different block-structure due to non momentum conservation

Reduced translational invariance in K_y

- A closer look at some short range hopping processes





- for FCI: hopping amplitudes depend on position of centre of mass / K_y

Can we map many body states using the Wannier basis?

- Direct equivalence of many body wavefct. as proposed by Qi

$$\mathcal{H}^{\mathrm{FQHE}}|\Psi\rangle = E_0|\Psi\rangle \qquad \mathcal{H}^{\mathrm{FCI}}|\Phi\rangle = E_0'|\Phi\rangle$$

$$|\Psi\rangle = \sum_{\alpha=\{n_{k_y}\}} \gamma_\alpha \prod_{k_y} (\hat{c}^\dagger)^{n_{k_y}} |vac.\rangle \qquad |\Phi\rangle = \sum_{\alpha=\{n_{K_y}\}} \kappa_\alpha \prod_{K_y} (\hat{c}^\dagger)^{n_{K_y}} |vac.\rangle$$

- Does not really work... $\kappa_\alpha \neq \gamma_\alpha$ [but some overlap: e.g. Laughlin *N=10 - O~0.8*]

- But: can now write both states in single Hilbert space with the same overall structure (indexed by K_y, enlarging the space for the torus)

- Can study adiabatic continuity between the two groundstates:

$$\mathcal{H}(x) = (1-x)\mathcal{H}^{\mathrm{FQHE}} + x\mathcal{H}^{\mathrm{FCI}}$$

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$\mathcal{H}^{\mathrm{FC}}$

$|\Psi\rangle =$ ... $^{K_y}\,|vac.\rangle$

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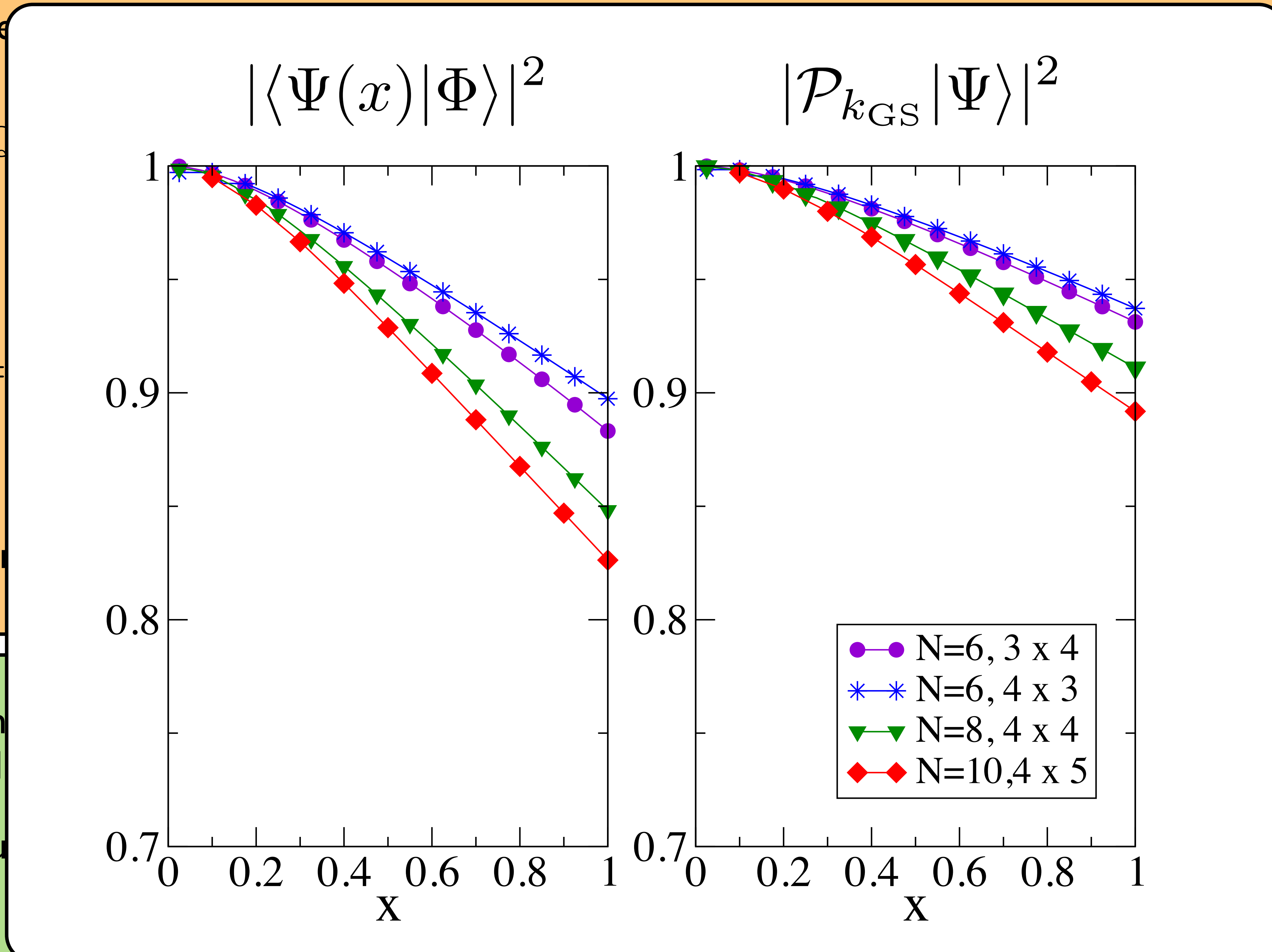

