From Majorana to parafermion quantum wires

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Joint work with: Jason Alicea, UC Irvine
David Clarke, UC Riverside $\rightarrow$ UC Irvine

Support:
DARPA QuEST

Ising model (of all things)

1D quantum Ising chain:

$$H = -J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z - h \sum_{j=1}^{L} \sigma_j^x$$

Jordan—Wigner transformation:

$$\boxed{\gamma_{2j-1} = \sigma_j^z \prod_{i<j} \sigma_i^x, \qquad \gamma_{2j} = \sigma_j^y \prod_{i<j} \sigma_i^x}$$

γ's are *Majorana* operators:

$$\gamma_j^2 = 1, \quad \gamma_j^\dagger = \gamma_j, \qquad \gamma_j \gamma_k = -\gamma_k \gamma_j$$

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Hamiltonian after Jordan—Wigner transformation:

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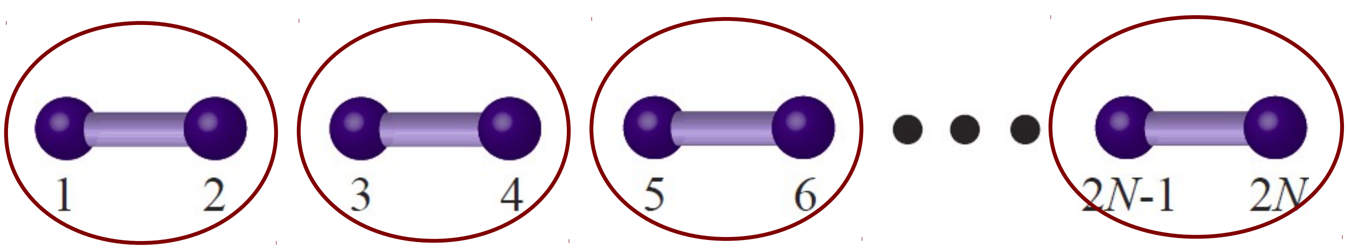
$J = 0:$



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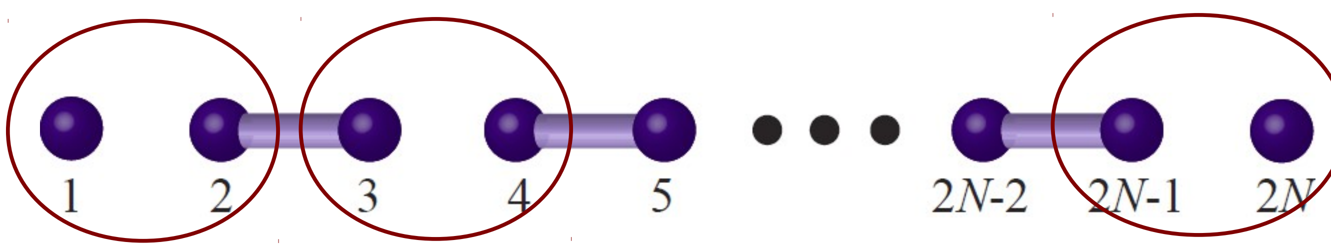


Anyons in 1D: Majorana wires

1D spinless p-wave superconductor(Kitaev 2001):

$$H = \mu \sum_{x=1}^{N} c_x^\dagger c_x - \sum_{x=1}^{N-1} \left(t c_x^\dagger c_{x+1} + |\Delta| e^{i\phi} c_x c_{x+1} + h.c. \right)$$

$$\mu = 0$$
$$t = |\Delta|$$

$$c_x = \frac{1}{2} e^{-i\frac{\phi}{2}} \left(\gamma_{B,x} + i\gamma_{A,x} \right)$$

Anyons in 1D: Majorana wires

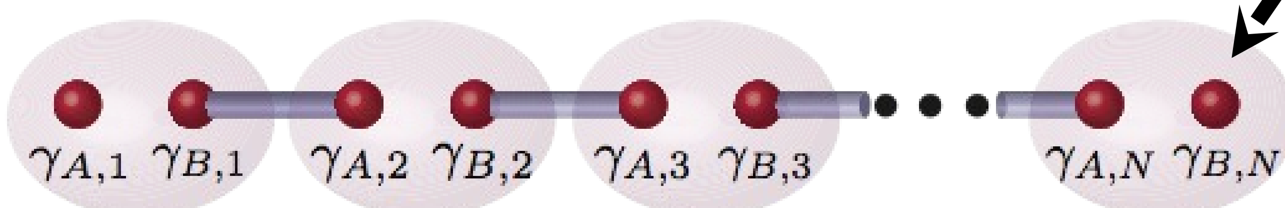
1D spinless p-wave superconductor(Kitaev 2001):

$$H=\mu\sum_{x=1}^{N}c_x^\dagger c_x-\sum_{x=1}^{N-1}\left(tc_x^\dagger c_{x+1}+|\Delta|e^{i\phi}c_xc_{x+1}+h.c.\right)$$

$$\boxed{\begin{array}{ll}\mu=0 & \\ & c_x=\frac{1}{2}e^{-i\frac{\phi}{2}}(\gamma_{B,x}+i\gamma_{A,x}) \\ t=|\Delta| & \end{array}}$$

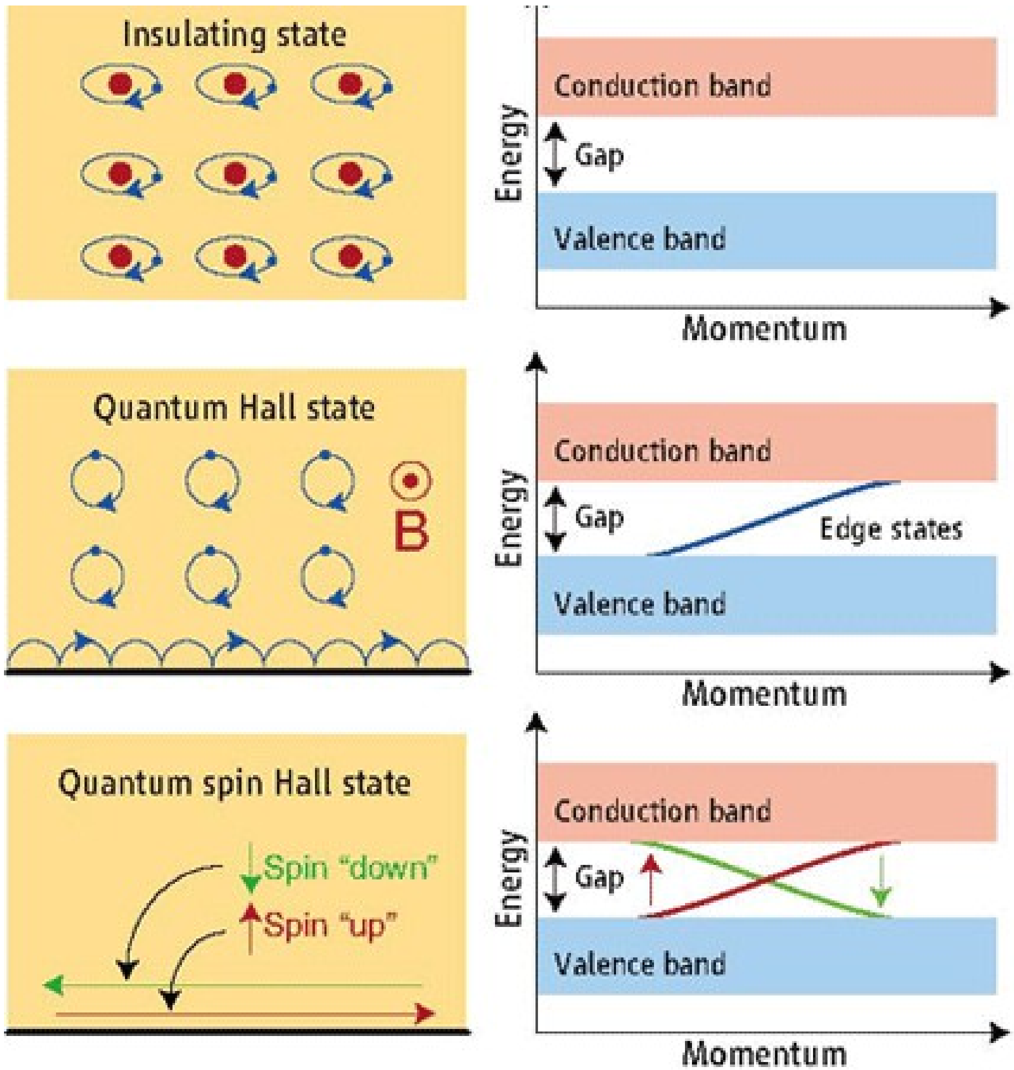
$$\Rightarrow \quad H=-it\sum_{x=1}^{N-1}\gamma_{B,x}\gamma_{A,x+1}$$

Unpaired Majorana fermions at the ends!



Many pictures: courtesy of J. Alicea

Realization in topological insulator edges



Kane & Mele, 2005; Bernevig, Hughes, Zhang, 2006; Fu & Kane, 2008

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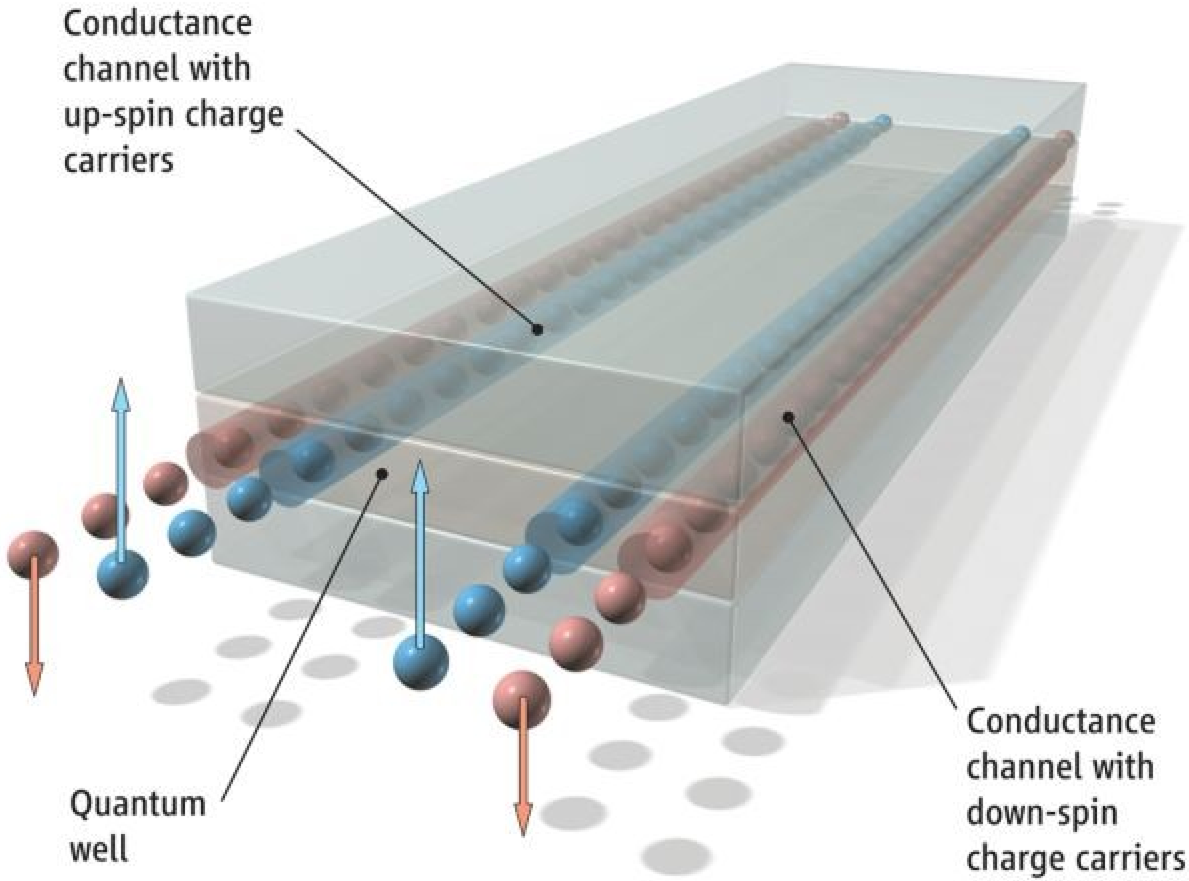

