# Heat equation approach to geometric changes in the torus Laughlin-state

Nordita

Stockholm, August 15, 2012

Alexander Seidel

Washington University in St. Louis

CMI Center for Materials Innovation

Collaborators:

Zhenyu Zhou (WashU)

Zohar Nussinov (WashU)

NSF

# Motivation: Parent Hamiltonians for FQH states

Landau-level projected local interactions

Prime example: $V_1$ Haldane pseudo-potential

1st quantized

$$\psi_{1/3}(z_1, \ldots, z_N)$$

$$\hat{V}_1 = P_{\mathrm{LLL}}\, \nabla^2 \delta^2(r_1 - r_2)\, P_{\mathrm{LLL}}$$

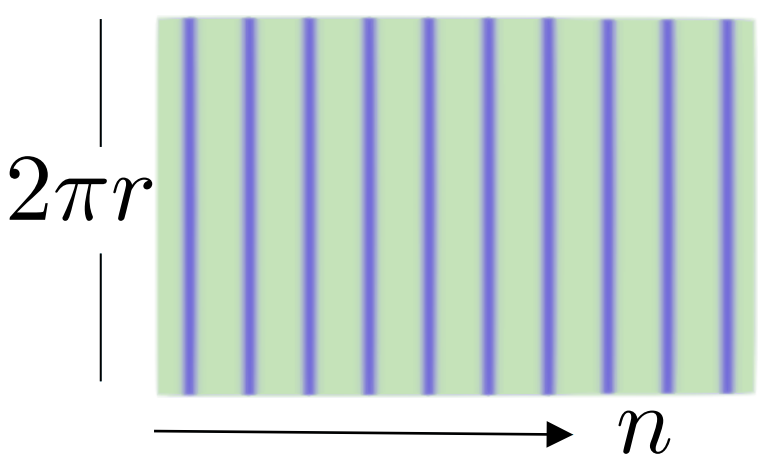
2nd quantized

$$\langle 0 | c_{n_1} \ldots c_{n_N} | \psi_{1/3} \rangle$$

$$\hat{V}_1 = \sum_R Q_R^\dagger Q_R$$

$$Q_R = \sum_x x \exp(-x^2/r^2)\, c_{R-x} c_{R+x}$$

(cylinder)



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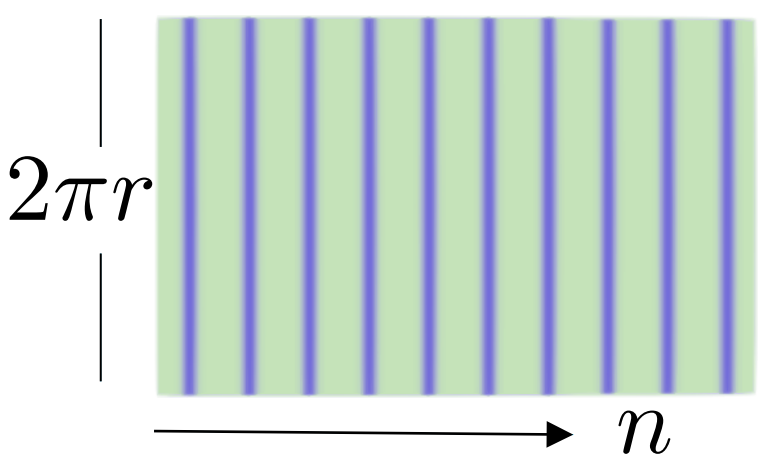


1. Translationally invariant 1D ``lattice'' model
   Use in flat band solid state systems:
   H. Wang, V. W. Scarola, PRB '11
   X.-L. Qi, PRL '11

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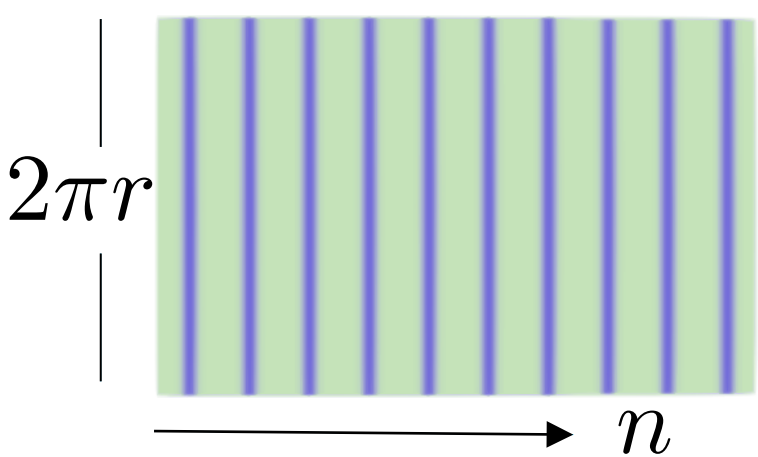


1. Translationally invariant 1D ``lattice'' model
2. ``Frustration free''

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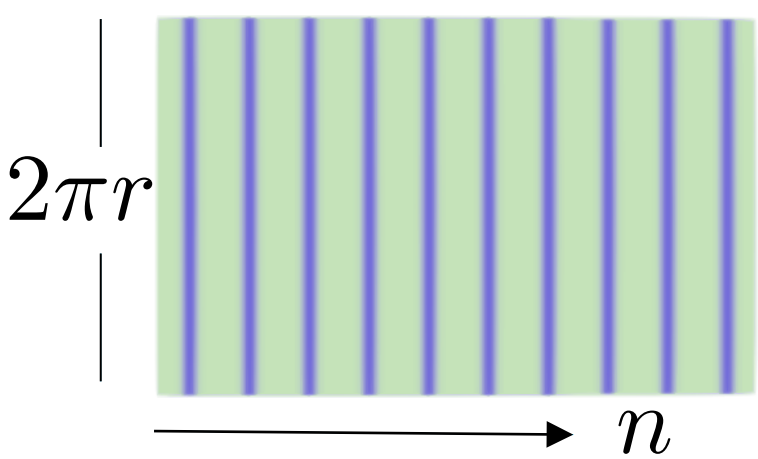



1. Translationally invariant 1D ``lattice'' model
2. ``Frustration free''
3. Give description of physics ``in Hilbert space''

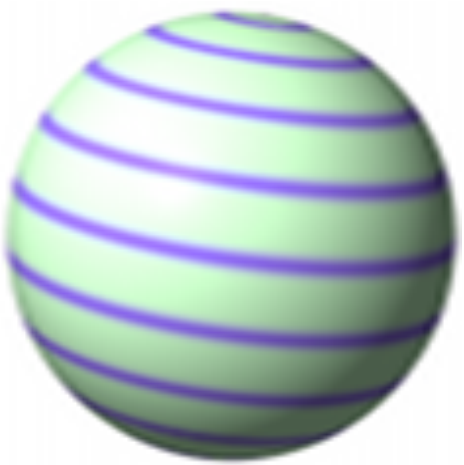
FDM Haldane, APS talk, March '12

FDM Haldane, PRL '11

R-Z Qiu, FDM Haldane, X Wan, K Yang, S Yi, PRB 12

# Motivation: Fock space decomposition of QH states

Example: Laughlin state on the sphere (Haldane PRL 83 )



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$n_k$

$$\psi_{1/3} = \prod_i v_i^{N_\Phi} \prod_{k<l} \left( \frac{u_k}{v_k} - \frac{u_l}{v_l} \right)^3$$

$$= \sum_{\{n_k\}} C_{\{n_k\}} \prod_k u_k^{n_k} v_k^{N_\Phi - n_k}$$

$\phi_n \propto u^n v^{N_\Phi - n}$

$u = e^{i\phi/2} \cos\theta/2 \quad v = e^{-i\phi/2} \sin\theta/2$

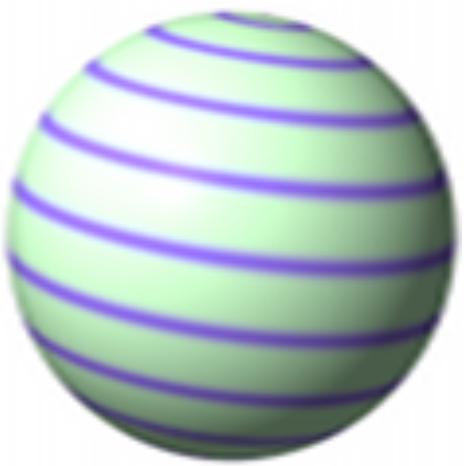
$$\langle 0 | c_{n_1} \ldots c_{n_N} | \psi_{1/3} \rangle = C_{\{n_k\}} \times \text{normalization}$$

The $C_{\{n_k\}}$ are known recursively by the general connection between various QH states and Jack polynomials.