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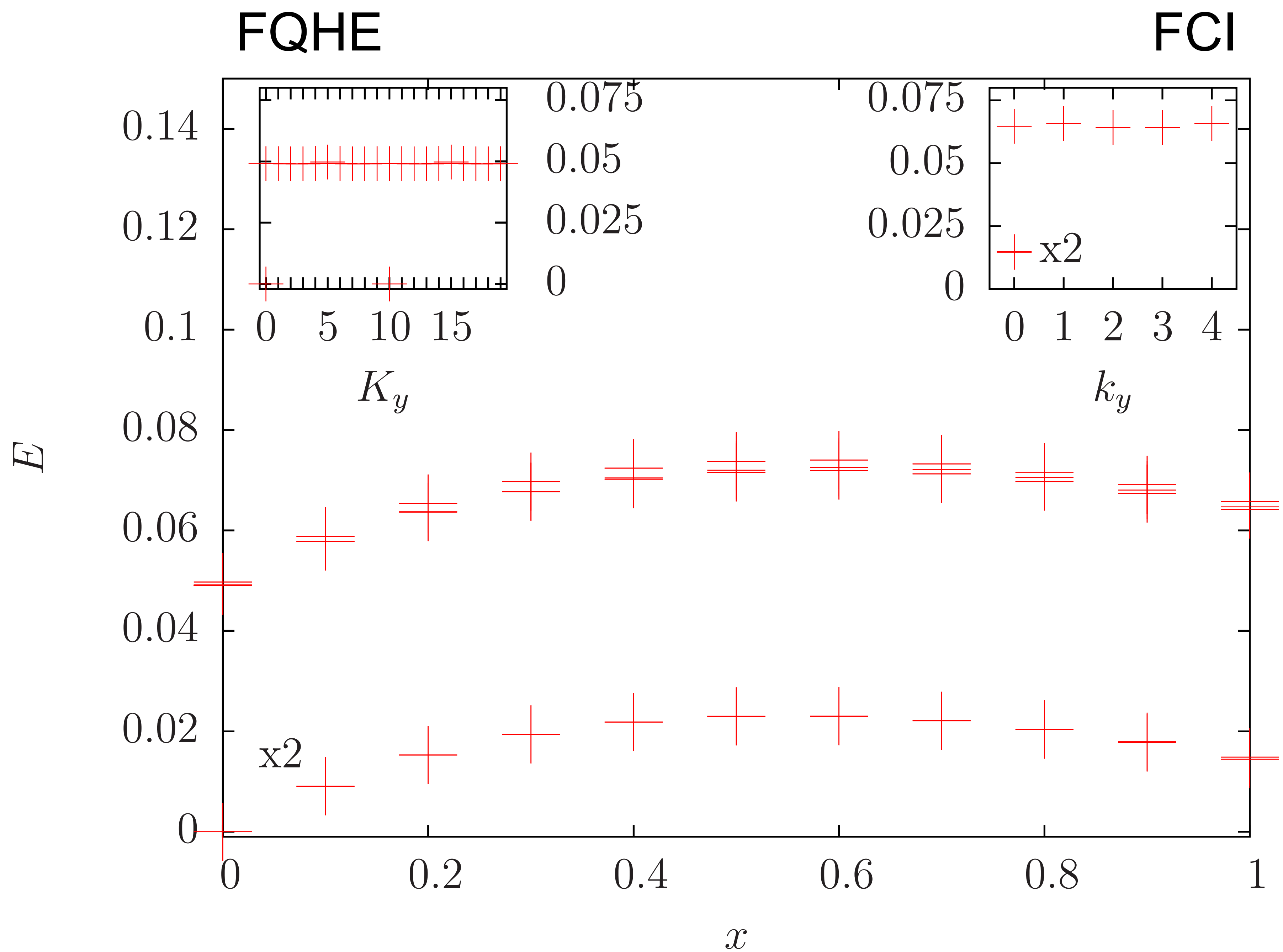
Adiabatic continuation in the Wannier basis

- Spectrum as function of x for N=10 bosons on a 4 x 5 lattice, filling factor $\nu = 1/2$