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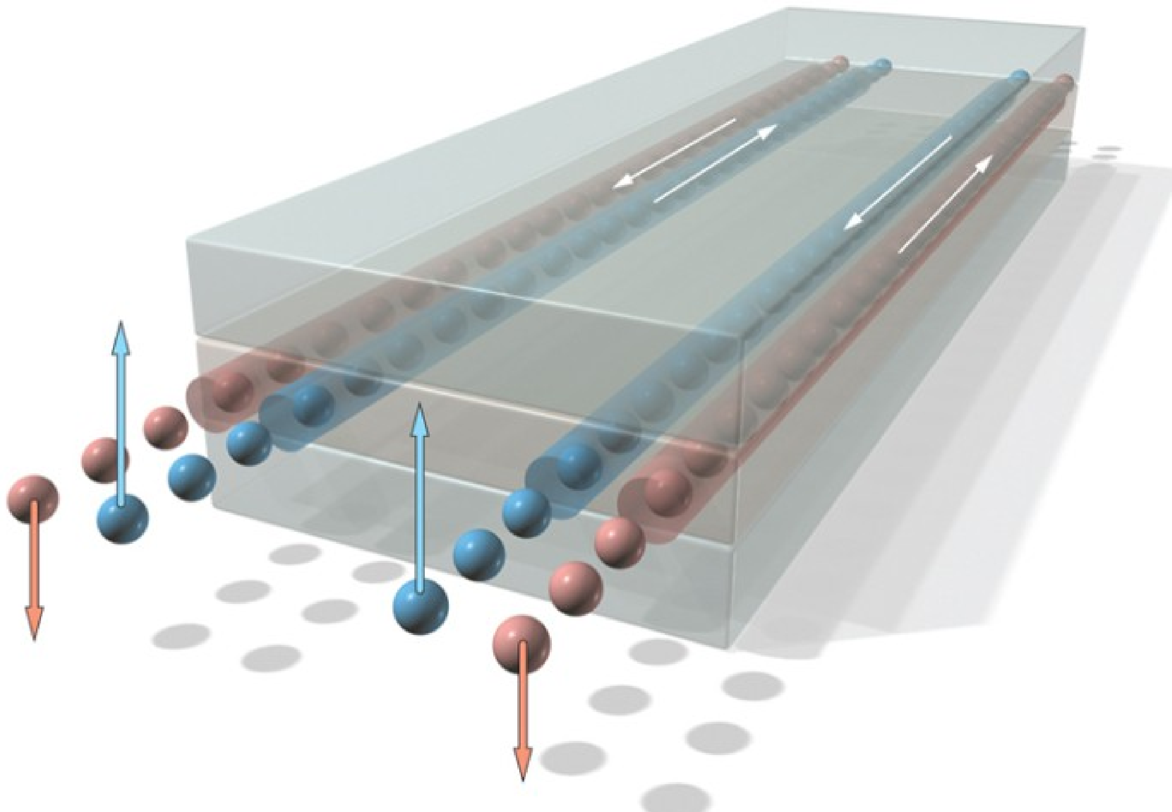

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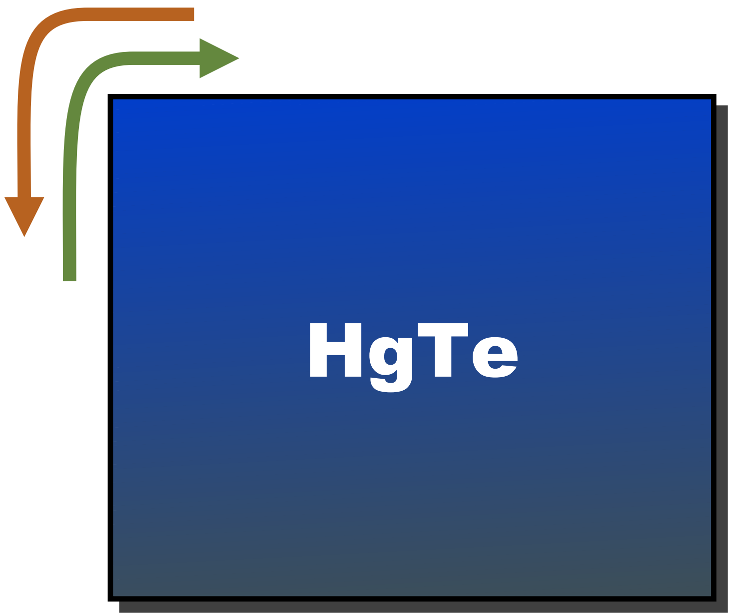
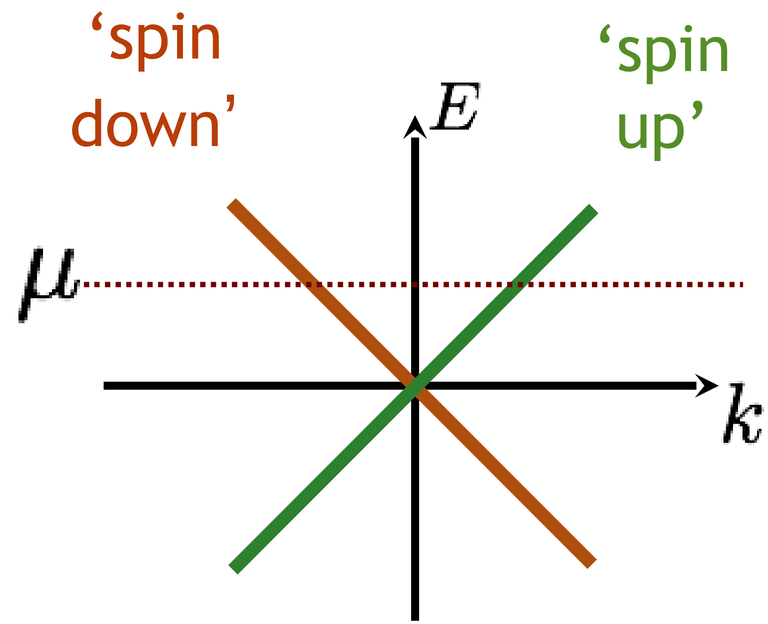


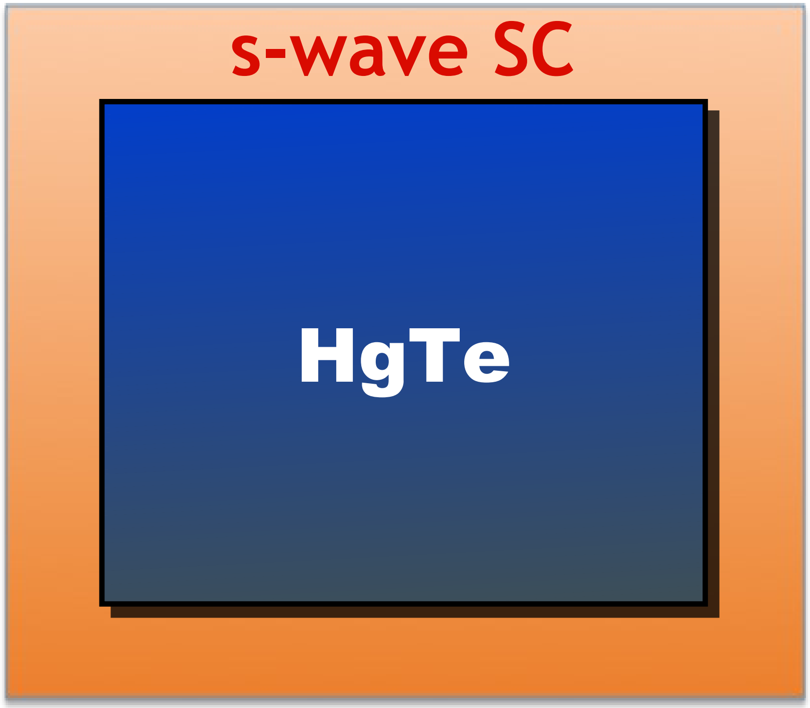


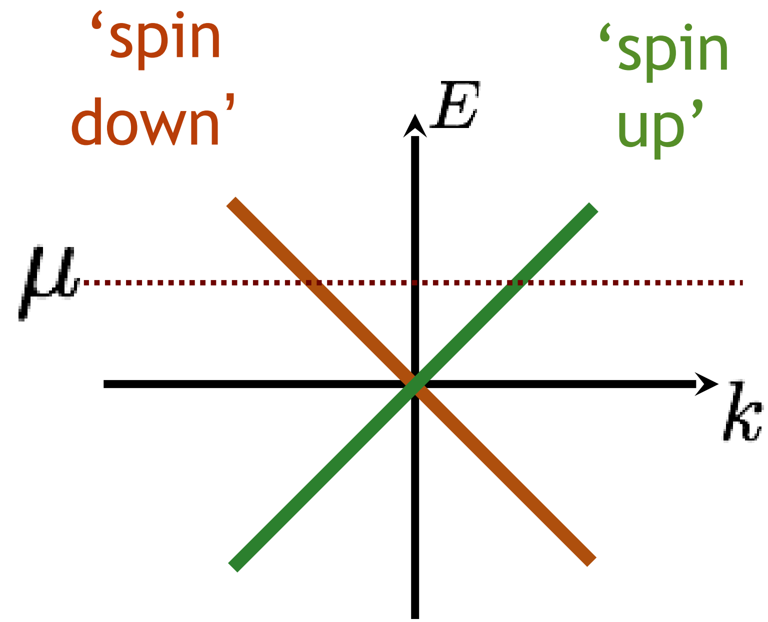
$$H_{\text{edge}} = \int dx[-\mu(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L) - i\hbar v(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L)]$$

1D and effectively 'spinless'! Just need superconductivity...

Fu & Kane, 2008

Realization in topological insulator edges





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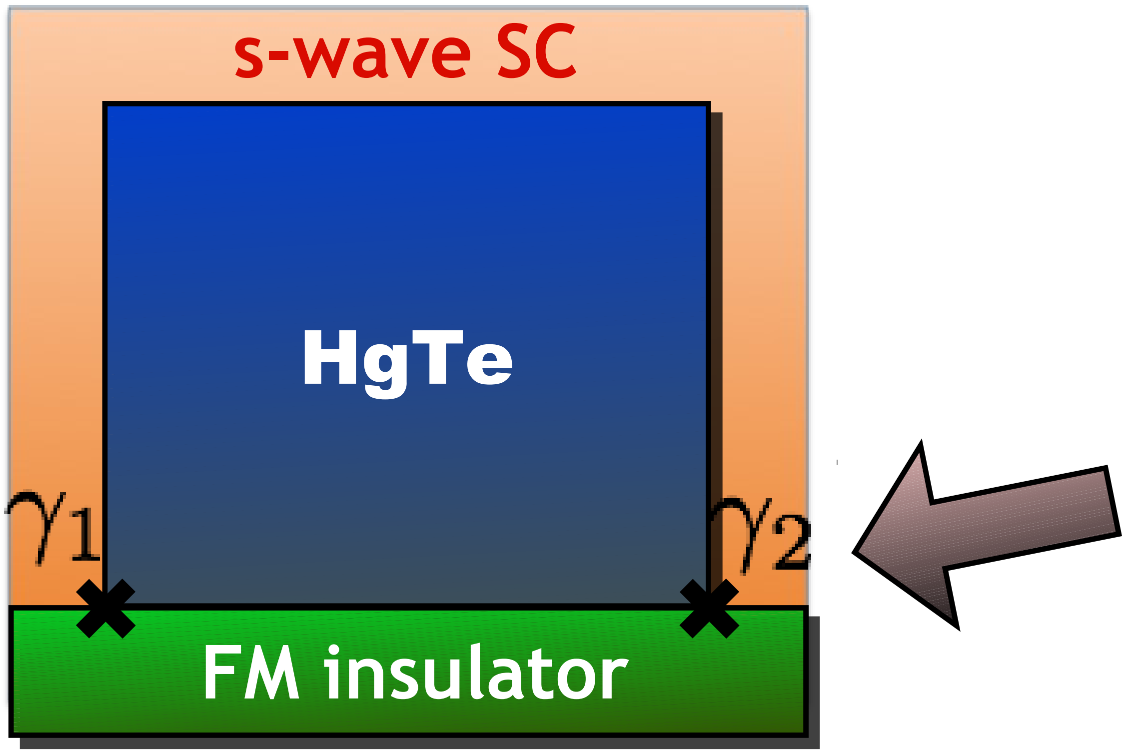
$$+[\Delta \psi_R \psi_L + h.c.]$$

Fu & Kane, 2008

Realization in topological insulator edges

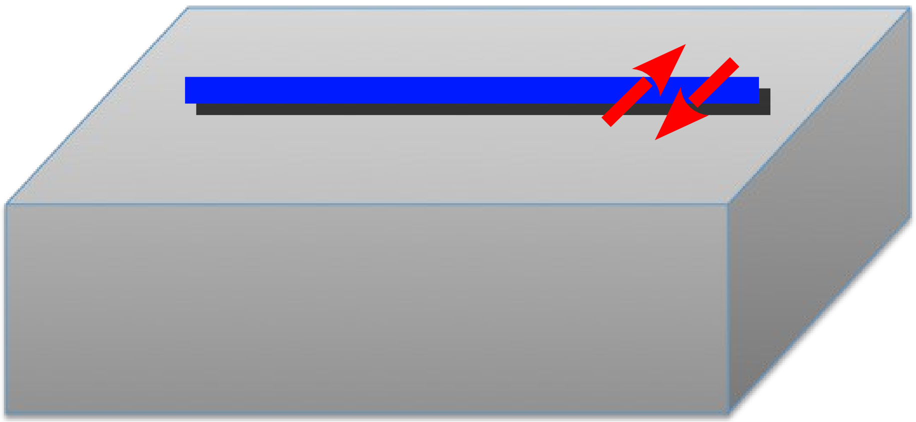


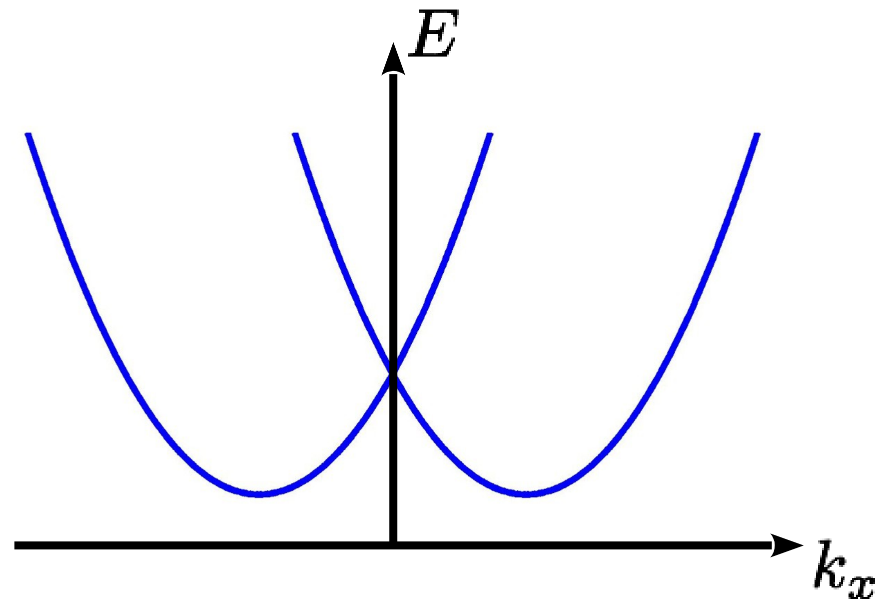
“Terminating” the SC wire by a magnetic gap: Majorana zero modes localised at the ends