B.A. Bernevig,
F.D.M. Haldane,
PRL 08, PRB 08

# Motivation: Fock space decomposition of QH states

Example: Laughlin state on the sphere (Haldane PRL 83 )



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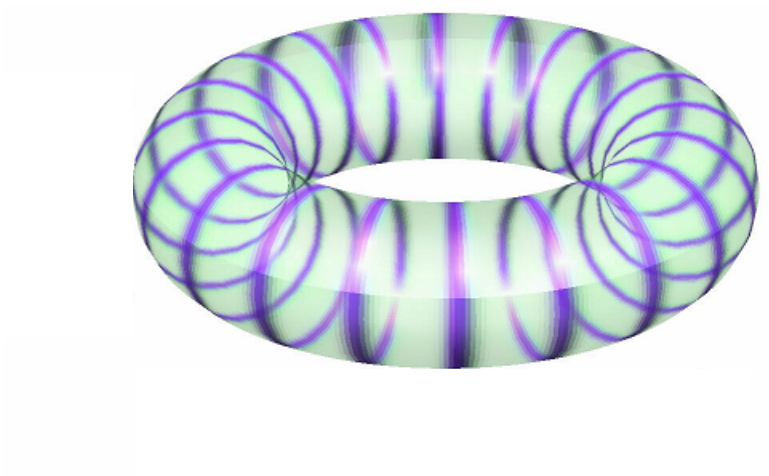
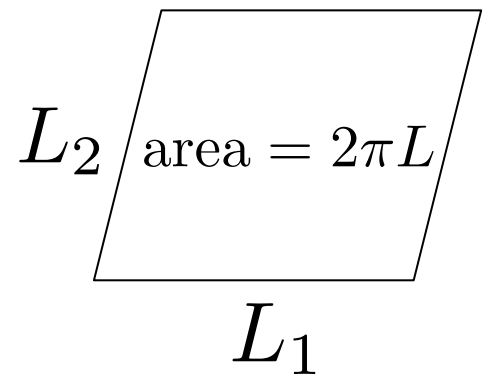
1001001001001001001001001001001001001

"inward squeezing"

# Motivation: Fock space decomposition of QH states

Example: Laughlin state on the torus (Haldane, Rezayi, PRB 85 )






$$\tau = L_2/L_1$$

$$\mathrm{Im}\, L_2 > 0$$

$$\psi_{1/q}(z_1 \ldots z_N) = \exp(-\frac{1}{2}\sum_k y_k^2)\, F_{\ell=0\ldots q-1}(z_1 + \ldots + z_N) \prod_{i<j} \theta_1(\frac{z_i - z_j}{L_x}, \tau)^q$$

$$\hat{V}_1 = \sum_{0 \le R \in \frac{1}{2}\mathbb{Z} < L} Q_R^\dagger Q_R$$

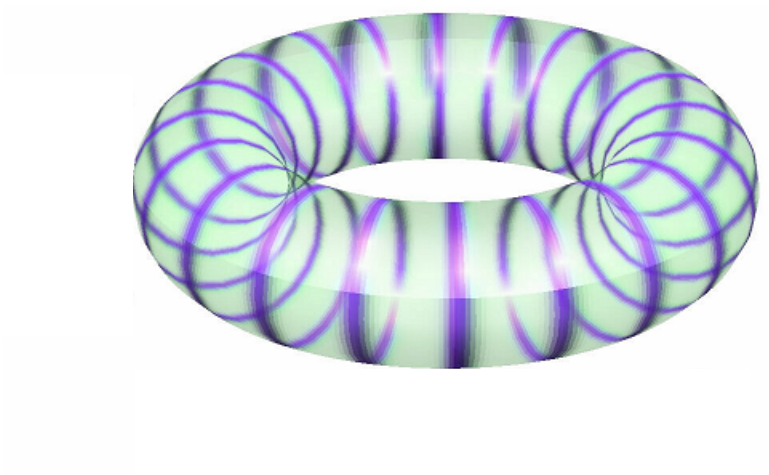
$$Q_R = \sum_{\substack{0<x<L/2 \\ x+R\in\mathbb{Z}}} \sum_{m\in\mathbb{Z}} (x+mL) \exp[\frac{2\pi i\tau}{L}(x+mL)^2]\, c_{R-x} c_{R+x}$$

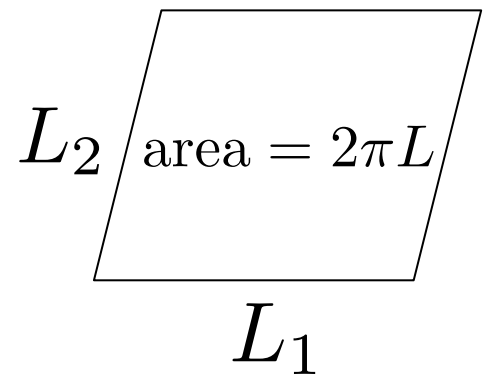
$$c_n \equiv c_{n+L}$$

Periodized version of the cylinder interaction

# Motivation: Fock space decomposition of QH states

Example: Laughlin state on the torus






$$\tau = L_2/L_1$$

$$\mathrm{Im}\, L_2 > 0$$

$$1001001001001001 \xrightarrow[\tau]{} \langle 0 | c_{n_1} \ldots c_{n_N} | \psi_{1/3} \rangle$$

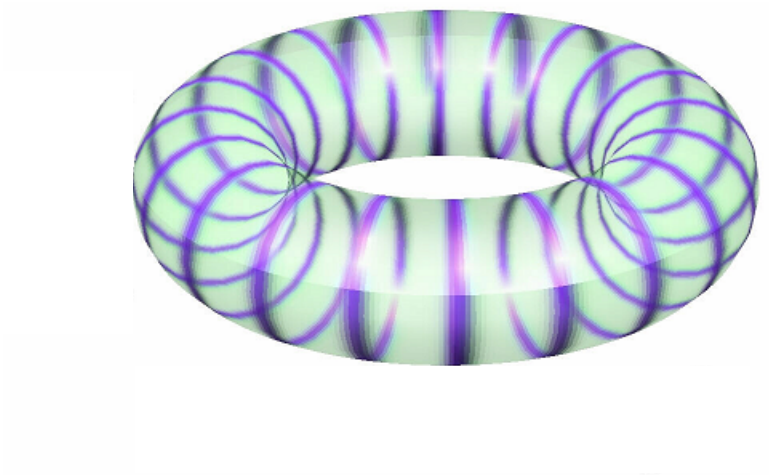
"thin torus limit"