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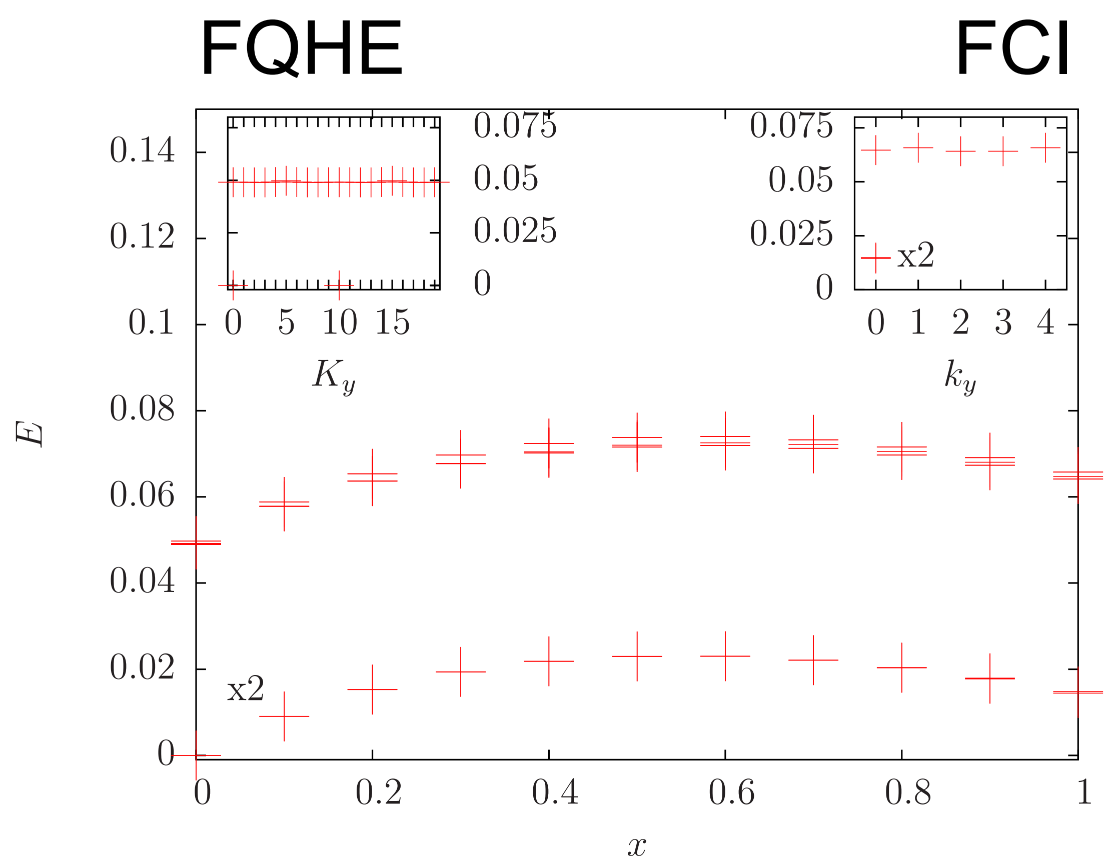
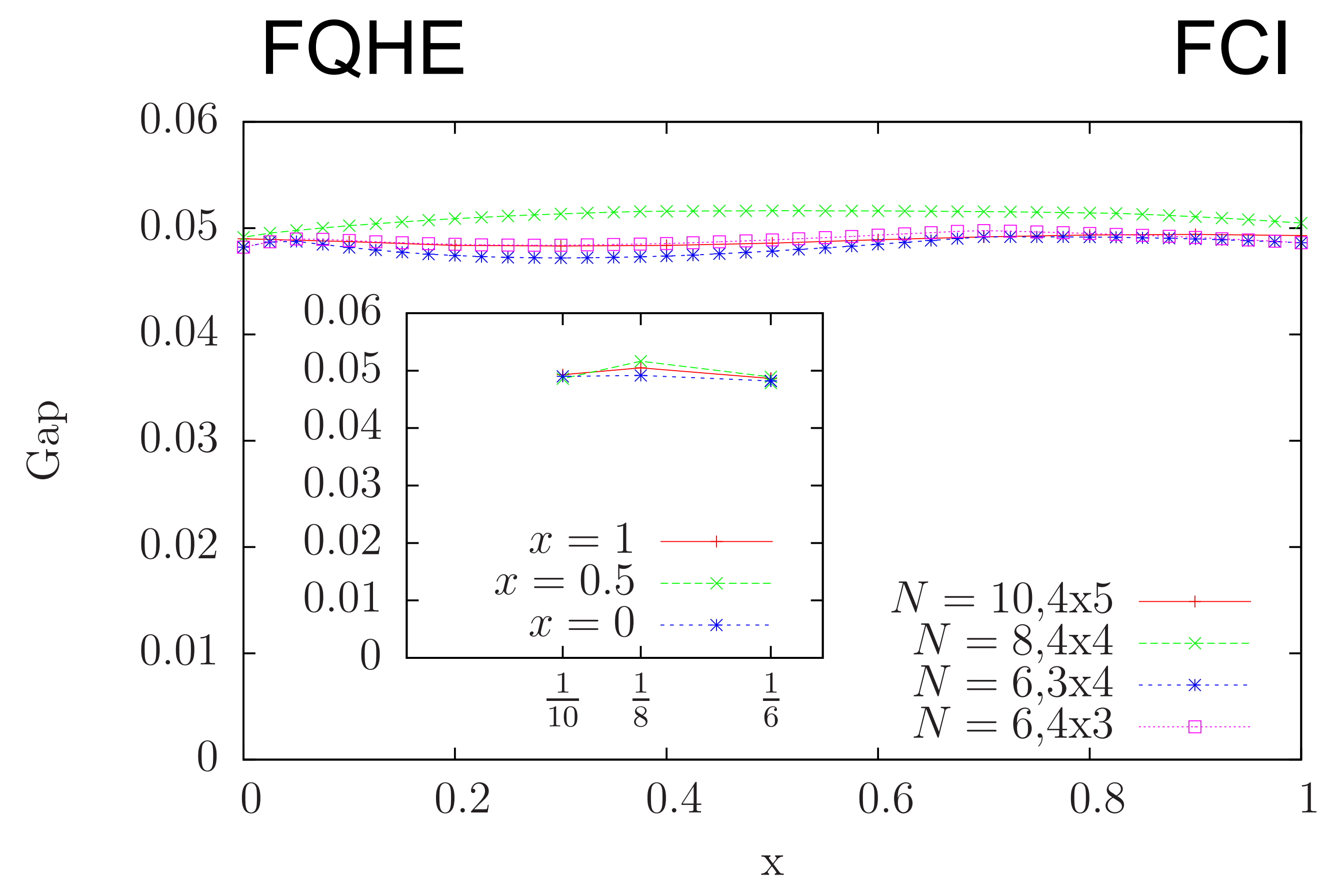


- Gap for different system sizes & aspect ratios:



