Fractional Topogical Insulators

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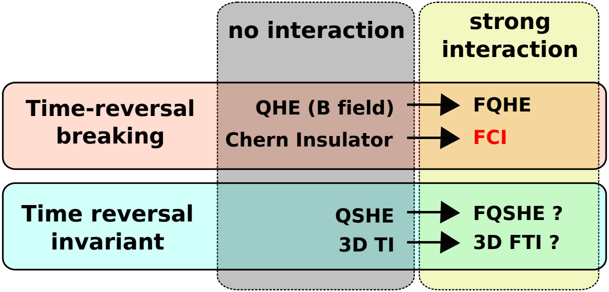
Motivations : FTI

A rich physics emerge when turning on strong interaction in QHE

What about Topological insulators ?



Outline

- Fractional Chern Insulators
- Entanglement spectroscopy
- Overlaps and $C > 1$
- FTI with time reversal symmetry

Fractional Chern Insulator

Interacting Chern insulators

- A Chern insulator is a zero magnetic field version of the QHE (Haldane, 88)
- Topological properties emerge from the band structure
- At least one band is a non-zero Chern number C, Hall conductance $\sigma_{xy} = \frac{e^2}{h}|C|$
- Basic building block of 2D $\mathbb{Z}_2$ topological insulator (half of it)
- Is there a zero magnetic field equivalent of the FQHE ? $\rightarrow$ Fractional Chern Insulator
- Here we will focus on the $C = \pm 1$.

From CI to FCI

To go from IQHE to FQHE, we need to :

- consider a single Landau level
- partially fill this level, $\nu = N/N_\Phi$
- turn on repulsive interactions

From CI to FCI

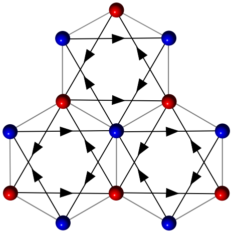
To go from IQHE CI to FQHE FCI, we need to :

- consider a single Landau level
  consider a single band
- partially fill this level, $\nu = N/N_\Phi$
  partially fill this band, $\nu = N/N_{\text{unit cells}}$
- turn on repulsive interactions
  turn on repulsive interactions

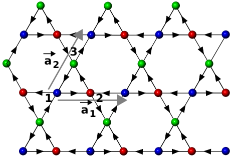
What QH features should we try to mimic to get a FCI ?

- Several proposals for a CI with nearly flat band that may lead to FCI
- But "nearly" flat band is not crucial for FCI like flat band is not crucial for FQHE (think about disorder)

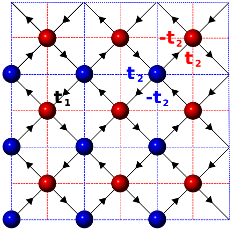
Four (almost) flat band models



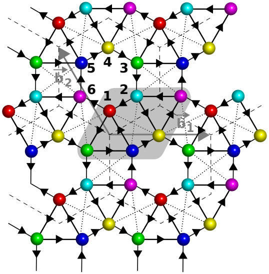
Haldane model,
Neupert et al. PRL (2011)



Kagome lattice,
E. Tang et al. PRL (2011)

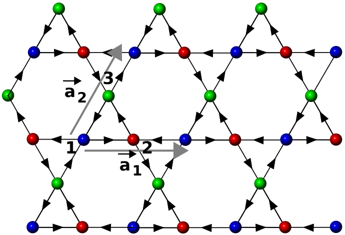


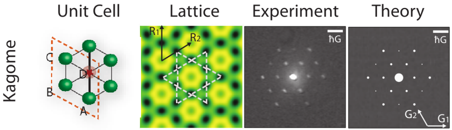
Checkerboard lattice,
K. Sun et al. PRL (2011).



Ruby lattice, PRB (2011)

The Kagome lattice model





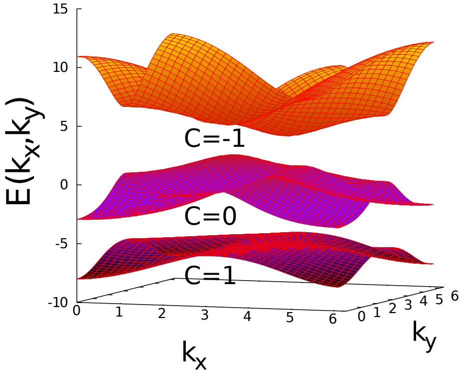
Jo et al. PRL (2012)

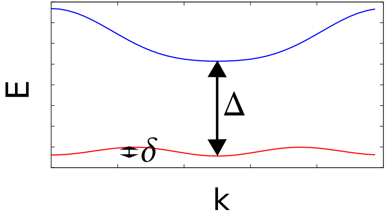
- three atoms per unit cell, spinless particles
- lattice can be realized in cold atoms
- only nearest neighbor hopping $e^{i\varphi}$
- three bands with Chern numbers $C = 1,\ C = 0$ and $C = -1$

$$\mathcal{H}(\mathbf{k}) = -t_1 \begin{bmatrix} 0 & e^{i\varphi}(1+e^{-ik_x}) & e^{-i\varphi}(1+e^{-ik_y}) \\ & 0 & e^{i\varphi}(1+e^{i(k_x-k_y)}) \\ \mathrm{h.c.} & & 0 \end{bmatrix}$$