Fu & Kane, 2008

Realization in 1D wires

1D spin-orbit-coupled wire (e.g. InAs)





$$H = \int dx \psi^\dagger \left[-\frac{\partial_x^2}{2m} - \mu - i\hbar v \partial_x \sigma^y \right] \psi$$

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)

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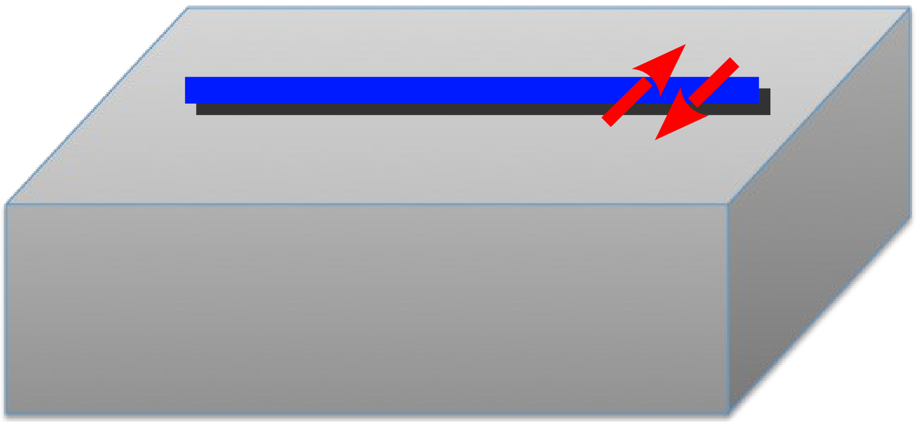

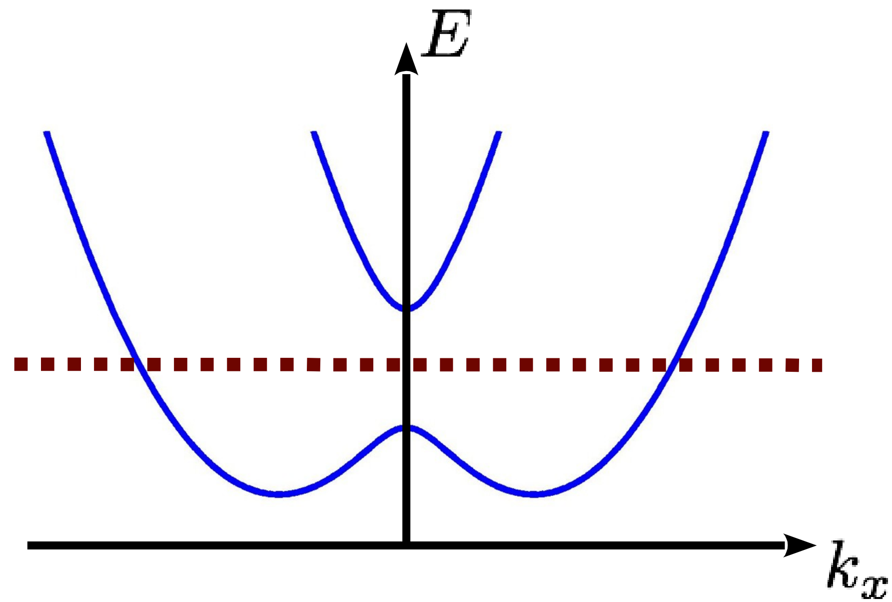

$$H = \int dx \psi^\dagger \left[-\frac{\partial_x^2}{2m} - \mu - i\hbar v \partial_x \sigma^y - \frac{g\mu_B B}{2}\sigma^z \right] \psi$$

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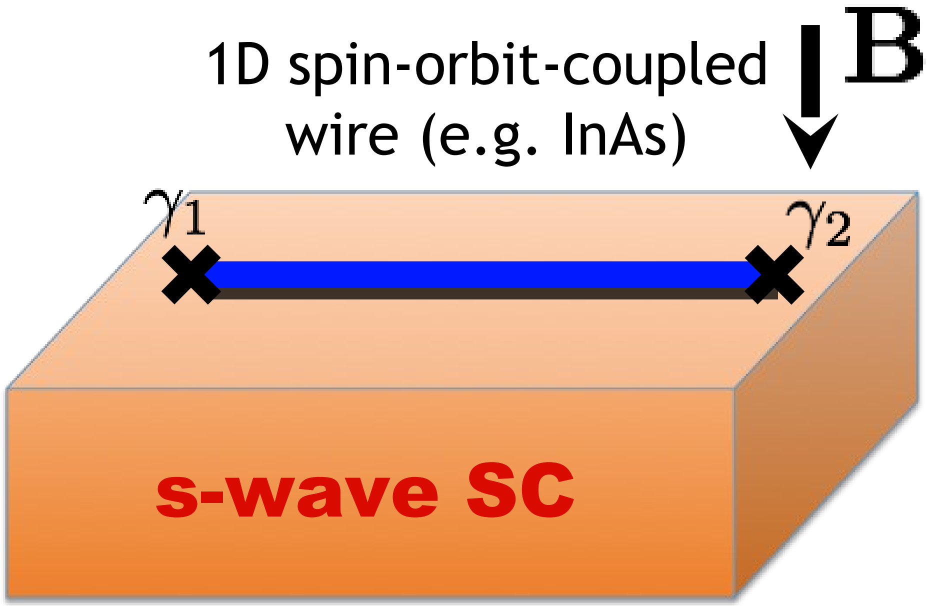
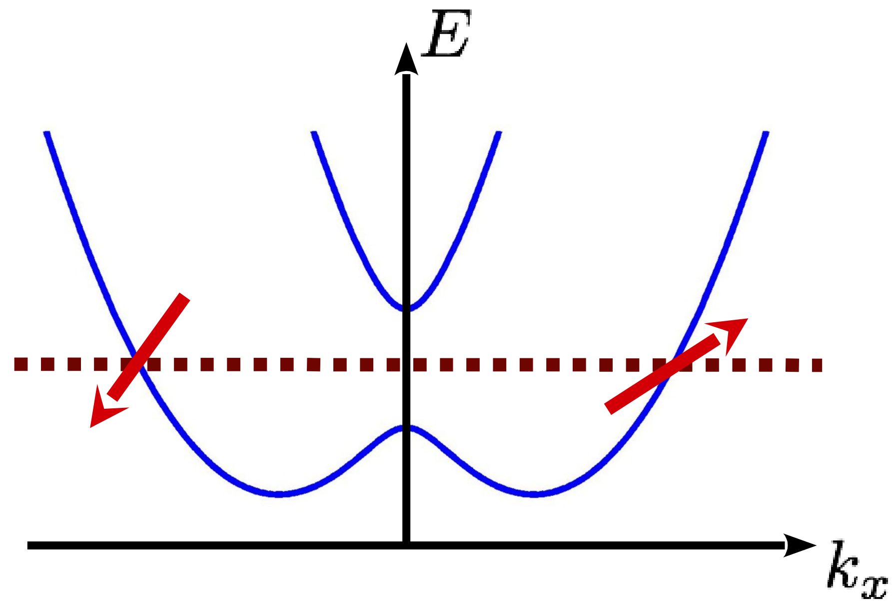
Realization in 1D wires





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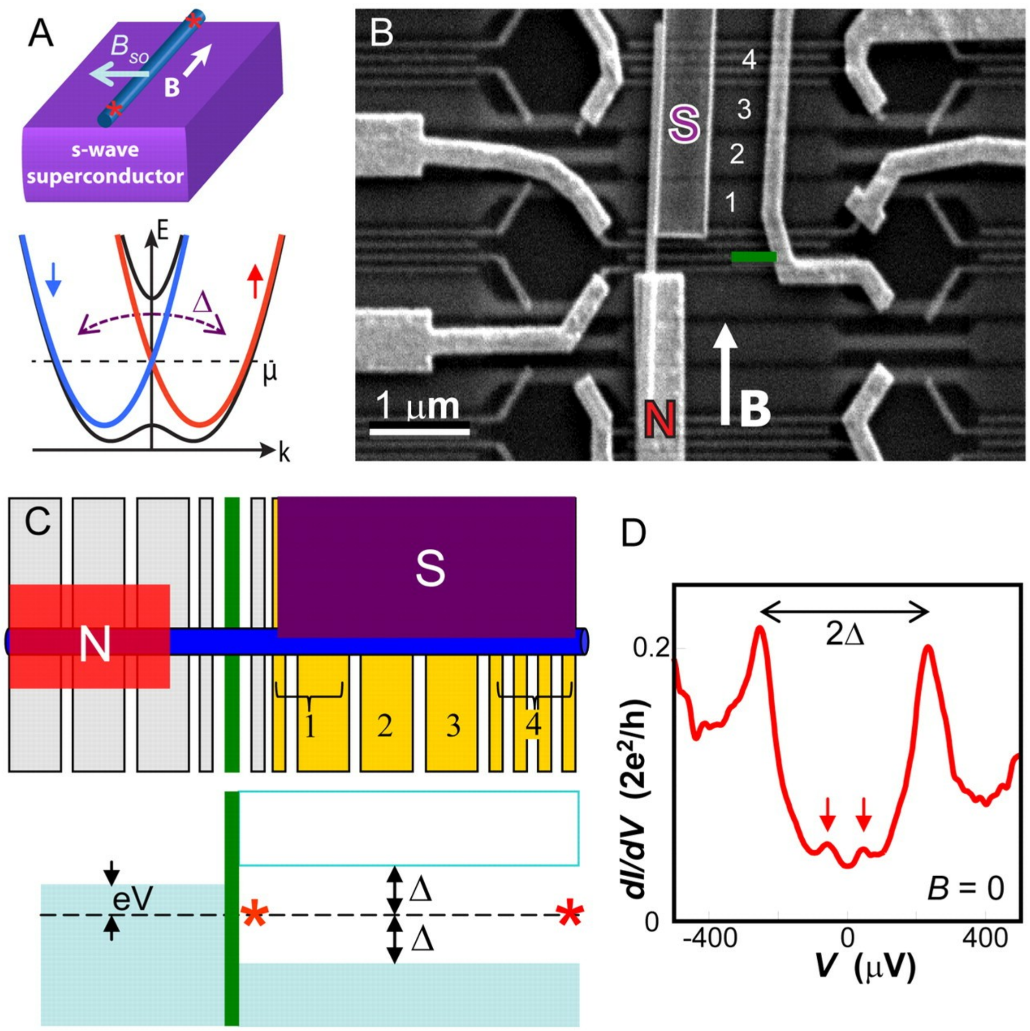
$$+ (\Delta \psi_\uparrow \psi_\downarrow + h.c.)$$

Generates a1D 'spinless' SC state with Majorana fermions!

(Lutchyn, Sau, Das Sarma 2010; Oreg, Refael, von Oppen 2010)

First possible experimental realization



Mourik et al., Science 2012 (Kouwenhoven's group, Delft) following proposals by Lutchyn, Sau & Das Sarma, 2010;
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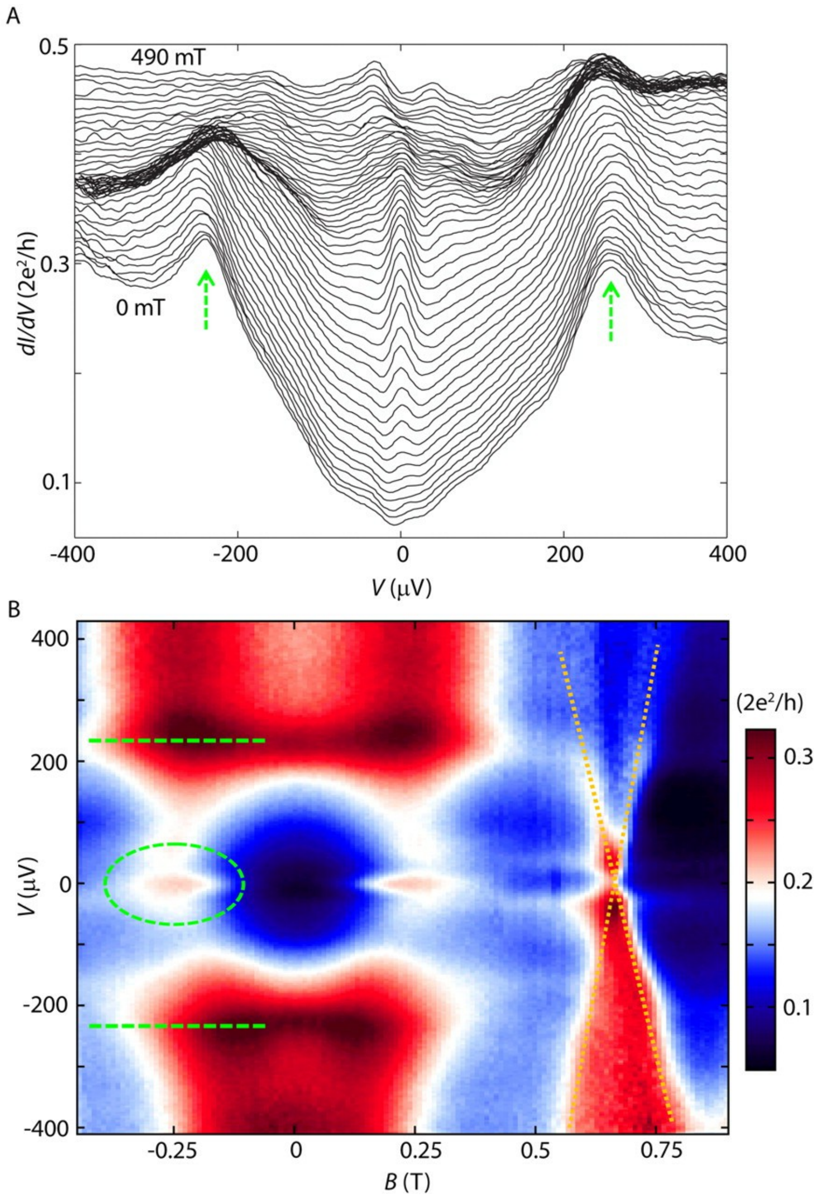