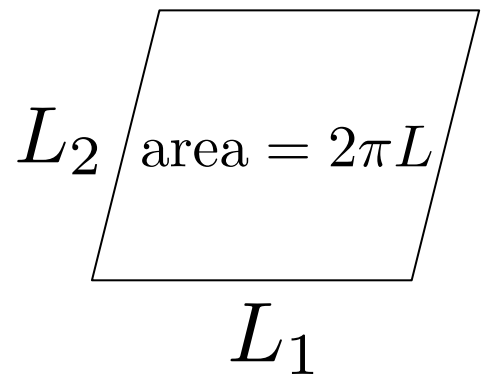
EJ Bergholtz, A Karlhede, PRL 94 '05

AS, H Fu, D-H Lee, JM Leinaas, JE Moore, PRL 95 '05

# Motivation: Fock space description of QH states

## Example: Laughlin state on the torus





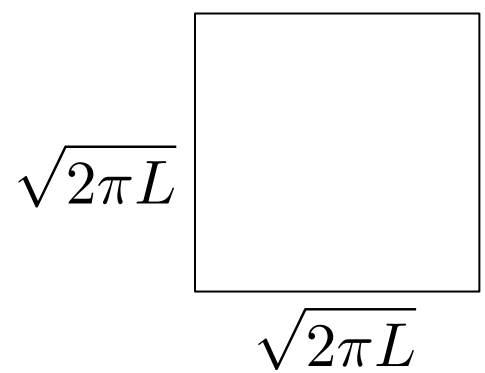

$$\tau = L_2/L_1$$

$$\mathrm{Im}\, L_2 > 0$$

$$g_{\mu\nu} = \mathrm{id}$$

Alternative view: change of metric

$$H = \frac{1}{2}\sum_{i=1}^{N} g^{\mu\nu}\pi_{i\mu}\pi_{i\nu} + P_{\mathrm{LLL}} \sum_{i<j} V(g_{\mu\nu}x_{ij}^{\mu}x_{ij}^{\nu})\, P_{\mathrm{LLL}}$$




$$1$$

$$g_{\mu\nu} \neq \mathrm{id}$$

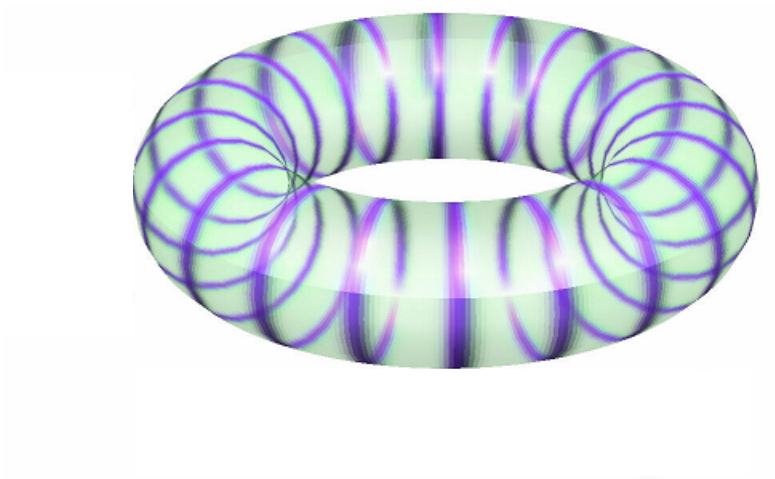
Connection with “Hall viscosity”:

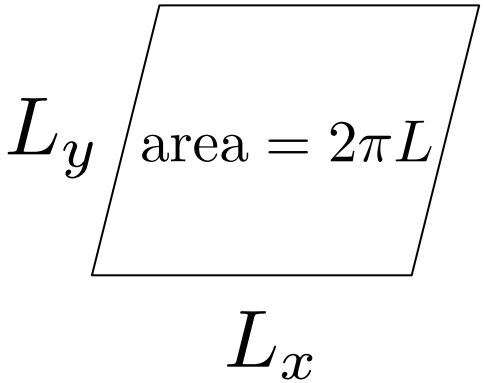
JE Avron, R. Seiler, PG Zograf PRL 95;
N. Read, PRB 09
FDM Haldane, arXiv:0906.1854
N. Read, EH Rezayi, PRB 10

# Outline

- Motivation
  - Understand structure of Fock space decomposition of torus Laughlin states
    - $\tau$-dependence
    - relation to root pattern
- Heat equation for $\tau$ - evolution of Laughlin states
  - 2-body operator as generator for $\tau$ -evolution of coefficients
  - presentation of torus-Laughlin state in terms of root pattern
- Application: Hall viscosity
- Conclusion

# $\tau$ -dependence of Laughlin state in Fock space






$$\tau = L_y / L_x$$

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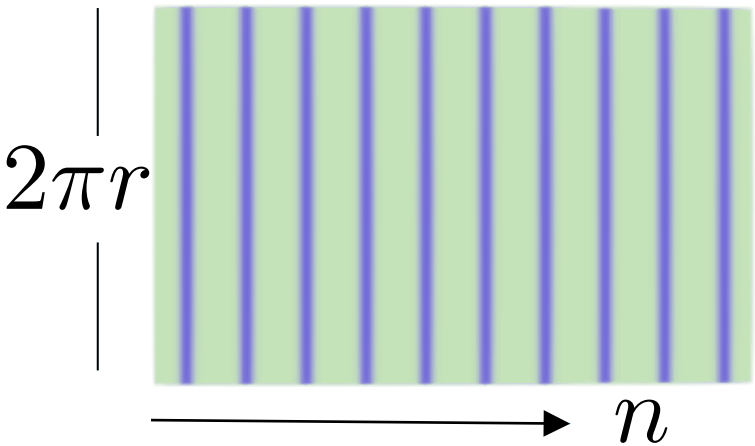
1001001001001001 $\xrightarrow{\tau}$ $\langle 0 | c_{n_1} \ldots c_{n_N} | \psi_{1/3} \rangle$

# A look at the cylinder




$$1001001001001001 \xrightarrow[r]{} \langle 0 | c_{n_1} \ldots c_{n_N} | \psi_{1/3} \rangle$$

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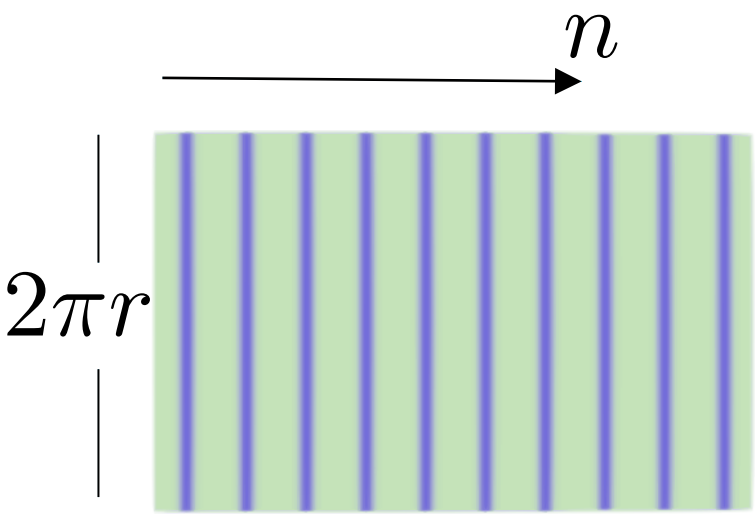


$$\phi_n = \xi^n e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}n^2/r^2}$$

$$\xi = e^{z/r}$$

$$\psi_{1/3} = \prod_{i<j} (\xi_i - \xi_j)^3 \times e^{-\frac{1}{2}\sum_k x_k^2}$$

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"inward squeezing"

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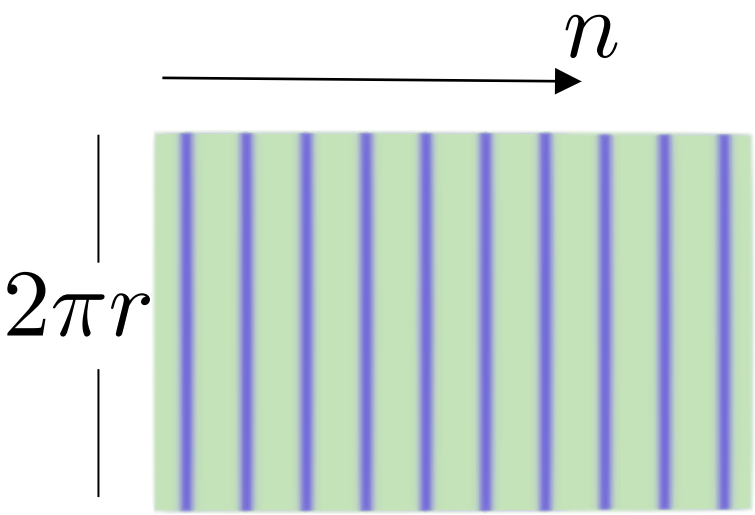