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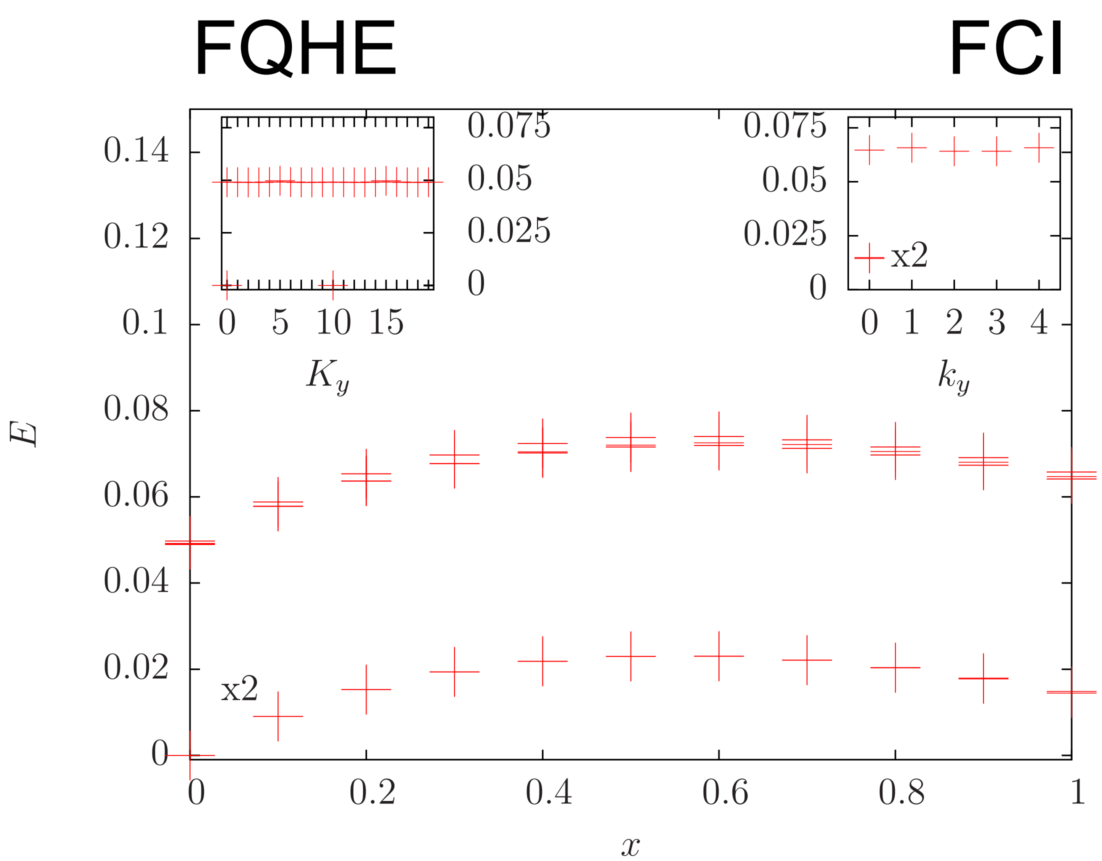

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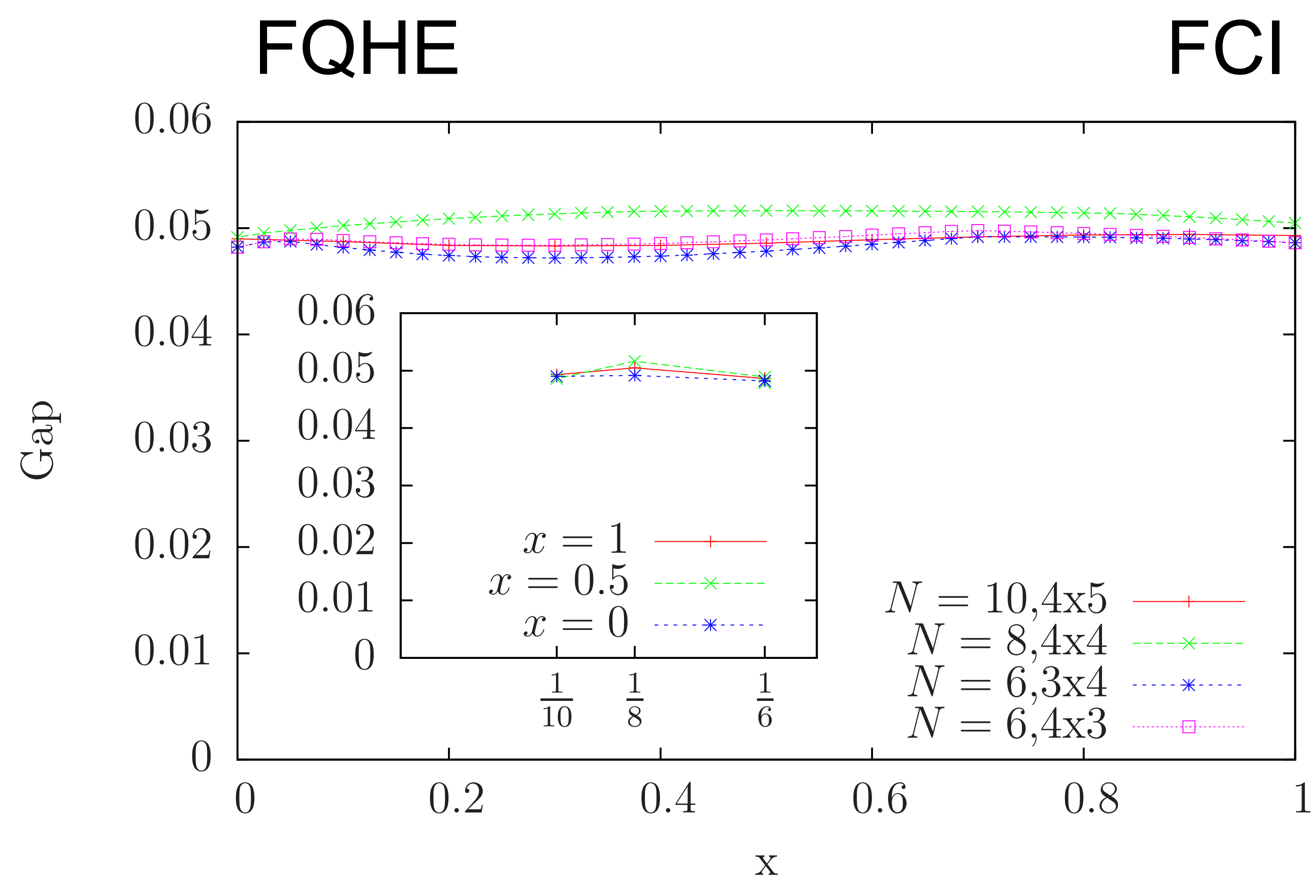