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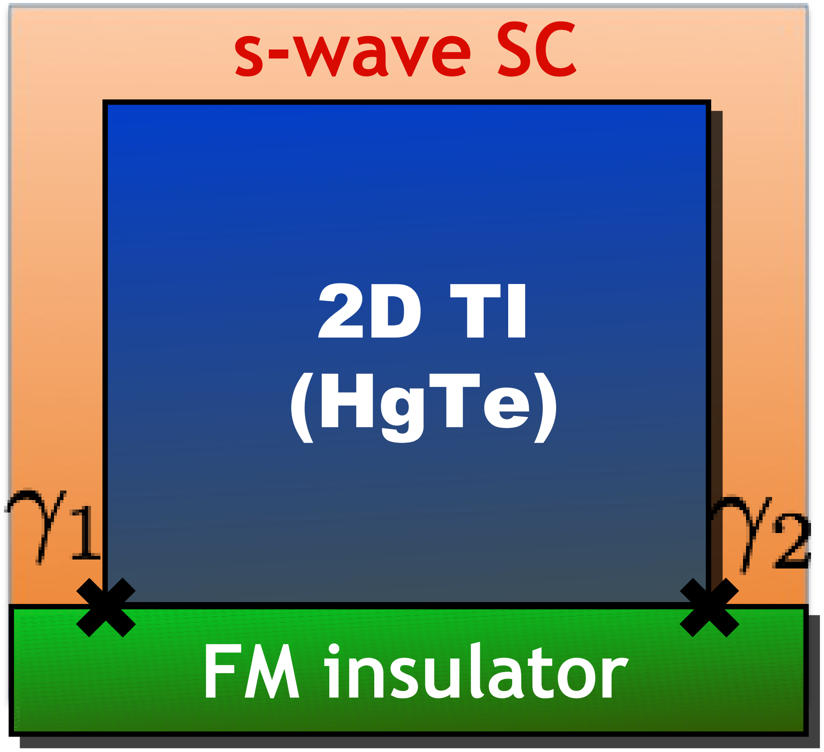
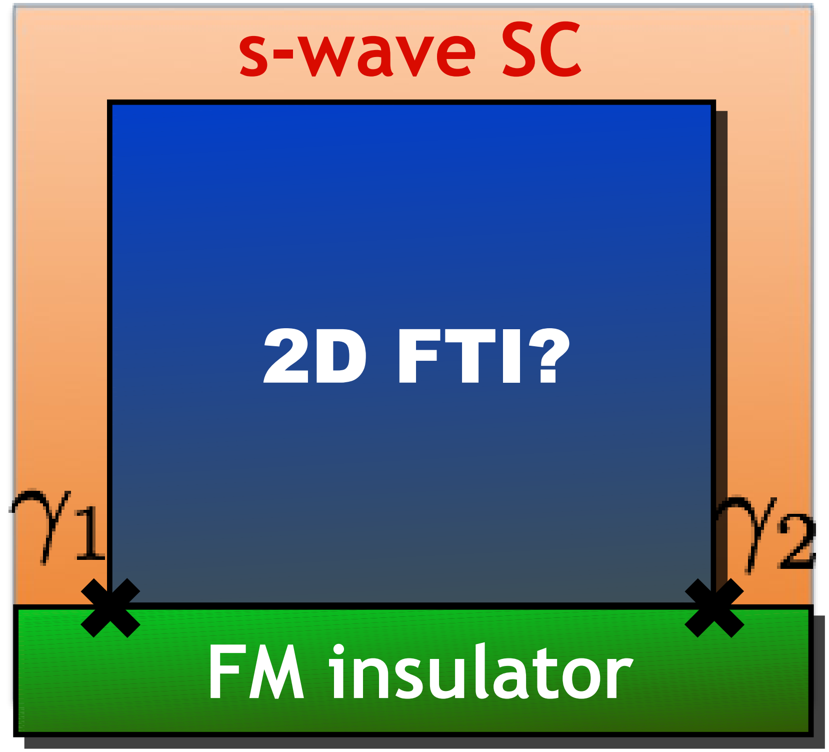
Back to topological insulator edges



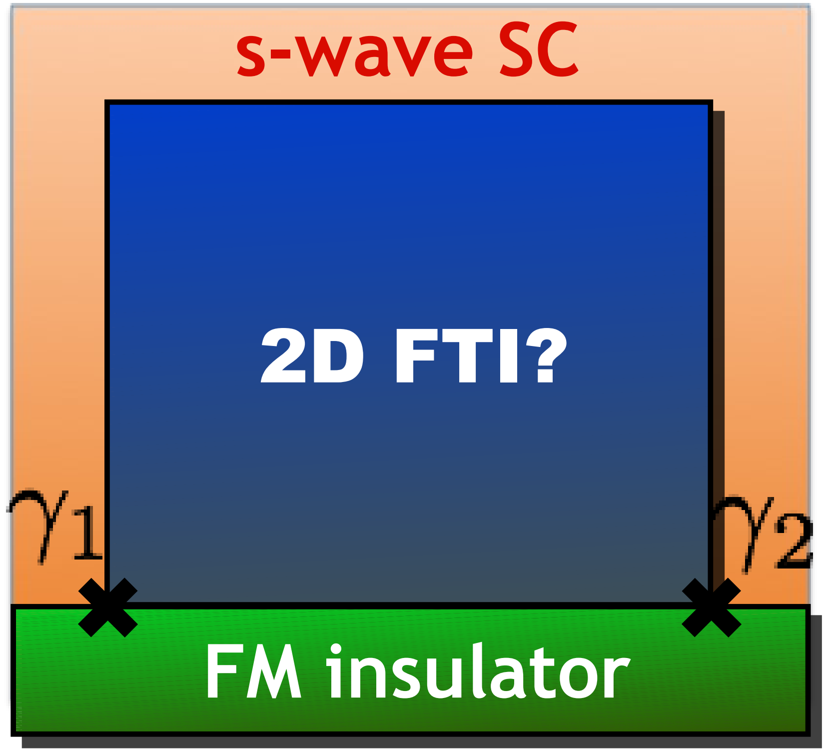
Fu & Kane, 2008

What about fractional TI edges?



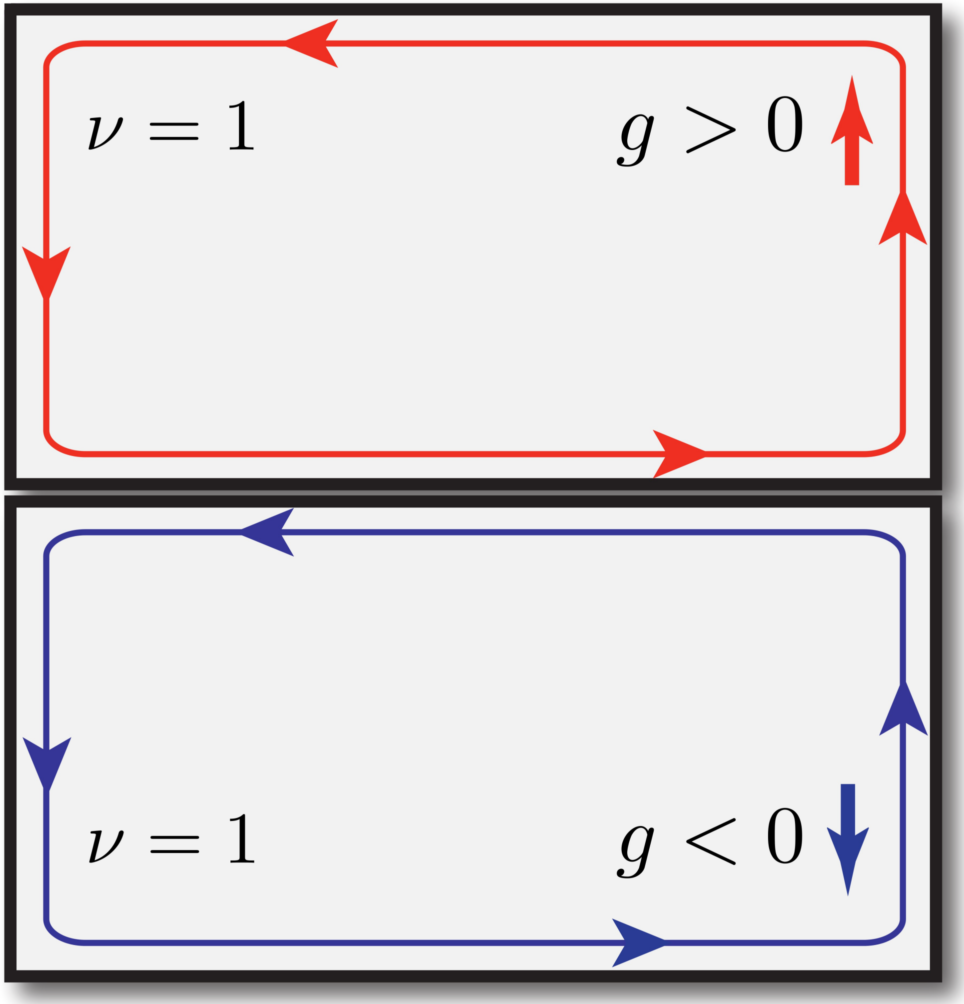
We could envision playing the same game with 2D fractional topological insulators (*à la* Levin & Stern, 2009), but…

What about fractional TI edges?



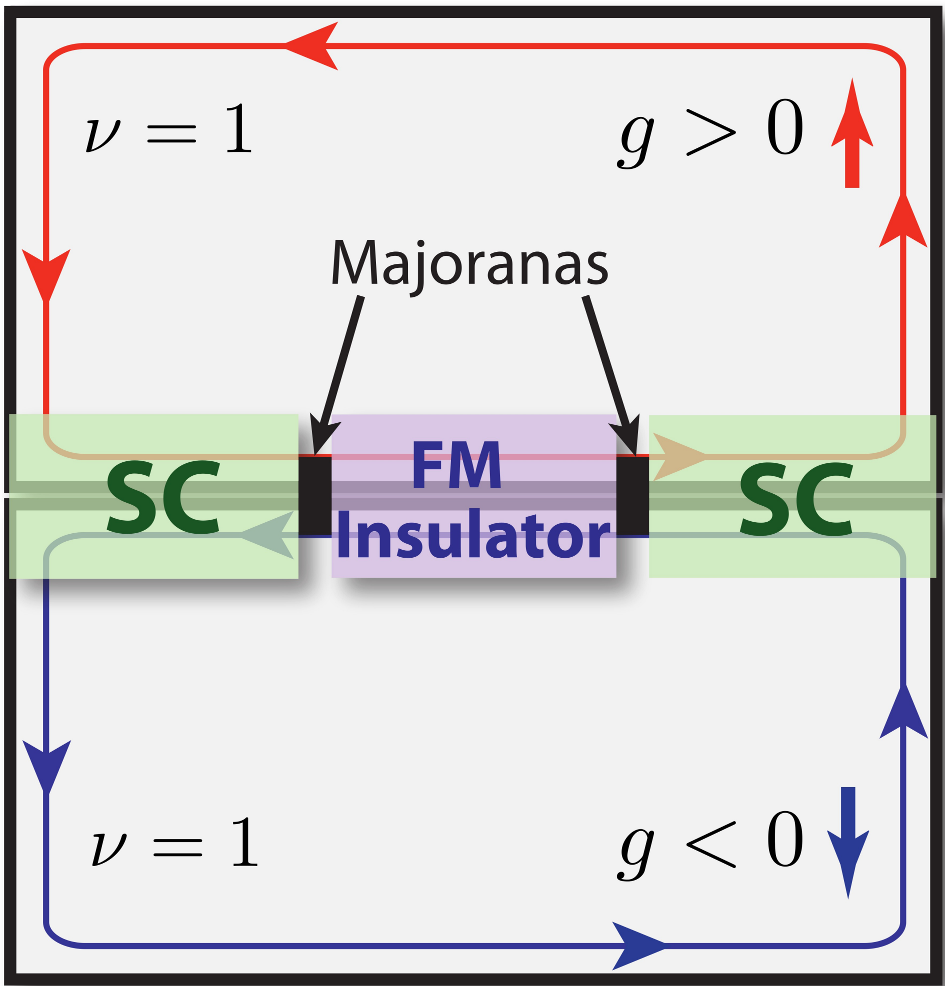
There are no known fractional topological insulators (yet).
But could we 'fake' the same physics elsewhere?

Realization in quantum Hall edges



Counter-propagating edge modes at the boundary between $g > 0$ and $g < 0$. The sign of g can be changed by stress.