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Rezayi & Haldane, PRB 94

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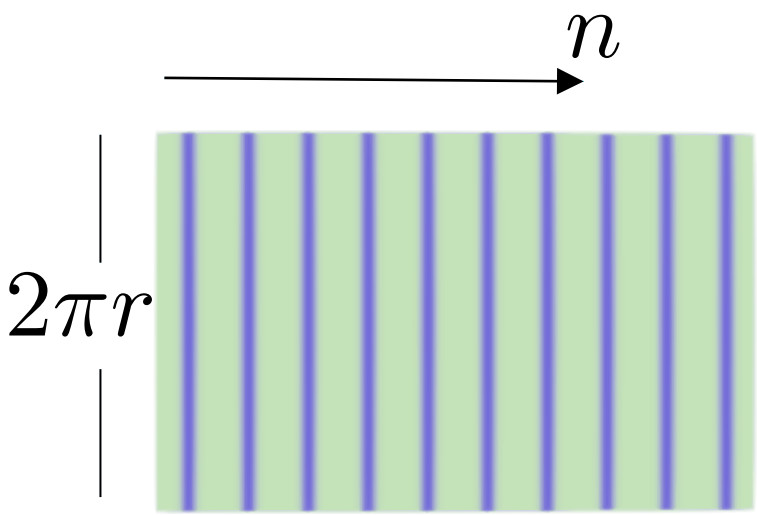


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$$Q_R = \sum_x x \exp(-x^2/r^2)\, c_{R-x} c_{R+x}$$

$$Q_R\, \psi_{1/3} = 0, \text{ all } R$$

Gives $r-$independent conditions on the $C_{\{n_k\}}$ 's.

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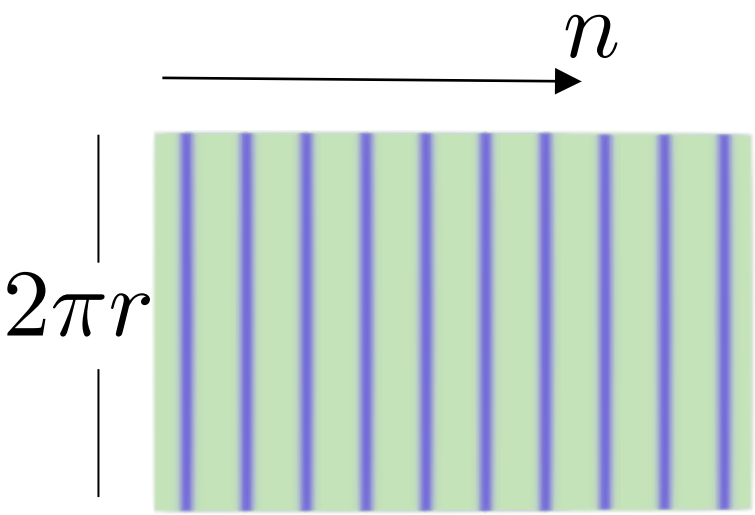


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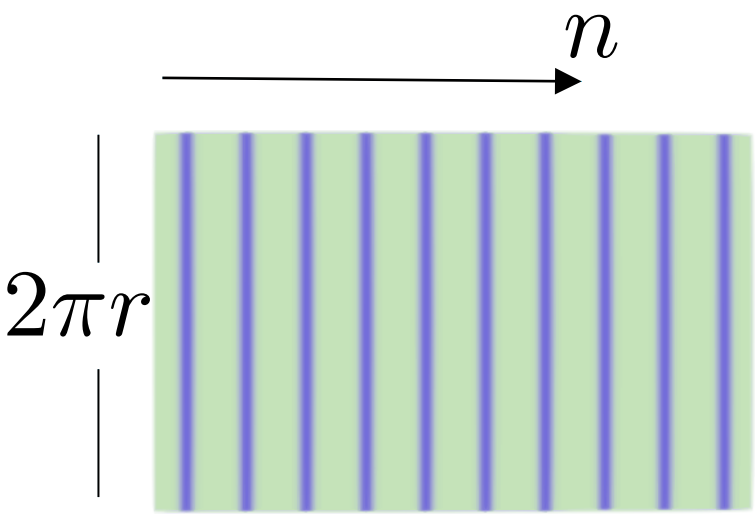
Geometric changes in the Fock space description of cylinder quantum Hall states are generated by a simple single-body operator:

$$G_{r^{-2}} = \frac{1}{2}\sum_n n^2 c_n^\dagger c_n$$

(This is a consequence of the polynomial structure and is not specific to the Laughlin state!)

# A look at the cylinder

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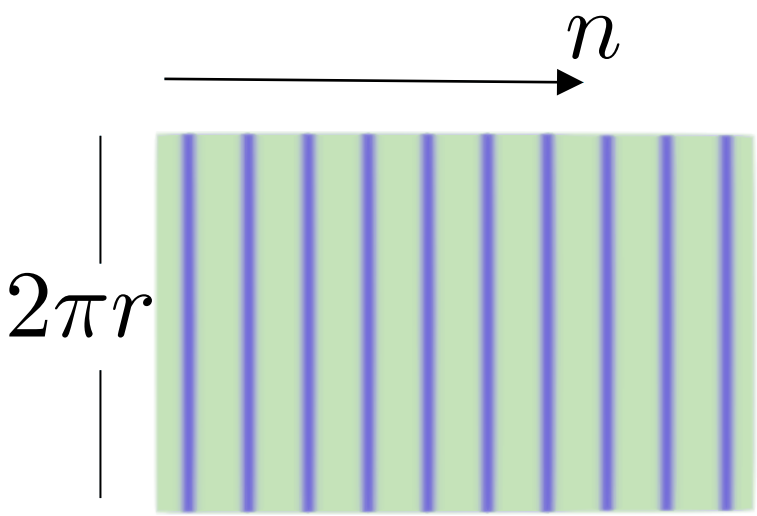


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$$\phi_n = \xi^n e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}n^2/r^2}$$

Landau level orbitals for different $r$ identified!

$$\xi = e^{z/r}$$

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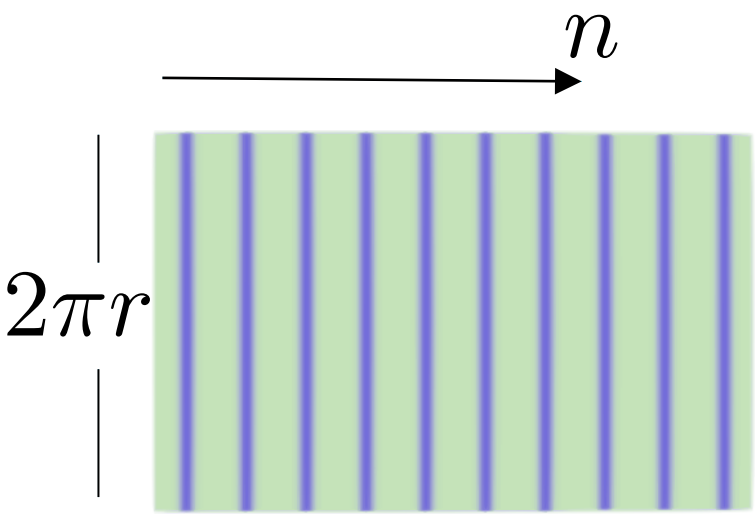