- We confirm the Laughlin state is adiabatically connected to the groundstate of the Haldane model

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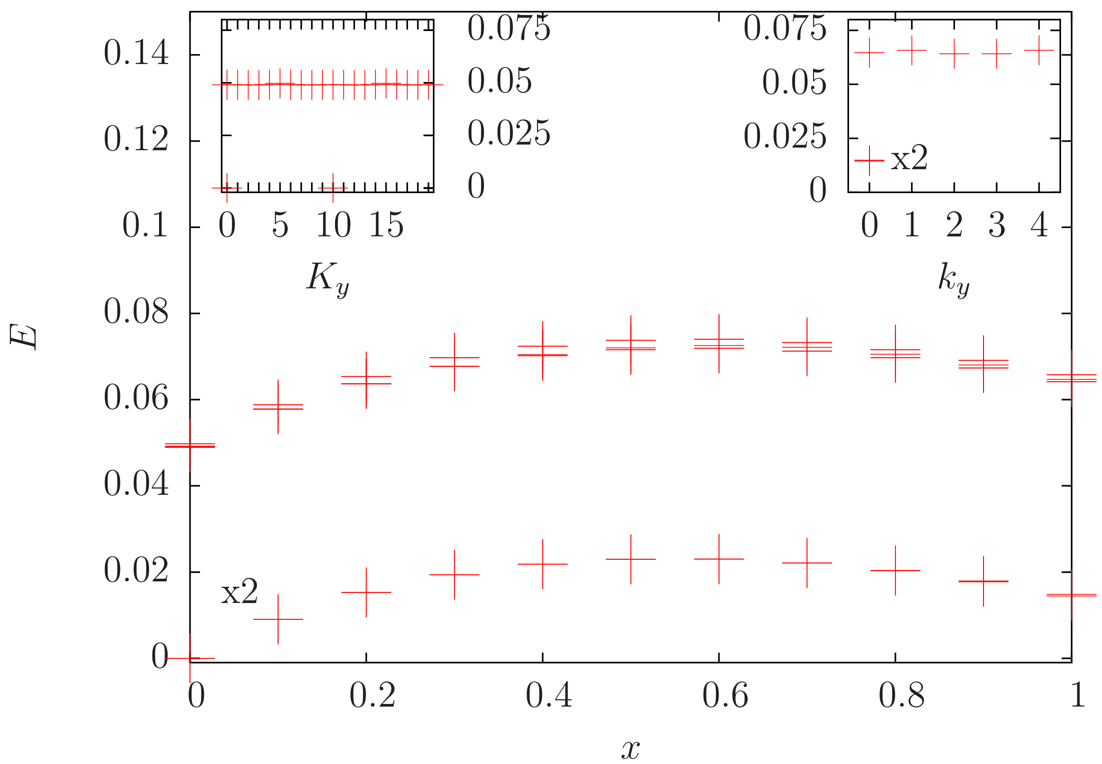
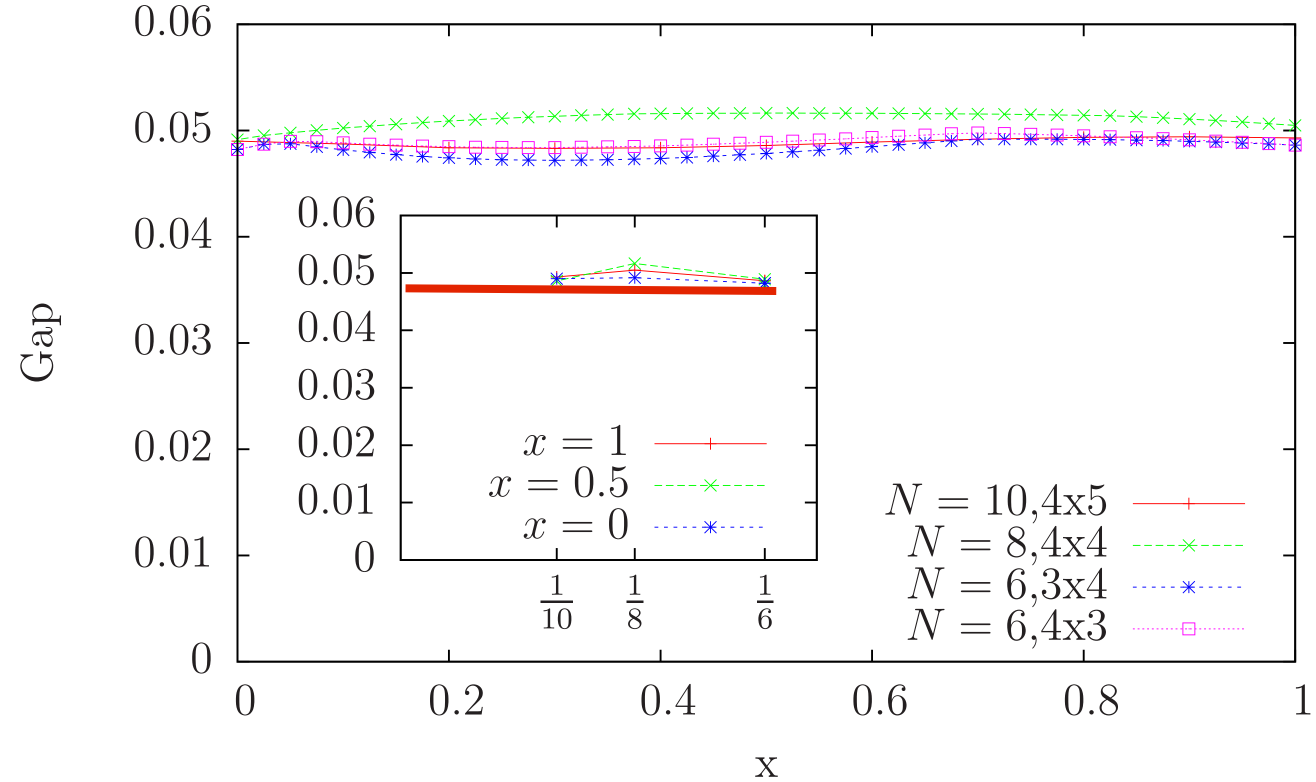
Adiabatic continuation in the Wannier basis