Realization in quantum Hall edges



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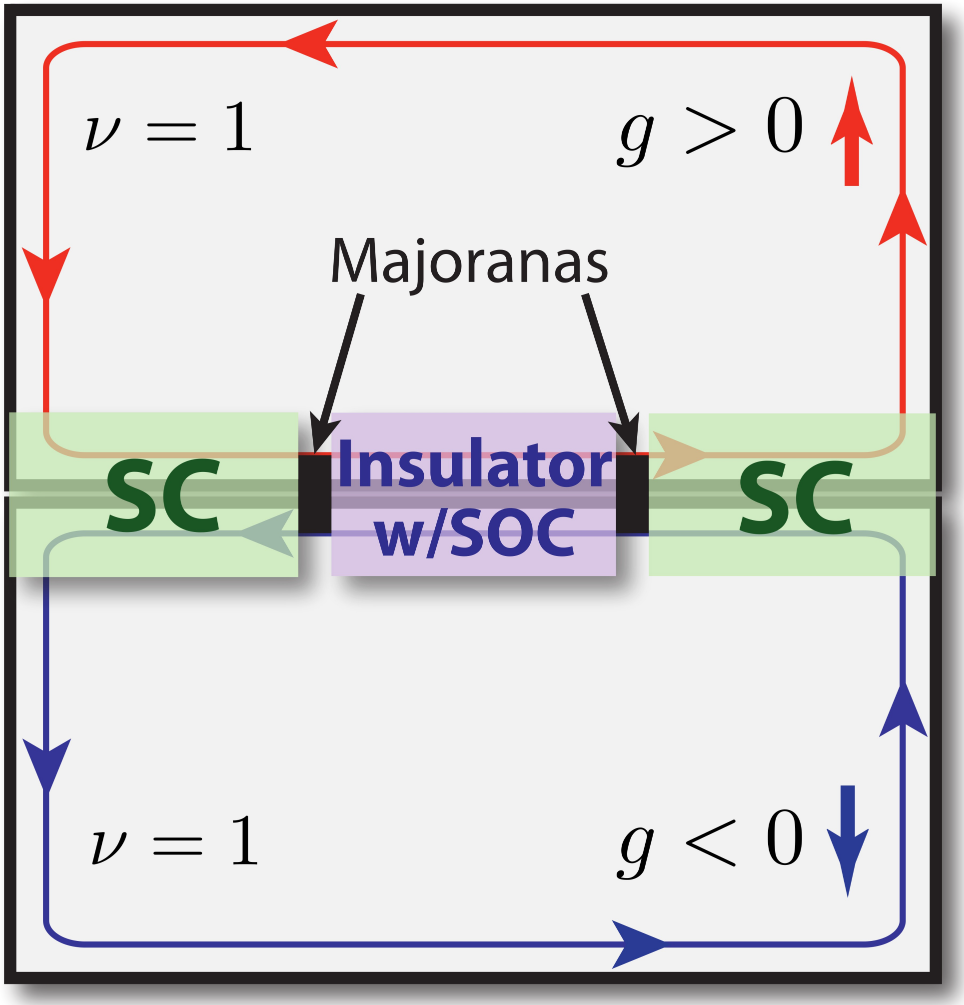

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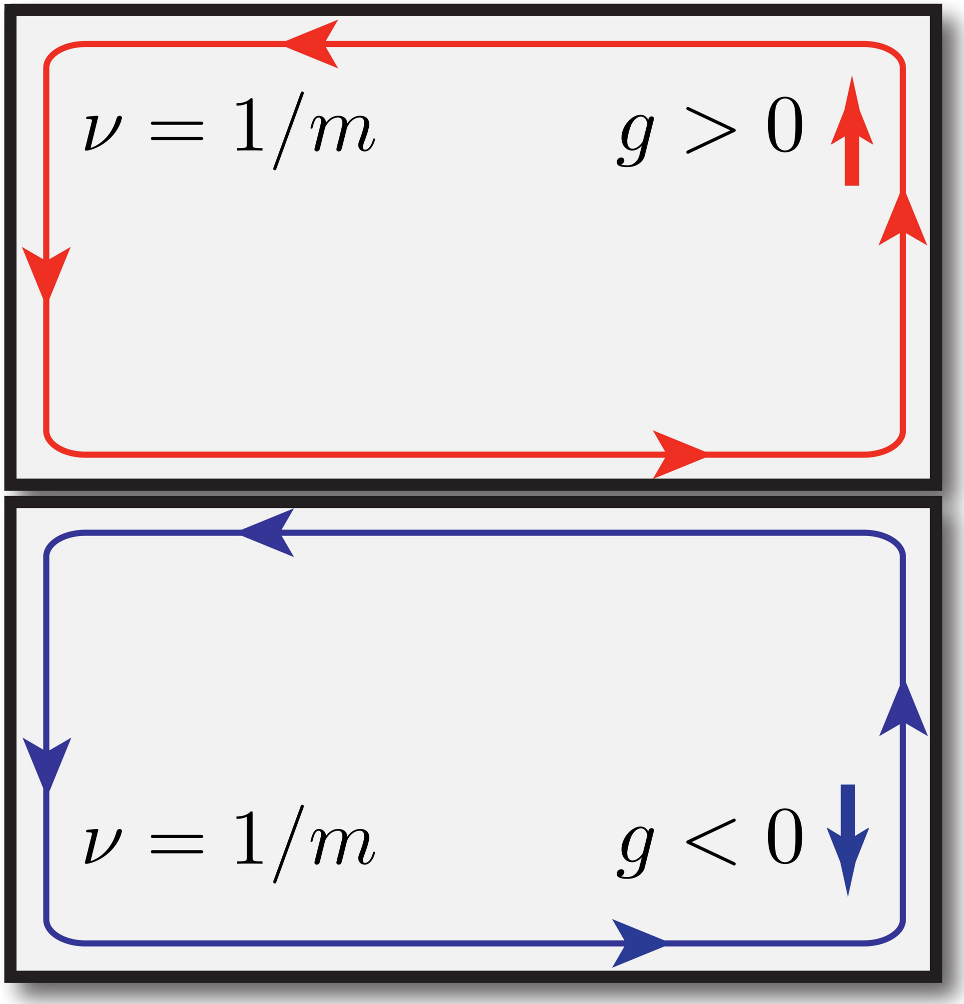

Counter-propagating ***fractionalised*** edge modes at the boundary between $g > 0$ and $g < 0$. The sign of g can be changed by stress.

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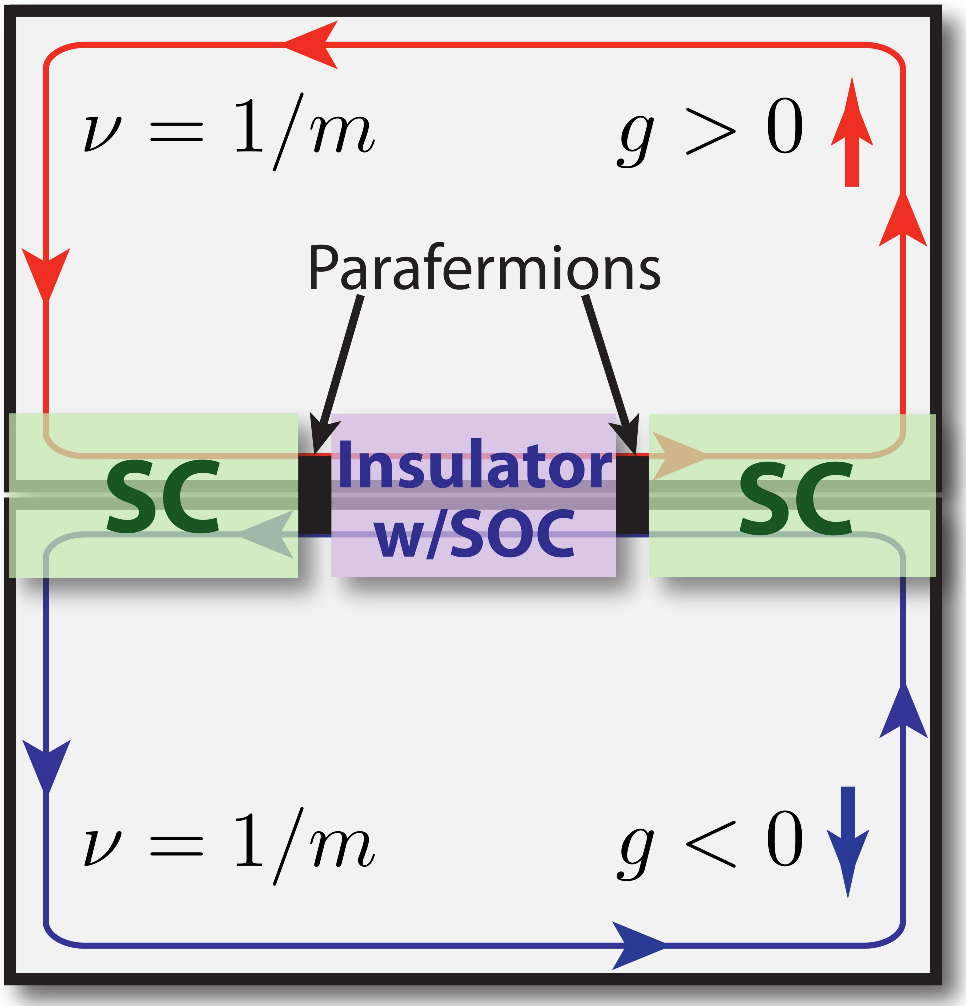

Counter-propagating ***fractionalised*** edge modes at the boundary between $g > 0$ and $g < 0$. The sign of g can be changed by stress.

Taking a cue from Stat Mech

1D quantum clock model (Fendley, unpublished):

$$H = -J \sum_{j=1}^{L-1} (\sigma_j^\dagger \sigma_{j+1} + H.c.) - h \sum_{j=1}^{L} (\tau_j^\dagger + \tau_j)$$

$$\sigma_j^N = 1 \qquad \sigma_j^\dagger = \sigma_j^{N-1}$$

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$$\sigma_j \tau_j = \tau_j \sigma_j e^{2\pi i/N}$$

N=2:
quantum Ising chain

$$\sigma \equiv \sigma^z$$

$$\tau \equiv \sigma^x$$

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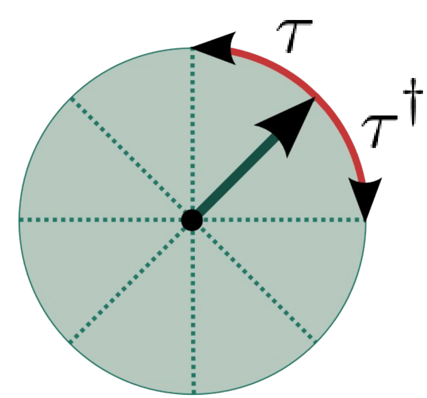
$N \neq 2$:
quantum clock

$$\sigma|q\rangle = e^{2\pi i q/N}|q\rangle$$

$$\tau^\dagger|q\rangle = |q+1\rangle$$



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Jordan—Wigner transformation:

$$\alpha_{2j-1} = \sigma_j \prod_{i<j} \tau_i, \qquad \alpha_{2j} = -e^{i\pi/N} \tau_j \sigma_j \prod_{i<j} \tau_i$$

α's are *parafermionic* operators:

$$\alpha_j^N = 1, \quad \alpha_j^\dagger = \alpha_j^{N-1}, \quad \alpha_j \alpha_k = \alpha_k \alpha_j e^{i\frac{2\pi}{N}\mathrm{sgn}(k-j)}$$

N=2: these are Majorana fermions $\left(\alpha_j^2 = 1,\ \ \alpha_j^\dagger = \alpha_j\right)$

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Hamiltonian after Jordan—Wigner transformation:

$$H = J\sum_{j=1}^{L-1}\left(e^{-i\frac{\pi}{N}}\alpha_{2j}^\dagger \alpha_{2j+1} + H.c.\right) + h\sum_{j=1}^{L}\left(e^{i\frac{\pi}{N}}\alpha_{2j-1}^\dagger \alpha_{2j} + H.c.\right)$$

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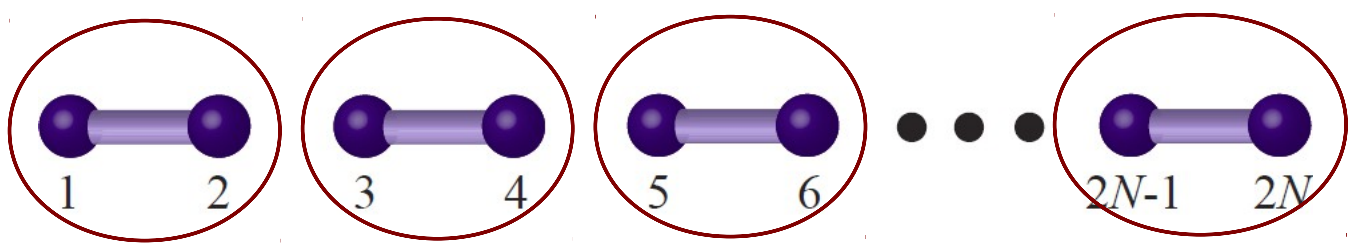
$J = 0 :$



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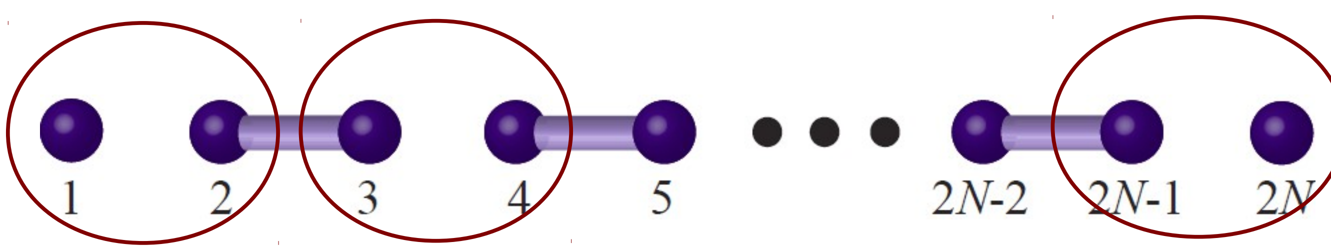