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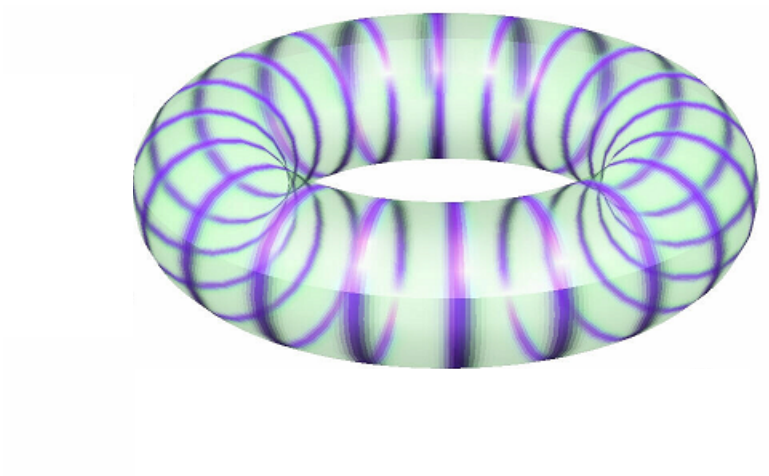
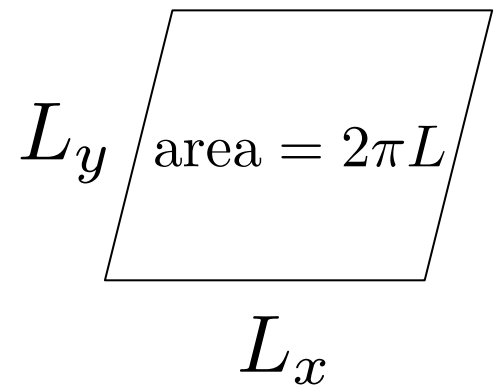
$$G_{r^{-2}} = \frac{1}{2}\sum_n n^2 c_n^\dagger c_n$$

Note, however, that this does <u>not</u> remain meaningful in the thin cylinder limit $r = 0$, in which the ket $|\psi_{1/3}(r)\rangle$ becomes $|100100100\ldots\rangle$.

# Back to the torus






$$\tau = L_y / L_x$$

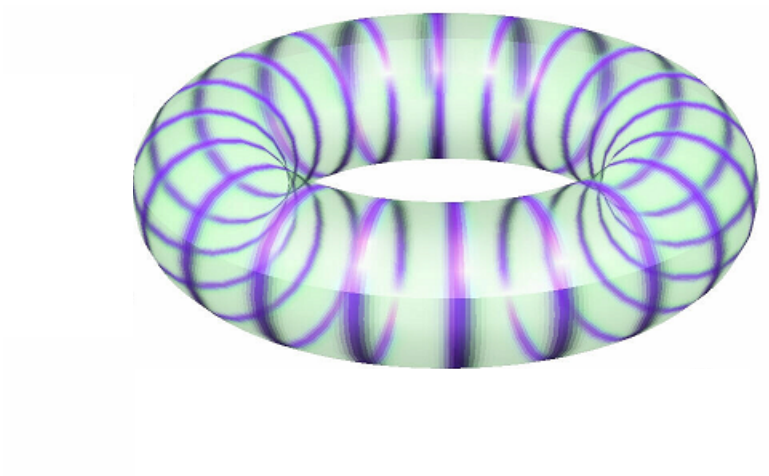
$$\mathrm{Im}\, L_y > 0$$

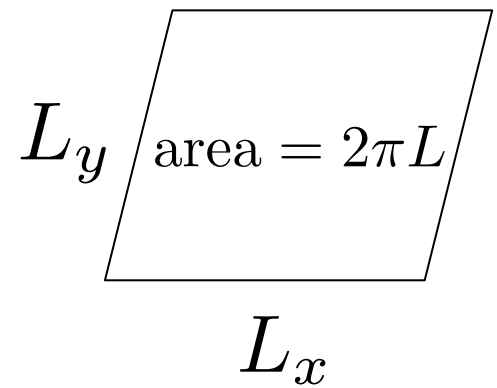
$$|\psi^{\ell}_{1/q}(\tau)\rangle = \sum_{\{n_k\}} C_{\{n_k\}}(\tau)\; c^{\dagger}_{n_N} \ldots c^{\dagger}_{n_1} |0\rangle$$

$$d|\psi^{\ell}_{1/q}(\tau)\rangle = d\tau\, G_{\tau} |\psi^{\ell}_{1/q}(\tau)\rangle$$

Want to define $G_{\tau}$ such that it generates the change with $\tau$ in the “guiding center” description of the Laughlin state.

# Back to the torus






$$\tau = L_y / L_x$$

$$\mathrm{Im}\, L_y > 0$$

$$|\psi^{\ell}_{1/q}(\tau)\rangle = \sum_{\{n_k\}} C_{\{n_k\}}(\tau)\; c^{\dagger}_{n_N} \ldots c^{\dagger}_{n_1} |0\rangle$$

$$d|\psi^{\ell}_{1/q}(\tau)\rangle = d\tau\, G_{\tau} |\psi^{\ell}_{1/q}(\tau)\rangle$$

We may assume (without loss of generality) that such $G_\tau$ is symmetric with respect to magnetic translations on the torus. This shows that unlike for the cylinder, this generator *cannot* be a single body operator!

(It would then have to be proportional to the particle number operator.)

# Differential equation for $\tau$ –dependence of torus Laughlin state

$$\psi_{1/q}(z_1 \ldots z_N, \tau) = \exp(-\frac{1}{2}\sum_k y_k^2)\, F_{\ell=0\ldots q-1}(\underbrace{z_1 + \ldots + z_N}_{Z}) \prod_{i<j} \theta_1(\frac{z_i - z_j}{L_x}, \tau)^q$$

$$F_\ell(Z) = \theta \begin{bmatrix} \frac{\ell}{q} + \frac{L-q}{2q} \\ -\frac{L-q}{2} \end{bmatrix} (qZ/L_x, q\tau)$$

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$$\partial_\tau \psi_{1/q} = e^{-\frac{1}{2} y_k y_k} \left( (\partial_\tau F_\ell) f_{rel} + F_\ell \partial_\tau f_{rel} \right)$$

Heat equation for center-of-mass factor:

$$\partial_\tau F_\ell(Z, \tau) = \frac{1}{4\pi i q} \partial_Z^2 F_\ell(Z, \tau)$$

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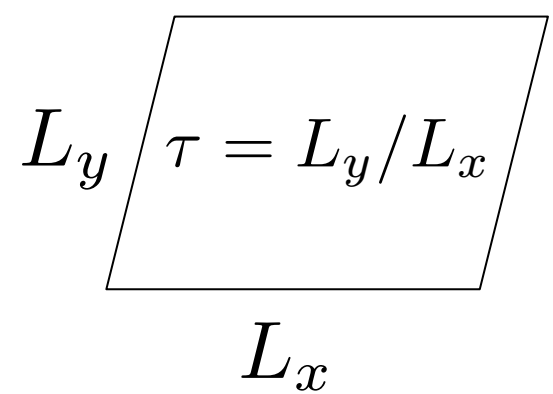
# Differential equation for $\tau$ –dependence of torus Laughlin state

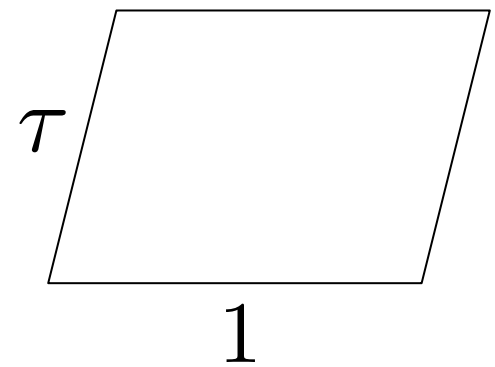
$$\partial_\tau \psi_{1/q}(\tau) = \left[ \frac{1}{4\pi i q} \partial_Z^2 + q \sum_{i<j} \frac{\partial_\tau \theta_1(z_i - z_j, \tau)}{\theta_1(z_i - z_j, \tau)} \right] \psi_{1/q}(\tau)$$

- RHS looks like a 2-body operator
- However: As written, the $\psi_{1/q}(\tau)$ don't really live in the same Hilbert space for different $\tau$ .
- Also: The differential equation still encodes the change of the Landau level basis as well as that of the expansion coefficients.