- Spectrum for N=10:



- Gap for different system sizes & aspect ratios:



- We confirm the Laughlin state is adiabatically connected to the groundstate of the Haldane model

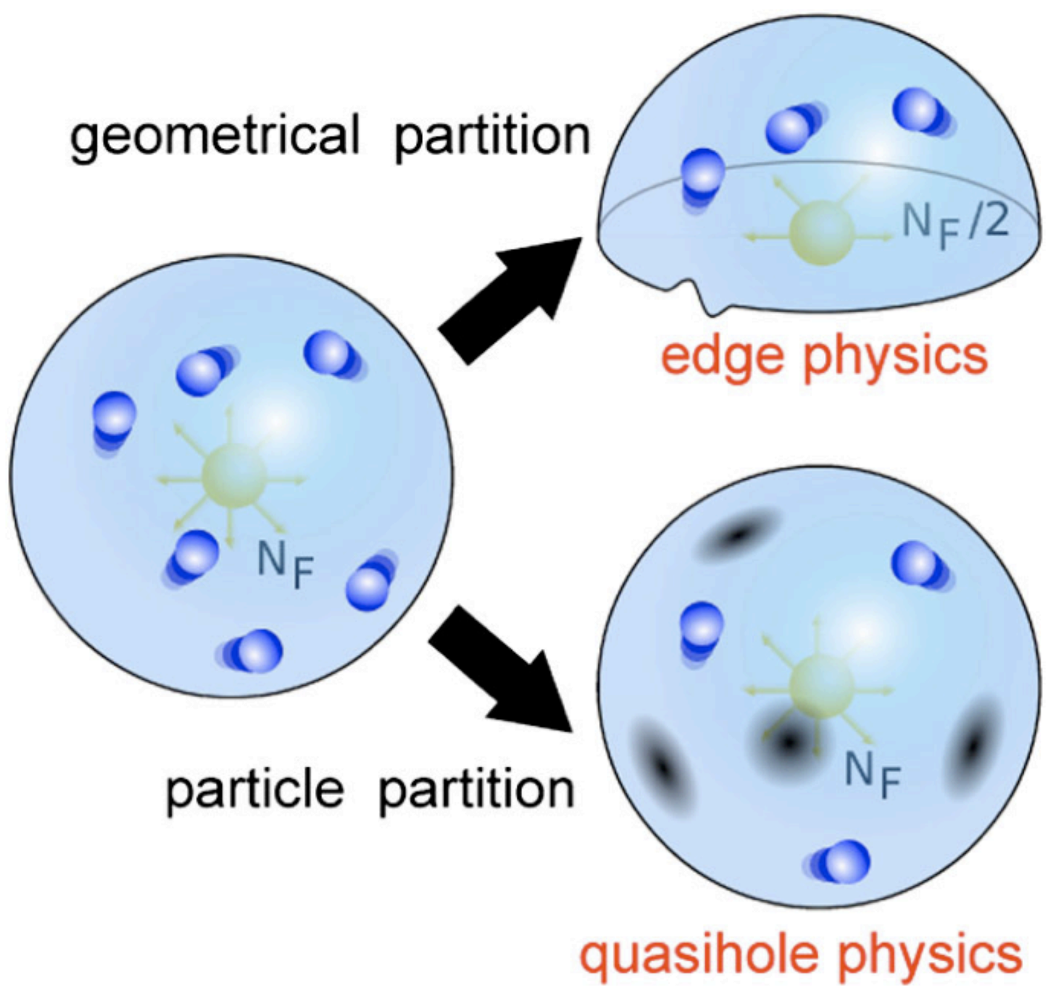
- Clean extrapolation to the thermodynamic limit - (unlike overlaps)