Parafermions vs Majoranas

Upshot:

Majorana Fermions:

$$\gamma^2 = 1$$

$$\gamma_y \gamma_x = -\gamma_x \gamma_y$$

Parafermions:

$$\alpha^N = 1$$

$$\alpha_y \alpha_x = \alpha_x \alpha_y e^{\frac{2\pi i}{N}\mathrm{sgn}(x-y)}$$

Majoranas ⟷ 1D quantum Ising model

Parafermions ⟷ 1D quantum Clock/Potts model

Paul Fendley, unpublished

Parafermions from quantum Hall edges

A Laughlin edge state at $\nu = 1/m$ is a natural starting point since

$$[\phi(x), \phi(y)] = i\frac{\pi}{m}\mathrm{sgn}(x-y)$$

and hence

$$e^{i\phi(x)}e^{i\phi(y)} = e^{i\phi(y)}e^{i\phi(x)}e^{i\frac{\pi}{m}\mathrm{sgn}(y-x)}$$

for chiral edge excitations of charge e/m.
Now, we have two counter-propagating modes, $\phi_{R/L}$, which obey

$$[\phi_{R/L}(x), \phi_{R/L}(y)] = \pm i\frac{\pi}{m}\mathrm{sgn}(x-y)$$

The electron fields are $\psi_{R/L} \sim e^{im\phi_{R/L}}$

Parafermions from quantum Hall edges

Change of variables: $\phi_{R/L} = \varphi \pm \theta$

Free Hamiltonian: $$\mathcal{H}_0 = \frac{mv}{2\pi} \int dx \left[(\partial_x \varphi)^2 + (\partial_x \theta)^2 \right]$$

Just need to show that a zero mode is bound at a domain wall between

$$\mathcal{H}'_{\mathrm{s}}(x) = \Delta(x)\psi_R\psi_L + H.c. \sim -\Delta(x)\cos(2m\varphi)$$

and

$$\mathcal{H}'_{\mathrm{m}}(x) = \mathcal{M}(x)\psi_R^\dagger\psi_L + H.c. \sim -\mathcal{M}(x)\cos(2m\theta)$$

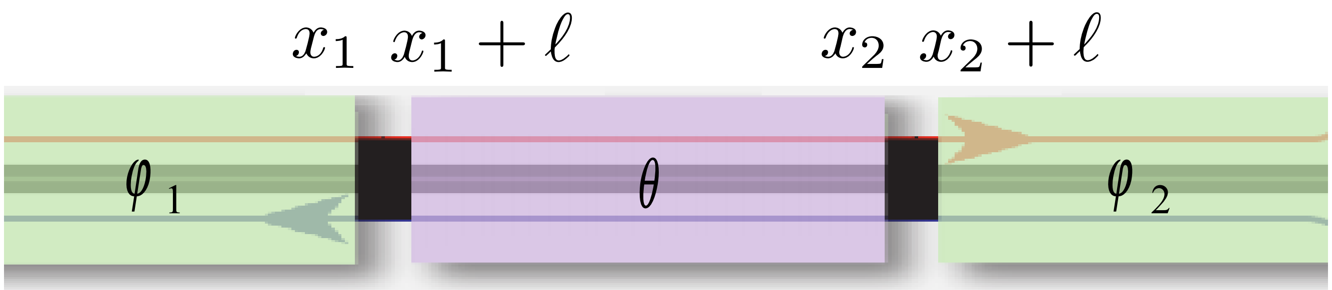
where $\psi_{R/L} \sim e^{im\phi_{R/L}}$

Parafermionic zero mode

Assuming strong tunnelling and pairing,

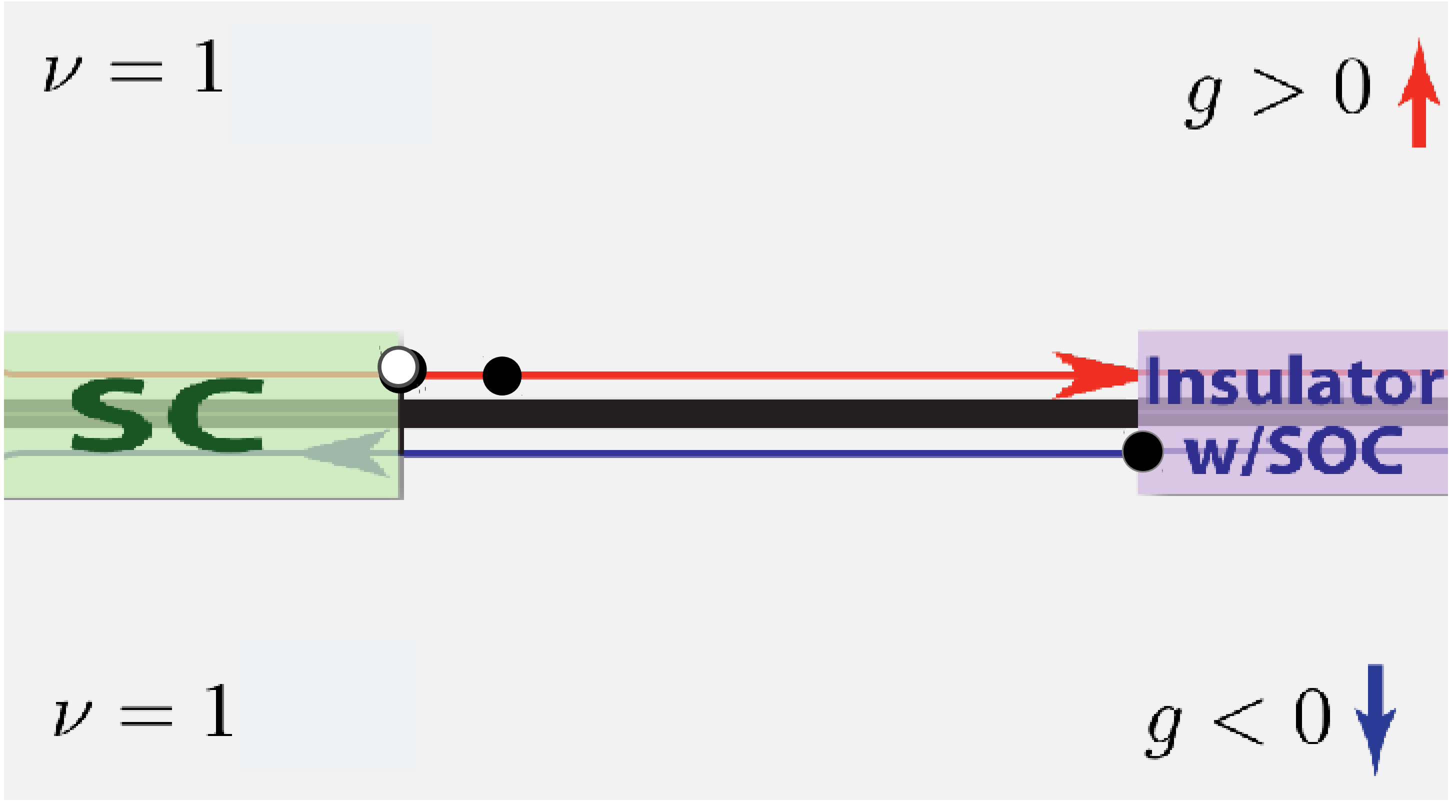
$\varphi = \dfrac{\pi n_\varphi}{m}$ under the superconductors

$\theta = \dfrac{\pi n_\theta}{m}$ under the SO coupled insulators

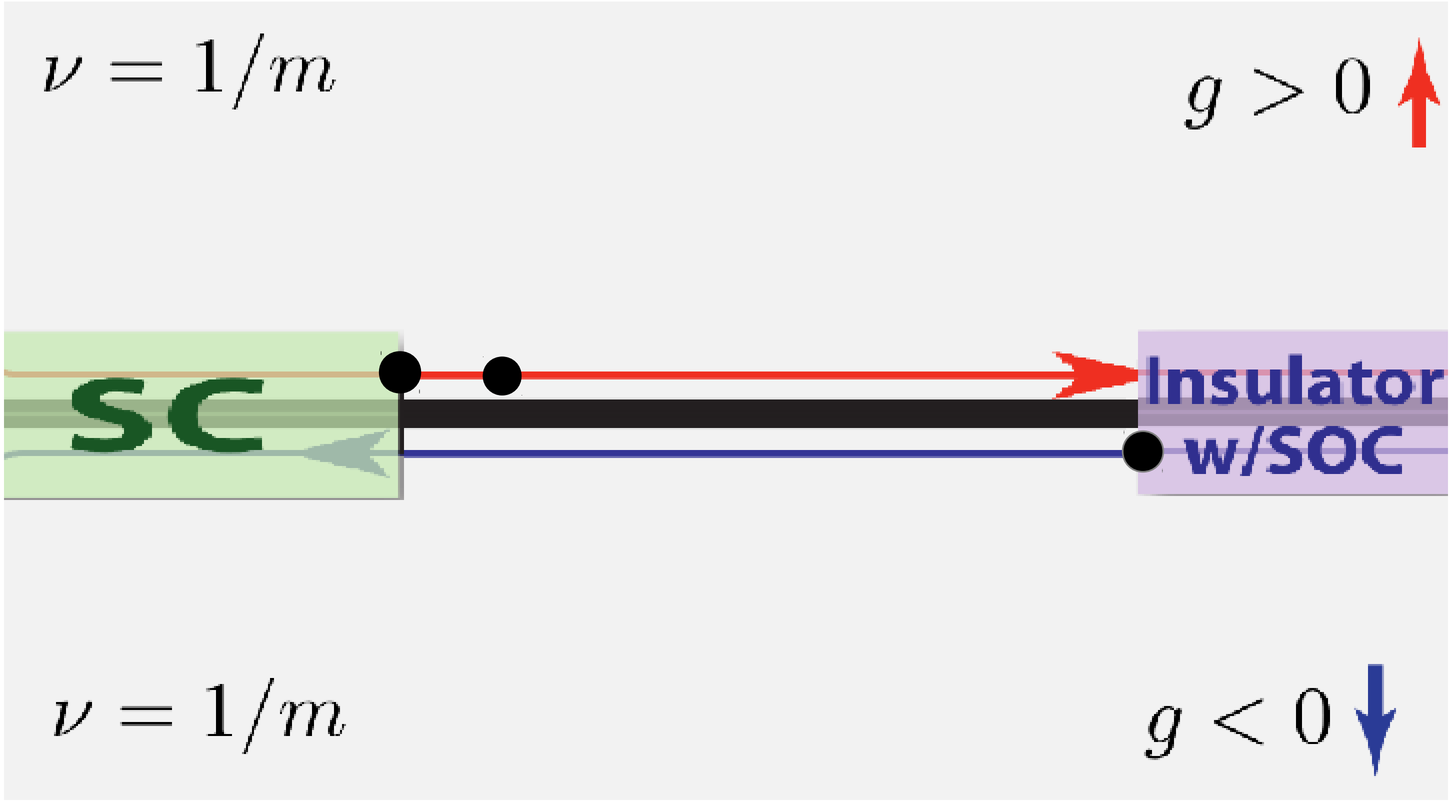


$$\alpha_j = e^{i\frac{\pi}{m}(\hat{n}_\varphi^{(j)}+\hat{n}_\theta)} \int_{x_j}^{x_j+\ell} dx \Big[e^{-i\frac{\pi}{m}(\hat{n}_\varphi^{(j)}+\hat{n}_\theta)} e^{i(\varphi+\theta)} + e^{-i\frac{\pi}{m}(\hat{n}_\varphi^{(j)}-\hat{n}_\theta)} e^{i(\varphi-\theta)} + H.c. \Big]$$

Majorana zero mode



Parafermionic zero mode



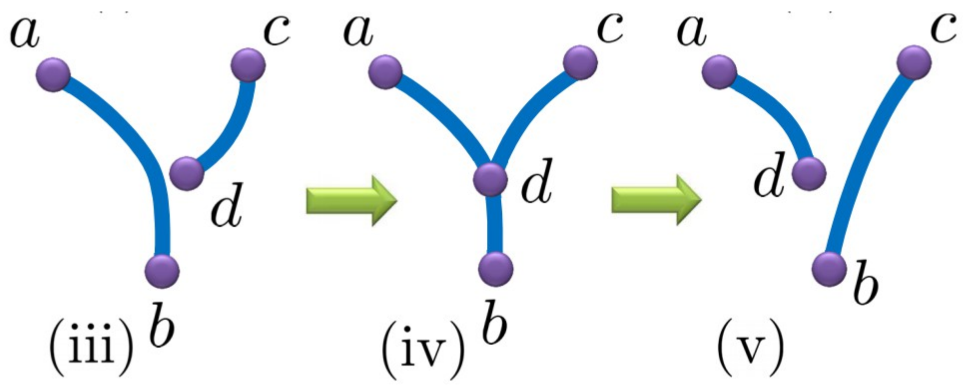
Braiding statistics in 1D?

d = 1

Exchange not well defined...