# Solve problem in 1D Hilbert space

$$0 < \mathrm{Im}\tau = 2\pi L l_b^2$$




$$\psi_{1/q}(z_1 \ldots z_N, \tau) = \exp(-\frac{1}{2}\sum_k y_k^2/l_B^2)\, F_{\ell=0\ldots q-1}(Z) \prod_{i<j} \theta_1(z_i - z_j, \tau)^q$$

View this as function of <u>real</u> variables in the interval [0,1] $(y_k \equiv 0)$.
For any $\mathcal{T}$ , the Laughlin state is thus a member of the Hilbert space of square-integrable functions over [0,1], endowed with scalar product

$$\langle \phi | \psi \rangle = \int_0^1 dx\, \phi^*(x)\psi(x)\,.$$

The following (un-normalized) basis of LLL orbitals remains orthogonal after restriction to 1D:

$$\chi_n(z) = e^{-\frac{y^2}{2l_B^2}}\, \theta \begin{bmatrix} n/L \\ 0 \end{bmatrix} (Lz, L\tau)$$

# Differential equation for $\tau$ –dependence of torus Laughlin state

$$\partial_\tau \psi_{1/q} = \left[ \frac{1}{4\pi i q} \partial_X^2 + q \sum_{i<j} \frac{\partial_\tau \theta_1(x_i - x_j, \tau)}{\theta_1(x_i - x_j, \tau)} \right] \psi_{1/q}$$

$$X = x_1 + \ldots + x_N$$

The operator on the RHS is now a well-defined 2-body operator acting within (a dense subspace of) the Fock space derived from square integrable functions on [0,1].

# Differential equation for $\tau$–dependence of torus Laughlin state

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$X = x_1 + \ldots + x_N$

$$\begin{tikzcd}
\mathcal{H} & & \mathcal{H} \\
\mathcal{L}_\tau \arrow[u, hook] \arrow[dr, leftrightarrow] & & \mathcal{L}_{\tau'} \arrow[u, hook] \arrow[dl, leftrightarrow] \\
& \mathcal{L} \arrow[d] & \\
& \mathcal{L} &
\end{tikzcd}$$

$c_n^\dagger \to N_n(\tau) c_n^\dagger$

# Differential equation for $\tau$–dependence of torus Laughlin state

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\end{tikzcd}$$

$$\tilde{G}_\tau = \sum_{mm'nn'} G_{mm'nn'}\, c_m^\dagger c_{m'}^\dagger c_n c_n'$$

$$c_n^\dagger \to N_n(\tau) c_n^\dagger$$

$$G_\tau = \sum_n \frac{\partial_\tau N_n(\tau)}{N_n(\tau)} c_n^\dagger c_n + \sum_{mm'nn'} \frac{N_m(\tau) N_{m'}(\tau)}{N_n(\tau) N_{n'}(\tau)} G_{mm'nn'}\, c_m^\dagger c_{m'}^\dagger c_n c_n'$$

# Differential equation for $\tau$ –dependence of torus Laughlin state

$$\partial_\tau \psi_{1/q} = \underbrace{\left[ \frac{1}{4\pi i q} \partial_X^2 + q \sum_{i<j} \frac{\partial_\tau \theta_1(x_i - x_j, \tau)}{\theta_1(x_i - x_j, \tau)} \right]}_{\Delta} \psi_{1/q}$$

$X = x_1 + \ldots + x_N$

$$\psi_{1/q} = I_\tau |\psi_{1/q}\rangle$$

$$\begin{array}{ll} \mathcal{H} & \\ \uparrow & \\ \mathcal{L}_\tau & I_\tau \\ \uparrow & \\ \mathcal{L} & \end{array}$$

# Differential equation for $\tau$ –dependence of torus Laughlin state

$$\partial_\tau \psi_{1/q} = \underbrace{\left[ \frac{1}{4\pi i q} \partial_X^2 + q \sum_{i<j} \frac{\partial_\tau \theta_1(x_i - x_j, \tau)}{\theta_1(x_i - x_j, \tau)} \right]}_{\Delta} \psi_{1/q}$$

$X = x_1 + \ldots + x_N$

$$\begin{array}{c} \mathcal{H} \\ \uparrow \\ \mathcal{L}_\tau \quad I_\tau \\ \uparrow \\ \mathcal{L} \end{array}$$

$$\psi_{1/q} = I_\tau |\psi_{1/q}\rangle$$

$$P_\tau \, \Delta \psi_{1/q} = P_\tau (\partial_\tau I_\tau) |\psi_{1/q}\rangle + I_\tau \tilde{G}_\tau |\psi_{1/q}\rangle$$

$0$ for $\tau$ imaginary

$$\underline{\underline{P_\tau \Delta P_\tau \, \psi_{1/q} = I_\tau \tilde{G}_\tau |\psi_{1/q}\rangle}}$$

# The generator $G_\tau$

$$P_\tau \Delta P_\tau \, \psi_{1/q} \quad = \quad I_\tau \tilde{G}_\tau |\psi_{1/q}\rangle \qquad (\tau \ \text{imaginary})$$

This implies that the matrix elements of $\tilde{G}_\tau$ are those of $\Delta$ , restricted to the lowest Landau level at $\tau$ .

$$\tilde{G}_\tau = \sum_{mm'nn'} G_{mm'nn'} \, c_m^\dagger c_{m'}^\dagger c_{n'} c_n \qquad \text{(+ arbitrary const. )}$$

$$G_{mm'nn'} = \frac{1}{2} \int_0^1 dx \int_0^1 dx' \, \chi_m^*(x) \chi_{m'}^*(x') \, \Delta \, \chi_{n'}^*(x') \chi_n^*(x)$$

The integrand is easily expanded in terms of plane waves, and so the integral readily expressed through (rapidly converging) multiple sums.

# The generator $G_\tau$

$$G_\tau = G_0 + \frac{1}{4\pi i q} G_1 + q G_2$$

$$G_0 = -\frac{1}{4\pi i L} \sum_l \frac{\mathcal{S}_l^2}{\mathcal{S}_l^0} c_l^\dagger c_l$$

$$\mathcal{S}_l^a = \sum_n (2\pi i [nL + l])^a e^{2\pi i L \tau (n + l/L)^2}$$

$$G_1 = (\frac{q}{L})^2 [\sum_l \mathcal{S}_l^2 c_l^\dagger c_l + \sum_{l_1 \neq l_2} \mathcal{S}_{l_1}^1 \mathcal{S}_{l_2}^1 c_{l_1}^\dagger c_{l_1} c_{l_2}^\dagger c_{l_2}]$$

$$G_2 = \frac{1}{2} \sum_{l_1 l_2 l_3 l_4} \frac{\Delta_{2, l_1 l_2 l_3 l_4}}{\sqrt{\mathcal{S}_{l_1}^0 \mathcal{S}_{l_2}^0 \mathcal{S}_{l_3}^0 \mathcal{S}_{l_4}^0}} c_{l_1}^\dagger c_{l_2}^\dagger c_{l_4} c_{l_3}$$

$$\Delta_{2, l_1 l_2 l_3 l_4} = \frac{2\pi}{i} \sum_{n \neq 0} (\frac{e^{i\pi\tau n}}{1 - e^{2i\pi\tau n}})^2 \sum_{n_1} e^{i\pi\tau L [(l_1 + n)/L + n_1]^2} (e^{i\pi\tau L (l_1/L + n_1)^2})^*$$
$$\sum_{n_4} (e^{i\pi\tau L [(l_4 + n)/L + n_4]^2})^* e^{i\pi\tau L (l_4/L + n_4)^2}$$

# The symmetrized generator $G_{\tau,\mathrm{sym}}$

Turns out $[G_\tau, T_x] = 0\,, \;\; T_y G_\tau T_y^\dagger \neq G_\tau$ .

We may just symmetrize:

$$G_{\tau,\mathrm{sym}} = \frac{1}{L}\sum_{n=0}^{L-1} T_y^n\, G_\tau (T_y^\dagger)^n$$

$G_{\tau,\mathrm{sym}}$ acts the same way on the q-fold degenerate Laughlin states as $G_\tau$

and

$$[G_{\tau,\mathrm{sym}}, T_x] = 0 = [G_{\tau,\mathrm{sym}}, T_y]$$

The L-1 linearly independent 2-body operators

$$D_n = G_{\tau,\mathrm{sym}} - T_y^n\, G_\tau (T_y^\dagger)^n \qquad n = 0 \ldots L-2$$

all satisfy

$$D_n |\psi_{1/q}^\ell\rangle = 0 \qquad \ell = 0 \ldots q-1$$

For q=3, we checked that this condition uniquely characterizes the $|\psi_{1/3}^\ell\rangle$ at filling factor $\nu = 1/3$ .