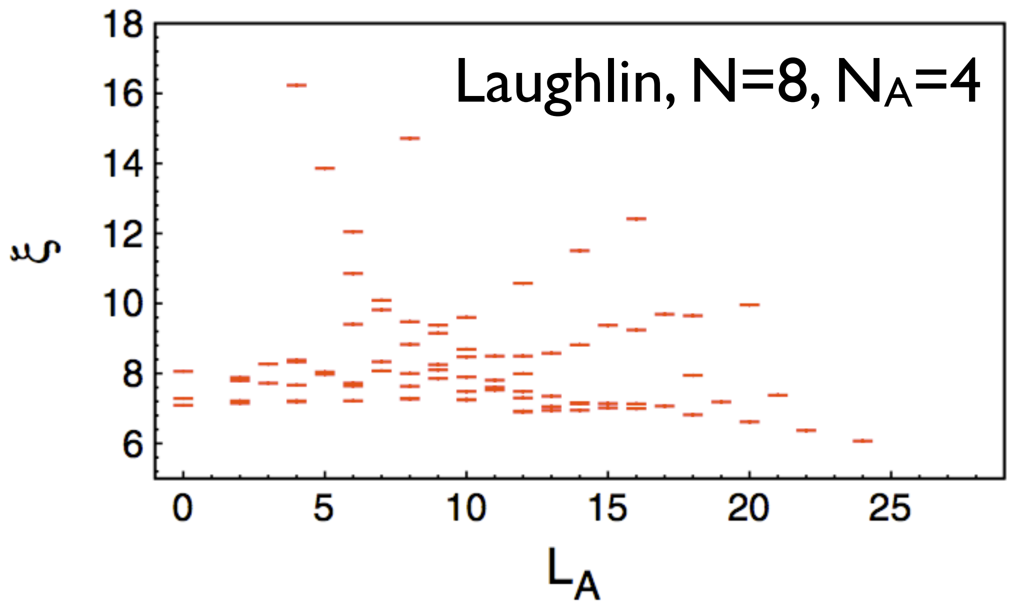
T. Scaffidi, GM, arxiv:1207.3539

Entanglement spectra and quasiparticle excitations

• Entanglement spectrum: arises from Schmidt decomposition of ground state into two groups A, B
=> Schmidt eigenvalues ξ plotted over quantum numbers for symmetries within each block

$$|\Psi\rangle = \sum_{\varpi} \sum_{i} e^{-\xi_{\varpi,i}/2} |\Psi^A_{\varpi,i}\rangle \otimes |\Psi^B_{\varpi,i}\rangle$$





Dominant (universal) eigenvalues of PES yield count of excited states - and their wavefunctions - from groundstate wavefunction only!

credit: Sterdyniak et al. PRL 2011

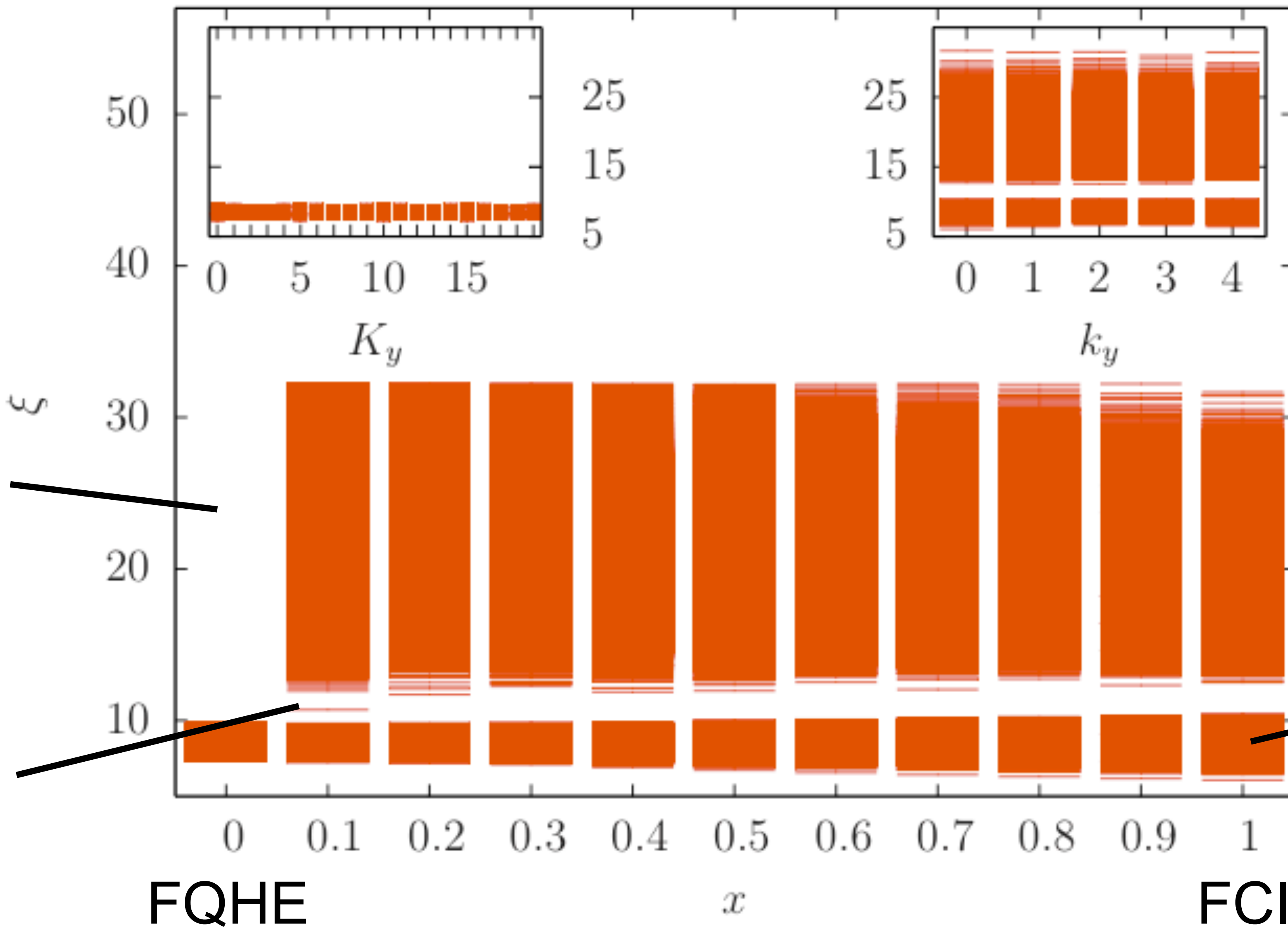
Adiabatic continuation of the entanglement spectrum