...because particles inevitably "collide"

Solution: cheat (use 2D networks with Y-junctions)



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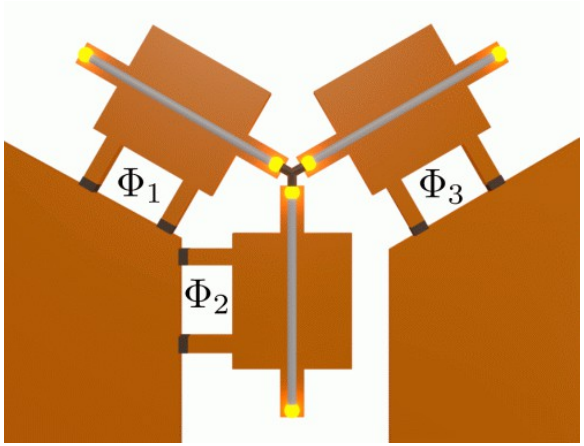
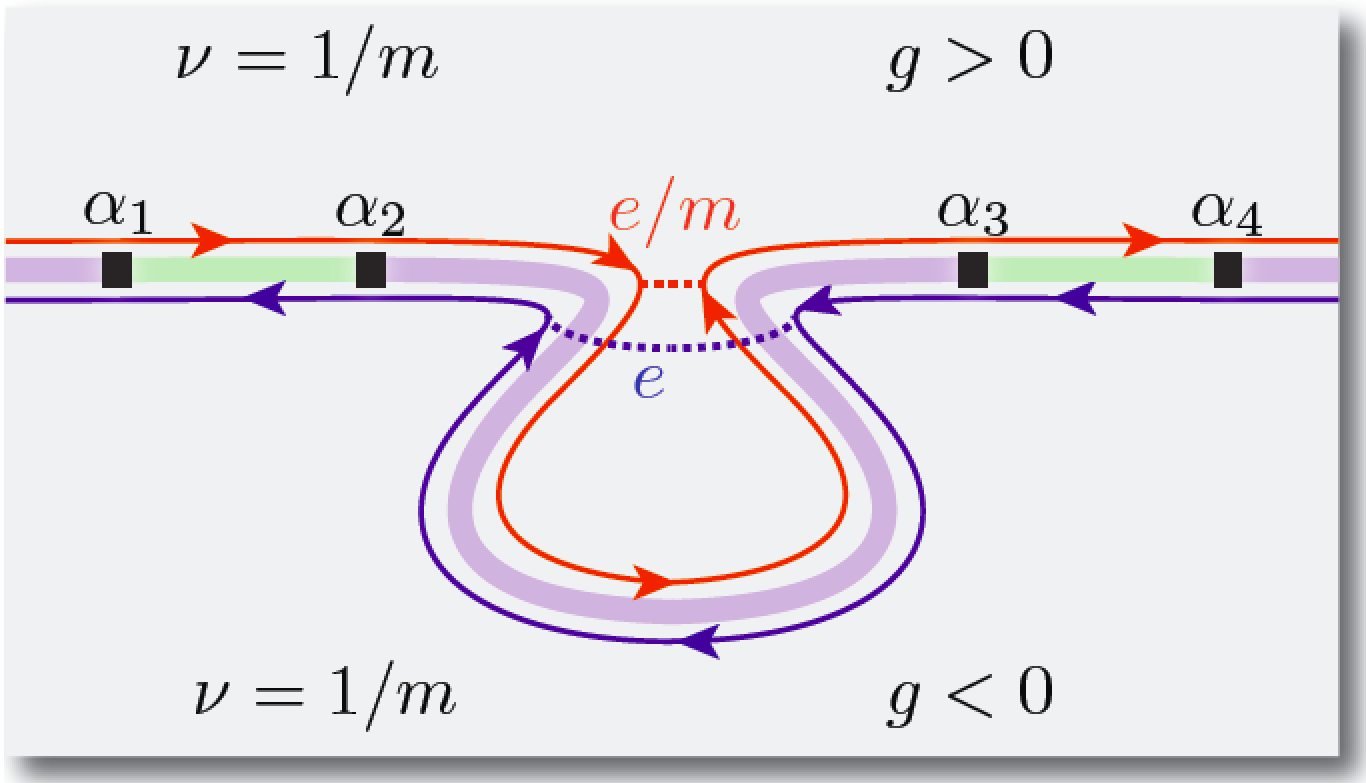


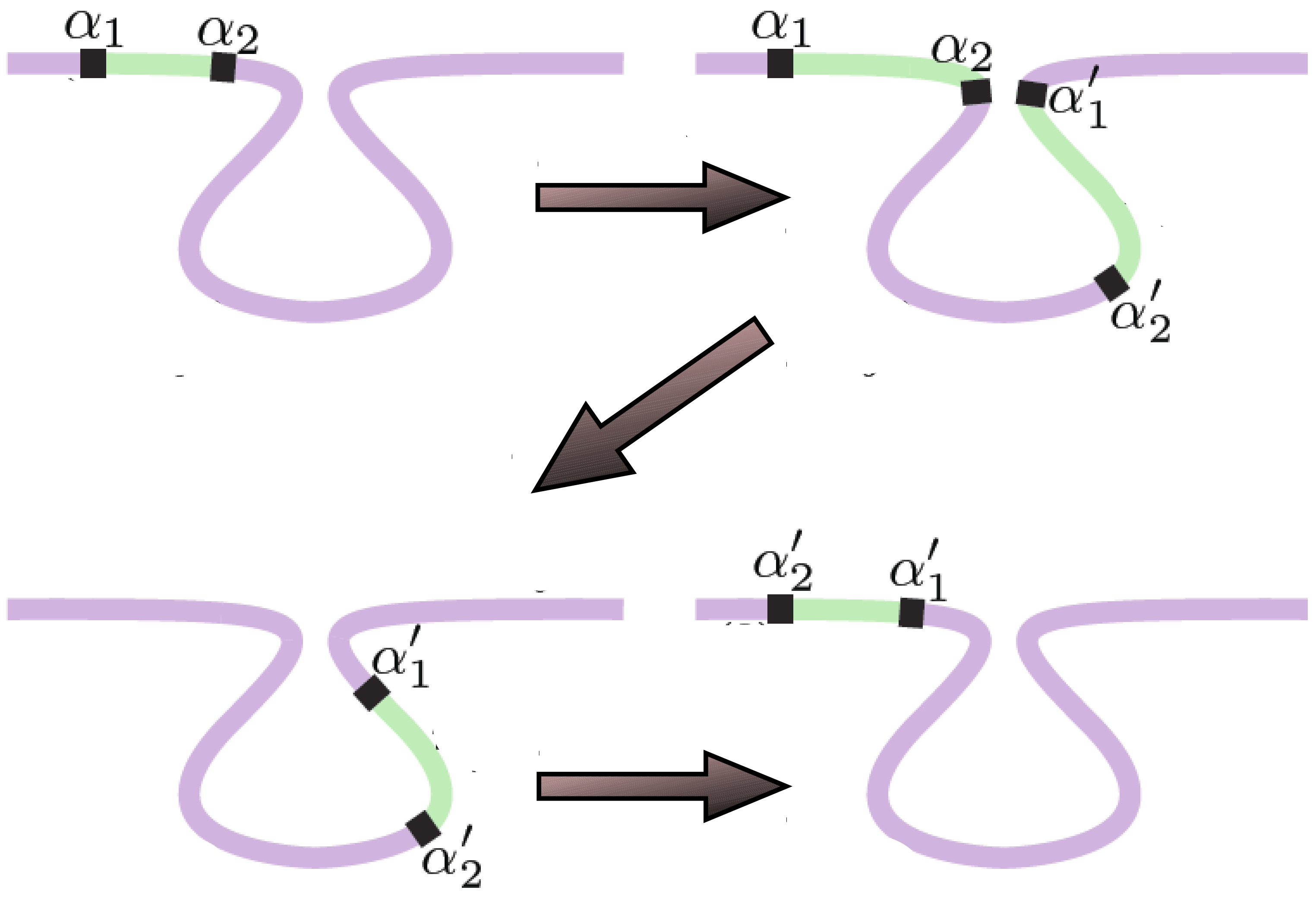
Exchanging end modes in our case

Apparent problem:

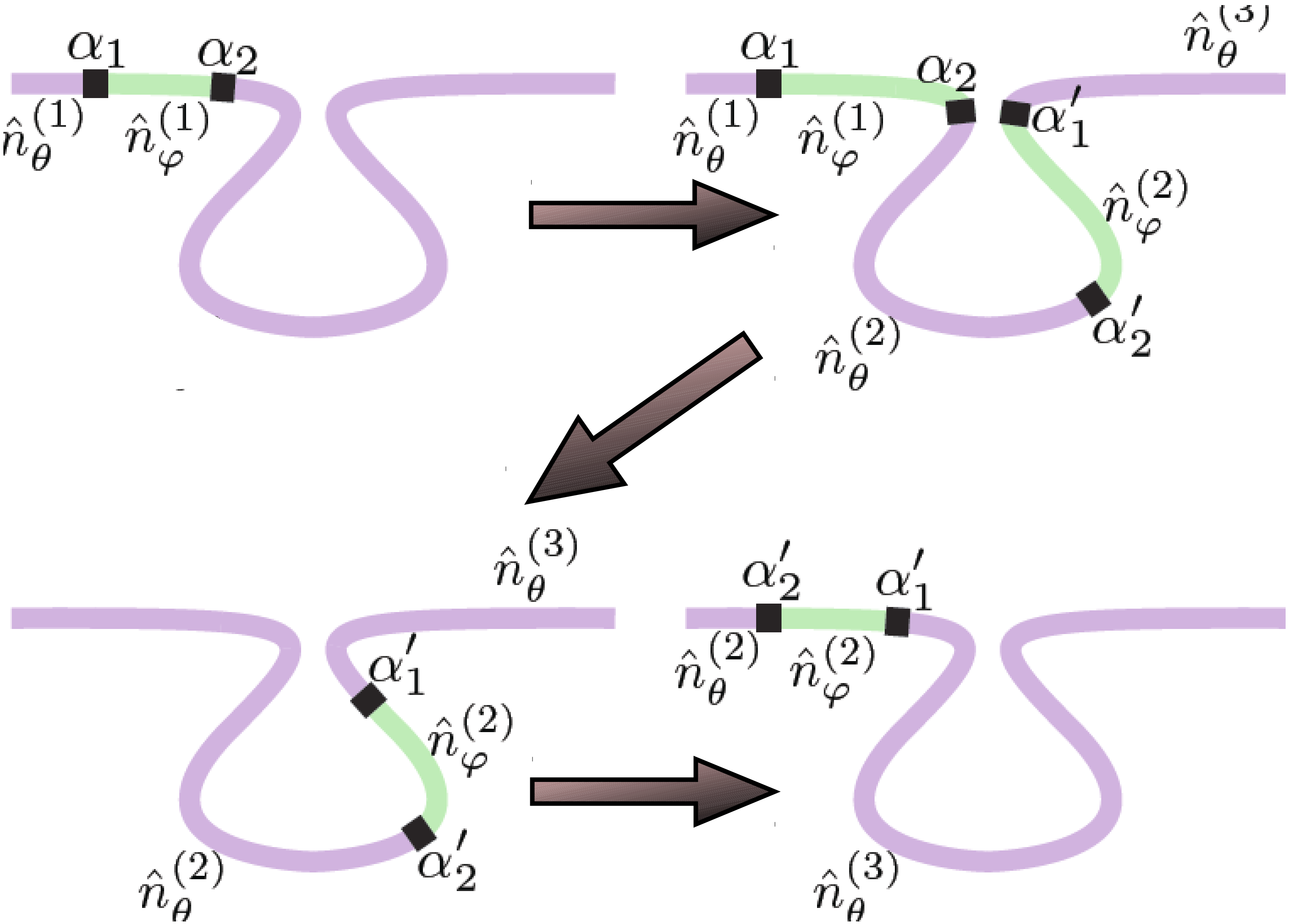
- We cannot have Y-junctions: our modes live on the domain walls..
- We can still exchange them:



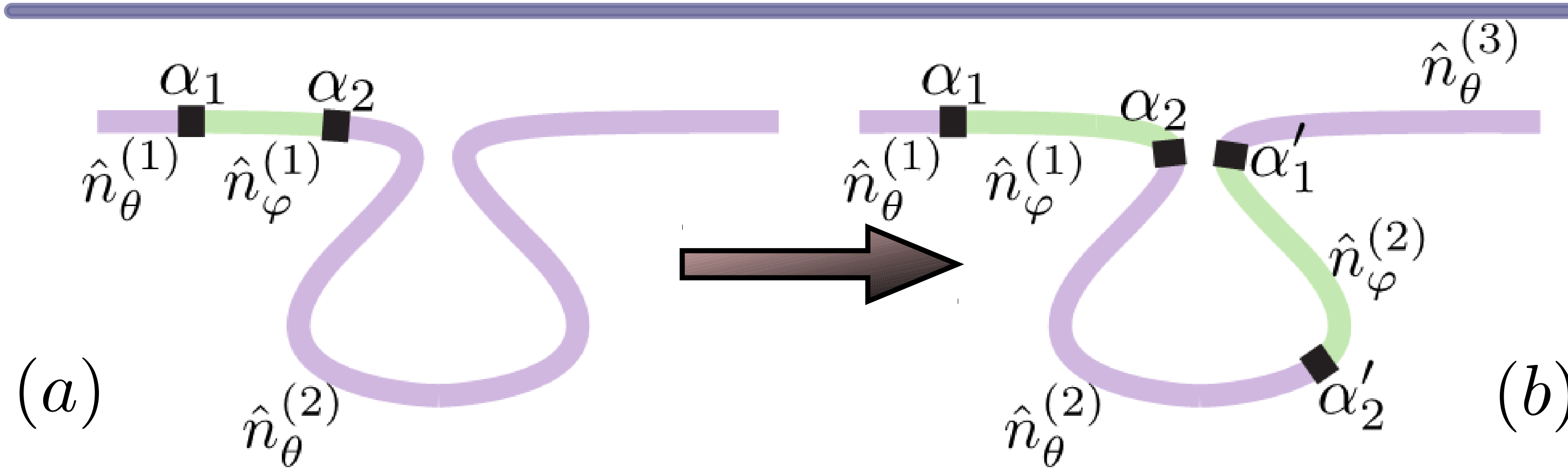
Exchanging end modes in our case



Exchanging end modes in our case

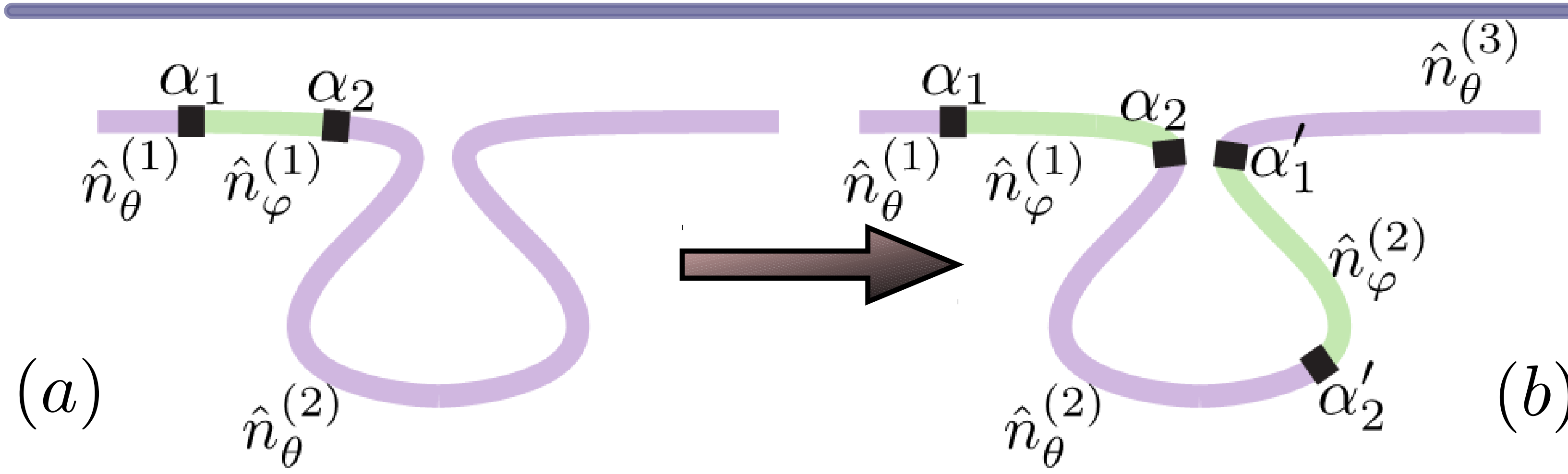


Exchanging end modes in our case



$$
\begin{aligned}
H_{a\to b} &= (t_J \alpha_2^\dagger \alpha_1' + H.c.) + (t \alpha_1'^\dagger \alpha_2' + H.c.) \\
&= -|t_J| \cos\left[\frac{\pi}{m}\left(\hat{n}_\varphi^{(2)} + \hat{n}_\theta^{(3)} - \hat{n}_\varphi^{(1)} - \hat{n}_\theta^{(2)}\right) + \beta\right] \\
&\quad -|t| \cos\left[\frac{\pi}{m}\left(\hat{n}_\theta^{(2)} - \hat{n}_\theta^{(3)}\right)\right]
\end{aligned}
$$

Exchanging end modes in our case



Integral of motion:

$$\chi \equiv e^{i\frac{\pi}{2m}}\alpha_2 \alpha_2'^{\dagger}\alpha_1' = e^{i\frac{\pi}{m}(\hat{n}_\varphi^{(1)}+\hat{n}_\theta^{(3)})}$$

Energy-minimizing condition:

$$\hat{n}_\varphi^{(2)} + \hat{n}_\theta^{(3)} - \hat{n}_\varphi^{(1)} - \hat{n}_\theta^{(2)} = k(\beta) \in \mathbb{Z}$$

Parafermion ZM Braiding

Upshot:

$$\alpha_1 \rightarrow e^{-i\frac{\pi}{m}k}\alpha_2$$

$$\alpha_2 \rightarrow e^{i\frac{\pi}{m}(1-k)}\alpha_1^\dagger\alpha_2^2$$

$m = 1$ (Majorana zero modes):

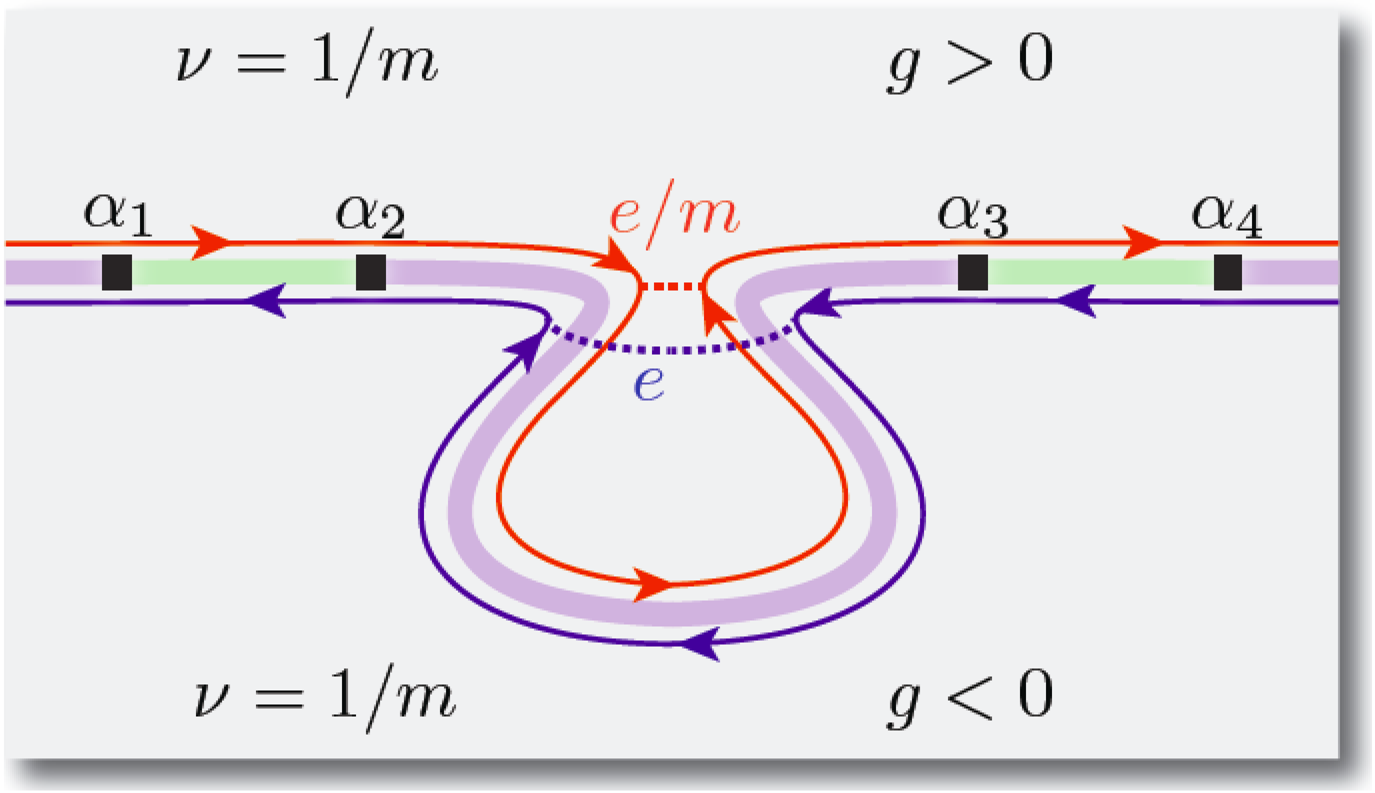
$$\gamma_1 \rightarrow \gamma_2$$

$$\gamma_2 \rightarrow -\gamma_1$$

Parafermion ZM Braiding

Important observation:

- If quasiparticles of both chiralities are allowed to tunnel, the braiding is not universal $\Rightarrow$ Potential problem for fractional TI!



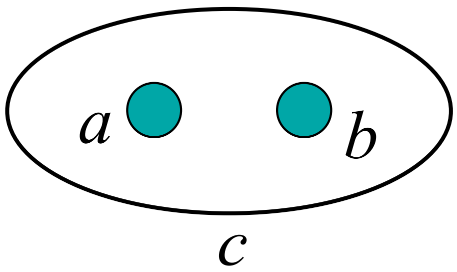
Topological Quantum Computation

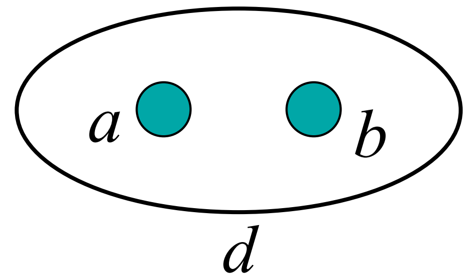
(Kitaev, Preskill, Freedman, Larsen, Wang)

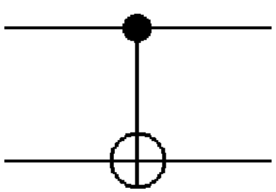
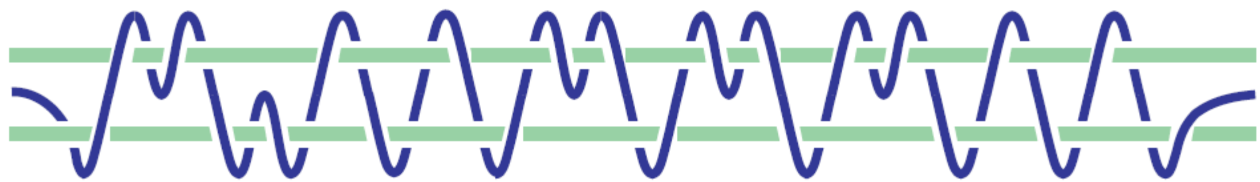
Things we need:

- Multidimensional Hilbert space where we can encode information → Qubits
- Ability to initialise and read-out a qubit
- Unitary operations → Quantum gates

Topological Quantum Computation

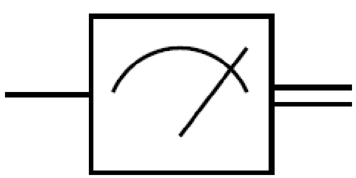
$|0\rangle \leftrightarrow$ 

$|1\rangle \leftrightarrow$ 

 $\leftrightarrow$ 

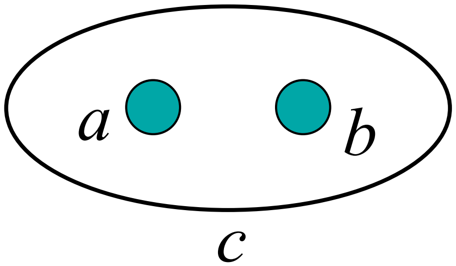
(Hormozi, Bonesteel, et. al.)

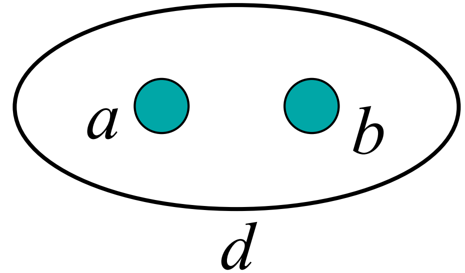
Or, perhaps use measurements to generate brading!
(Bonderson, Freedman, Nayak, 2009)

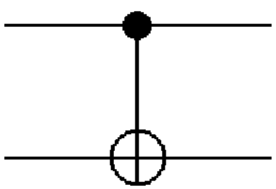
 $\leftrightarrow$ Interferometer

Bonderson, KS & Slingerland, PRL 2006, PRL 2007, Ann. Phys. 2008

Topological Quantum Computation

$|0\rangle \leftrightarrow$ 

$|1\rangle \leftrightarrow$ 

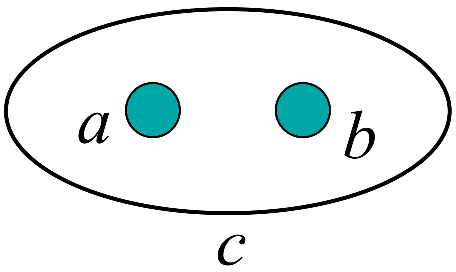
 $\leftrightarrow$ 

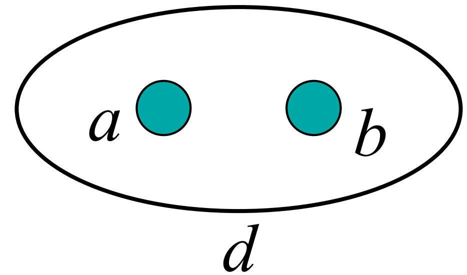
(Hormozi, Bonesteel, et. al.)

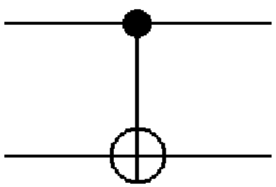
Or, perhaps use measurements to generate brading!
(Bonderson, Freedman, Nayak, 2009)

- Majorana zero modes are not universal!
  - No entangling gates with braiding alone
  - No phase gate

Topological Quantum Computation

$|0\rangle \leftrightarrow$ 

$|1\rangle \leftrightarrow$ 

 $\leftrightarrow$ 

(Hormozi, Bonesteel, et. al.)

Or, perhaps use measurements to generate brading!
(Bonderson, Freedman, Nayak, 2009)

- Parafermionic zero modes are still not universal...
  - Can do entangling gates!
  - No phase gate?

Conclusions

- Parafermionic zero modes can be localised in systems with counter-propagating fractionalised edge modes (FQHE, or fractional topological insulators)
  - Fractional Josephson effect with periodicity $4m\pi$
  - Zero-bias anomaly – similar to the Majorana case, but with fractionalised charge tunnelling
- Potential utility for quantum computing?

- D. Clarke, J. Alicea & KS, arXiv:1204.5479
- Parallel work:
  - N. Lindner, E. Berg, G. Refael & A. Stern, arXiv:1204.5733
  - M. Cheng, arXiv:1204.6084