# Generating $|\psi^{\ell}_{1/3}\rangle$ from thin torus limit

$$\frac{d}{d\tau}|\psi^{\ell}_{1/3}(\tau)\rangle = G_{\tau,\mathrm{sym}}|\psi^{\ell}_{1/3}(\tau)\rangle$$

It turns out that this is well behaved in the $\tau \to \infty$ limit. In particular, unlike in the cylinder case, $G_{\tau,\mathrm{sym}}$ has off-diagonal matrix elements that can generate the full Laughlin state at $\tau$ out of $|100100100100\ldots\rangle$.

We thus have

$$|\psi^{\ell}_{1/3}(\tau)\rangle = T_{\tau'}\exp\{\int_{\infty}^{\tau} d\tau\, G_{\tau',\mathrm{sym}}\}\,|100100100\ldots\rangle$$

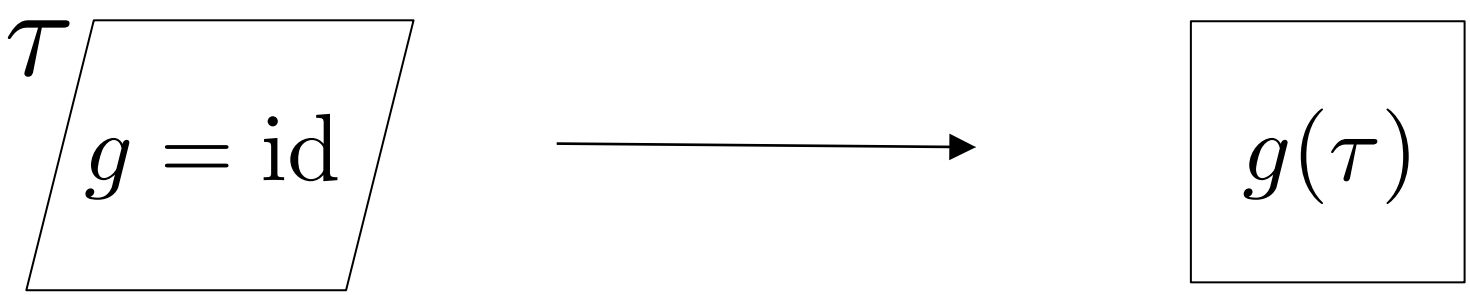
# Application: Hall viscosity

The non-dissipative, anti-symmetric part of the viscosity of a quantum Hall state is related to the adiabatic curvature on the space of background metrics.
(J. E. Avron, R. Seiler, P.G. Zograf PRL 95;  N. Read, E.H. Rezayi, PRB 10).

$$|\psi\rangle = |\psi(g_{\mu\nu})\rangle$$

$$g_{\mu\nu} = g_{\mu\nu}(\tau) := \Lambda(\tau)^T \Lambda(\tau)$$

$$\Lambda(\tau) = \begin{pmatrix} \tau_y^{-1/2} & \tau_x \tau_y^{-1/2} \\ 0 & \tau_y^{1/2} \end{pmatrix}$$



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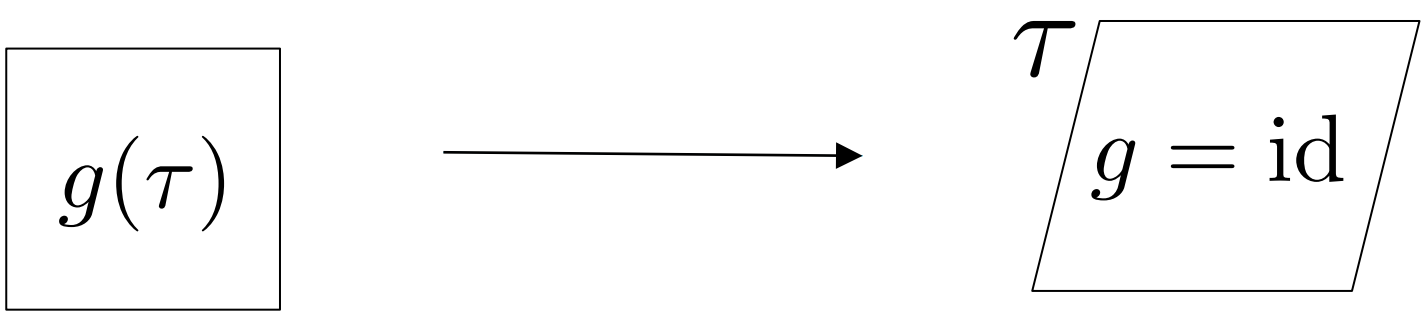
$$F = -2\,\mathrm{Im}\,\langle \partial_{\tau_x}\psi(g_{\mu\nu})|\partial_{\tau_y}\psi(g_{\mu\nu})\rangle$$

$$F = -\frac{V\eta^{(A)}}{\tau_y^2} \qquad \eta^{(A)} \;:\; \text{“Hall viscosity”}$$

$$\eta^{(A)} = \frac{1}{2}\,\bar{s}\bar{n}\hbar$$

N. Read, PRB 09



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$$\bar{s} = \frac{\mathcal{S}}{2} \quad , \qquad \mathcal{S}$$ : topological shift, ( $\mathcal{S} = 3$ for 1/3 Laughlin state)

# Application: Hall viscosity

$$F = -2\,\mathrm{Im}\,\langle \partial_{\tau_x}\psi(g_{\mu\nu})|\partial_{\tau_y}\psi(g_{\mu\nu})\rangle = \nabla_\tau \times A$$

$$\psi = \sum_{\{n_k\}} C_{\{n_k\}}|\{n_k\}\rangle_g$$

$$A = i\sum_{\{n_k\}}\left(|C_{\{n_k\}}|^2\ \underbrace{{}_g\langle\{n_k\}|\nabla_\tau|\{n_k\}\rangle_g}_{\text{const. contributing } 1/2 \text{ to } \bar{s}} + C^*_{\{n_k\}}\nabla_\tau C_{\{n_k\}}\right)$$

P. Lévay, J. Math. Phys. 95
N. Read, E.H. Rezayi, PRB 10

For the Laughlin state, we can of course relate the 2nd term to the operator $G_\tau$ itself.