Total #eigenvalues below entanglement gap
= 4x(201 + 200 + 200 + 200 + 200)

Total #eigenvalues below entanglement gap
= 804 + 800 + 800 + 800 + 800

$N = 10$

$N_A = 5$



'Infinite' entanglement gap for pure Laughlin state

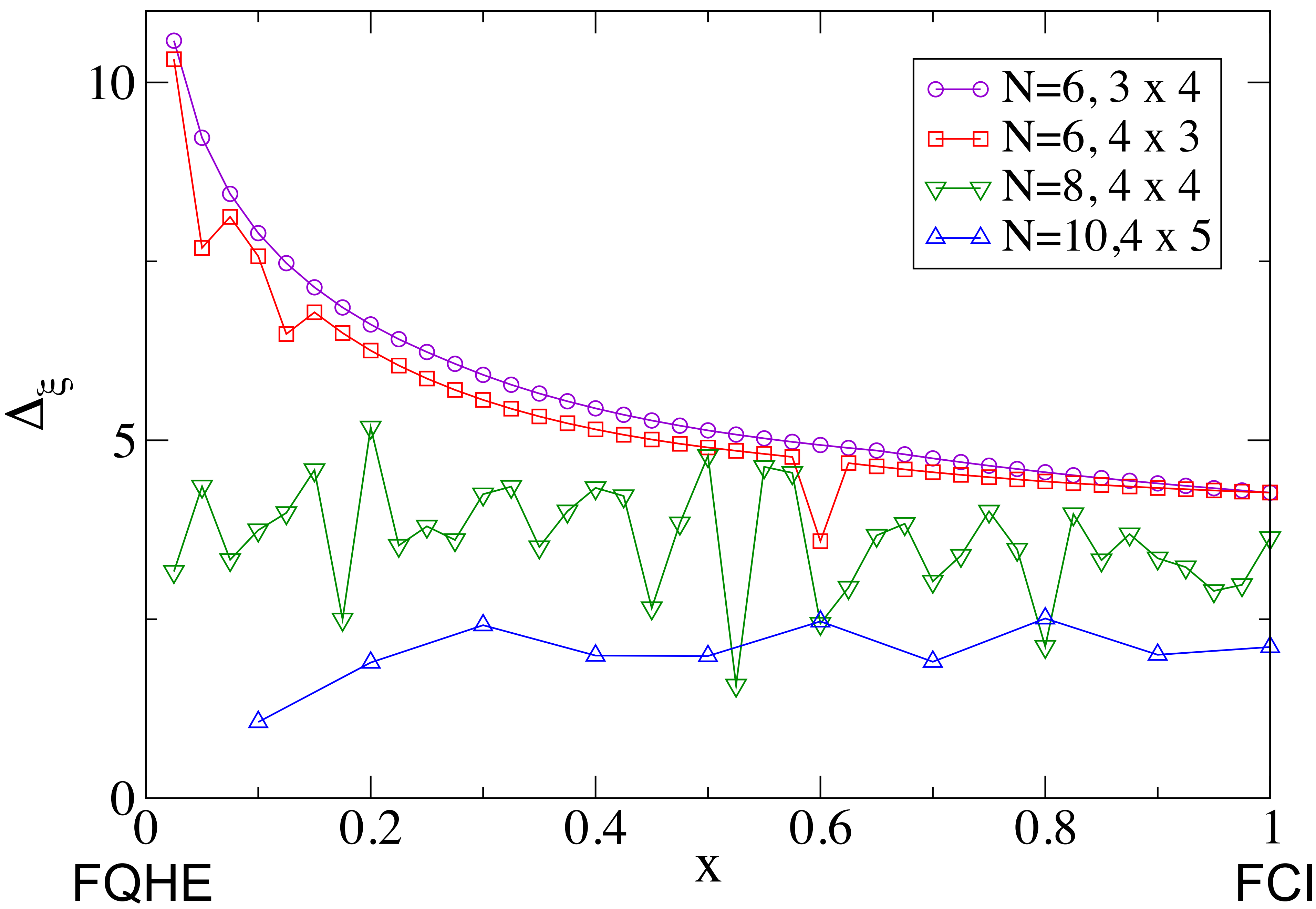
low-lying excited state

Same number of states for all x

Dictionary: FQHE FCI

$$|\Psi\rangle = \sum_{\varpi} \sum_{i} e^{-\xi_{\varpi,i}/2} |\Psi^{A}_{\varpi,i}\rangle \otimes |\Psi^{B}_{\varpi,i}\rangle$$

Finite size behaviour of entanglement gap



- The entanglement gap remains open for all values of the interpolation parameter x
- Finite size scaling behaviour is not so clear (sufficient numerical accuracy?)

Conclusions

• Wavefunctions of FCI's in the Wannier basis are similar but not identical to FQH states in the Landau gauge

• We demonstrated the adiabatic continuity of the ground states at $\nu=1/2$ using Qi's mapping between Wannier basis and FQH eigenstates

• FCI wavefunctions from Qi's construction not very accurate

several formal problem with Qi's Wannier states fixed by proper construction of gauge, see: Wu, Regnault, Bernevig, arxiv:1206.5773.

T. Scaffidi, GM, arxiv:1207.3539

Conclusions

Analytic continuation of FCI states to FQHE wavefunctions represents a new tool for the identification of strongly correlated phases in Chern bands

Can make a robust identification of phases which can be confidently extrapolated to the thermodynamic limit

T. Scaffidi, GM, arxiv:1207.3539