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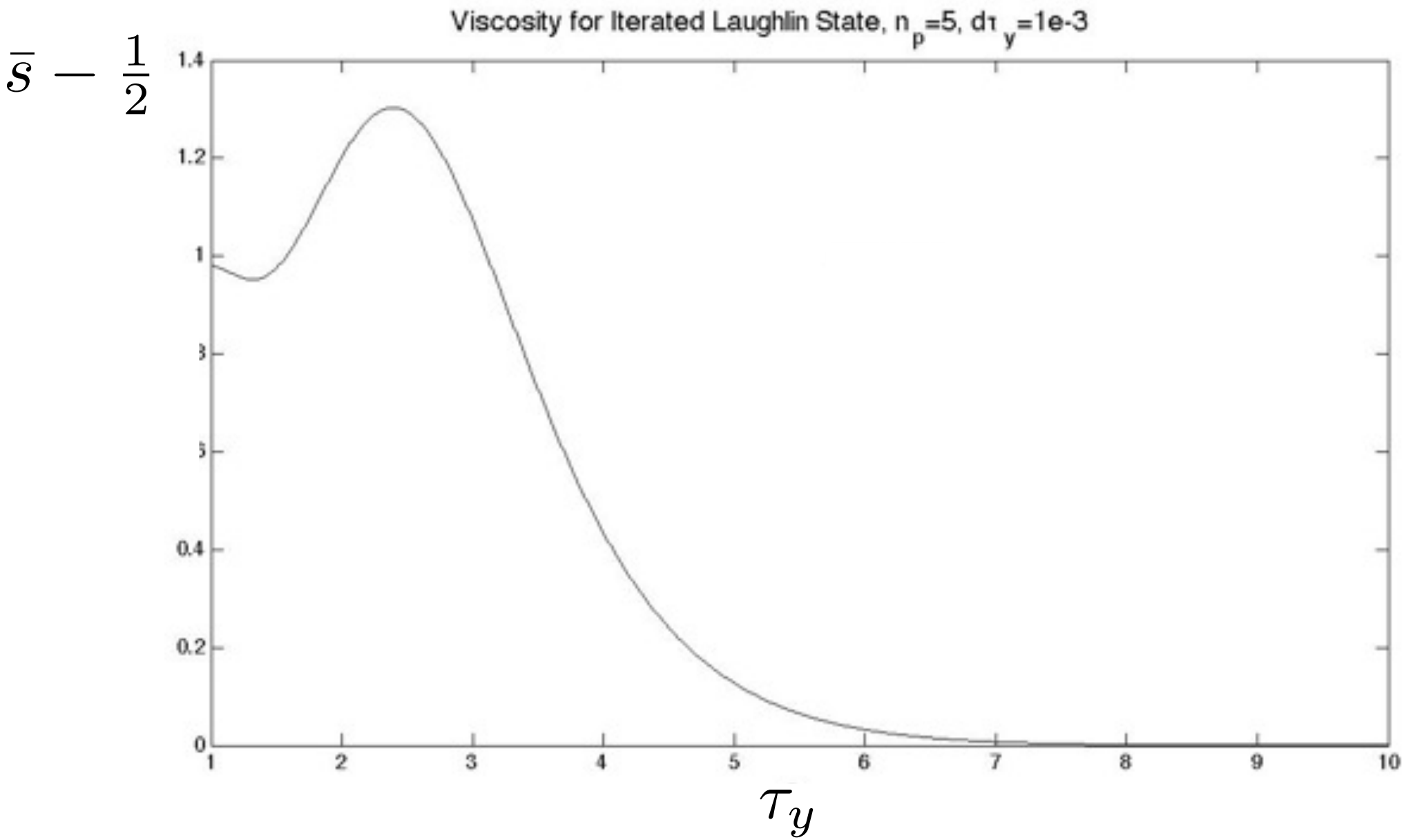
$$\bar{s} = \frac{2\eta^{(A)}}{\hbar\bar{n}} = -\tau_y^2 F/V = \frac{1}{2} + \frac{2\tau_y^2}{N}\,2\left(\langle\psi_{1/3}|G_\tau^\dagger G_\tau|\psi_{1/3}\rangle - |\langle\psi_{1/3}|G_\tau|\psi_{1/3}\rangle|^2\right)$$

# Application: Hall viscosity



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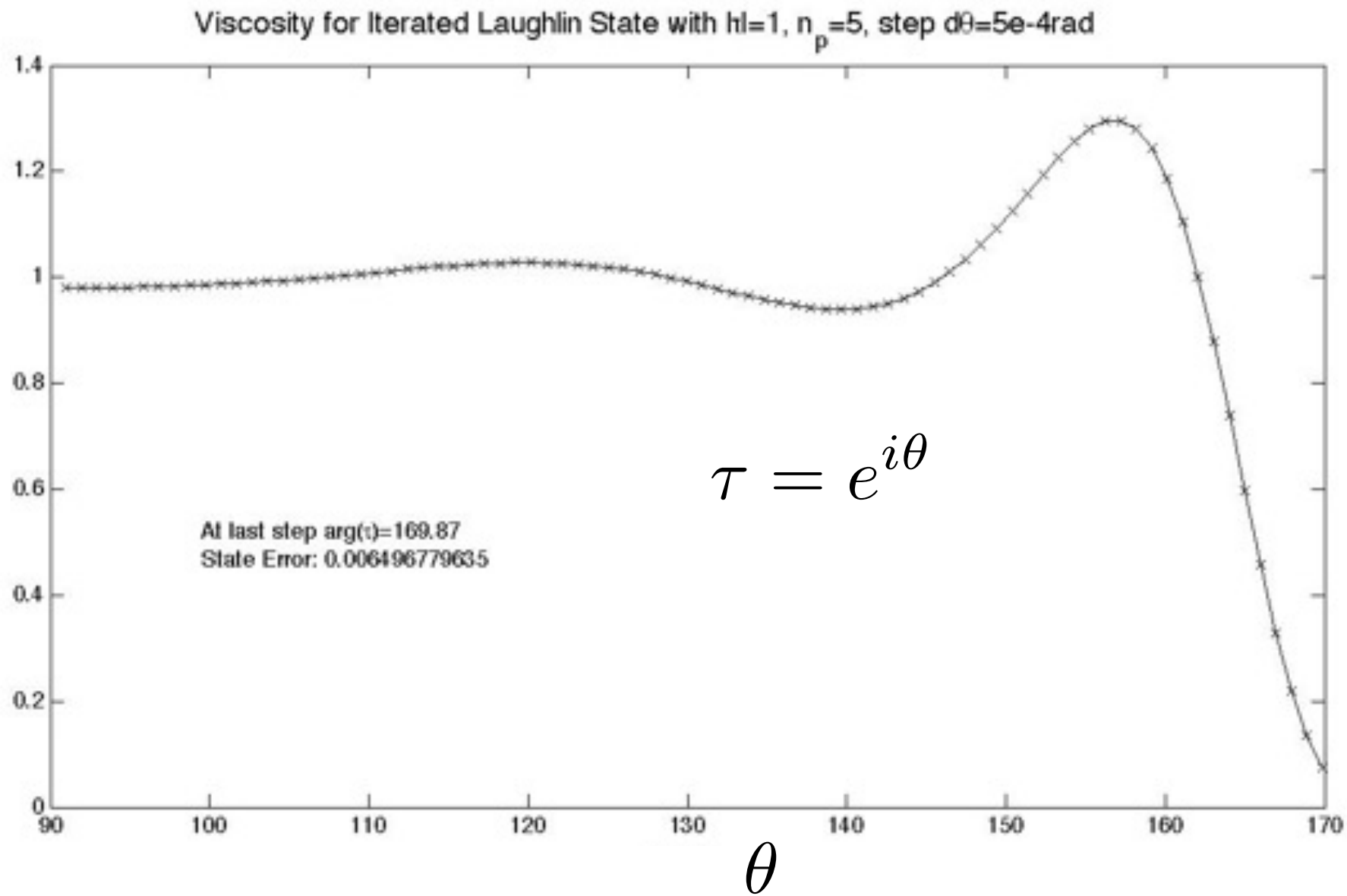
$$\bar{s} - \frac{1}{2}$$




Cf. N. Read, E.H. Rezayi, PRB 10

# Conclusions

◆ Changes in the occupation number basis description of the torus Laughlin state with modular parameter $\mathcal{T}$ are generated by a 2-body operator $G_\tau$.

◆This allows for the following presentation of the torus Laughlin state only in terms of the root pattern and a path-ordered exponential involving 2-body operators:

$$|\psi^{\ell}_{1/3}(\tau)\rangle = T_{\tau'} \exp\{\int_{\infty}^{\tau} d\tau\, G_{\tau'}\}\, |100100100\ldots\rangle_{\ell}$$

◆As an application we calculated the Hall viscosity of the 1/3 Laughlin state.

◆A new family of L-1 2-body operators was found that annihilates the torus Laughlin state. This property characterizes the topologically 3 degenerate ground states at $\nu = 1/3$ uniquely.

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(J. E. Avron, R. Seiler, P.G. Zograf PRL 95).

$$\eta^{(A)}_{abcd} = \frac{1}{V} F_{ab,cd}|_{g=\mathrm{id}}$$

$$F_{ab,cd} = 2\,\mathrm{Im}\,\langle \partial_{\lambda_{ab}}\psi(\lambda)|\partial_{\lambda_{cd}}\psi(\lambda)\rangle$$