$k_x = \mathbf{k}.\mathbf{a_1}, k_y = \mathbf{k}.\mathbf{a_2}$

The flat band limit





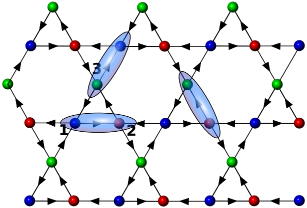
- $\delta \ll E_c \ll \Delta$ (E_c being the interaction energy scale)
- We can deform continuously the band structure to have a perfectly flat valence band
- and project the system onto the lowest band, similar to the projection onto the lowest Landau level

$$\mathcal{H}(\mathbf{k}) = \sum_{n=1}^{\text{nbr bands}} \mathcal{P}_n E_n(\mathbf{k})$$

$$\longrightarrow \quad \mathcal{H}^{FB}(\mathbf{k}) = \sum_{n=1}^{\text{nbr bands}} n\mathcal{P}_n$$

Two body interaction and the Kagome lattice

Our goal : stabilize a Laughlin-like state at $\nu = 1/3$.
A key property : the Laughlin state is the unique densest state that screens the short range repulsive interaction.



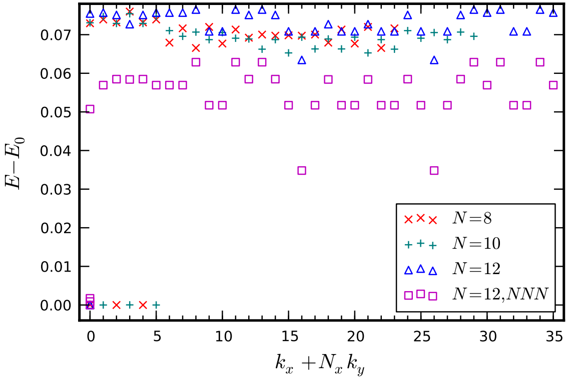
$$H_{\mathrm{int}}^{F} = U \sum_{<i,j>} : n_i n_j :$$

$$H_{\mathrm{int}}^{B} = U \sum_{i} : n_i n_i :$$

- A nearest neighbor repulsion should mimic the FQH interaction.
- We give the same energy penalty when two part are sitting on neighboring sites (for fermions) or on the same site (for bosons).
- On the checkerboard lattice : Neupert et al. PRL 106, 236804 (2011), Sheng et al. Nat. Comm. 2, 389 (2011), NR and BAB, PRX (2011)

The $\nu = 1/3$ filling factor

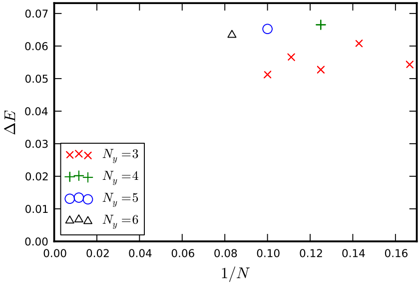
An **almost** threefold degenerate ground state as you expect for the Laughlin state on a torus (here lattice with periodic BC)



But 3fold degeneracy is not enough to prove that you have Laughlin-like physics there (a CDW would have the same counting).

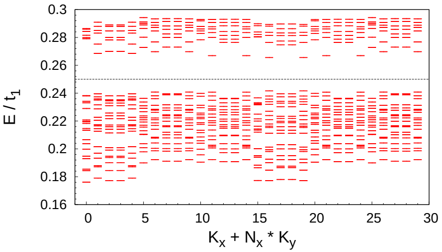
Gap

- Many-body gap can actually increase with the number of particles due to aspect ratio issues.
- Finite size scaling not and not monotonic reliable because of aspect ratio in the thermodynamic limit.
- The 3-fold degeneracy at filling 1/3 in the continuum exists for any potential and is not a hallmark of the FQH state. On the lattice, 3-fold degeneracy at filling 1/3 means more than in the continuum, but still not much



Quasihole excitations

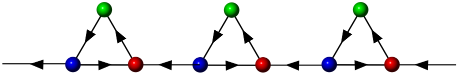
- The form of the groundstate of the Chern insulator at filling 1/3 is not exactly Laughlin-like. However, the universal properties SHOULD be.
- The hallmark of FQH effect is the existence of fractional statistics quasiholes.
- In the continuum FQH, Quasiholes are zero modes of a model Hamiltonians - they are really groundstates but at lower filling. In our case, for generic Hamiltonian, we have a gap from a low energy manifold (quasihole states) to higher generic states.

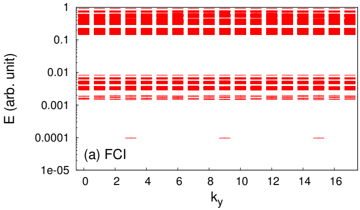


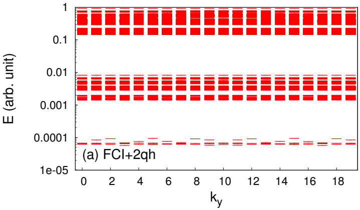
$N = 9,\ N_x = 5, N_y = 6$

The number of states below the gap matches the one of the FQHE !

The one dimensional limit : thin torus

- let's take $N_x = 1$, thin torus limit
- **the interacting system can be solved exactly for a specific model.**
- 
- the groundstate is just the electrostatic solution (1 electron every 3 unit cells)
- **a charge density wave** and not a Laughlin state



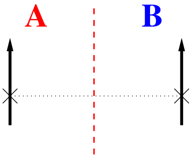


Can we differentiate between a Laughlin state and a CDW ?

Entanglement spectroscopy

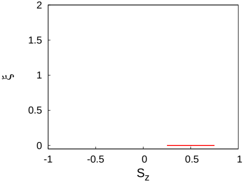
Entanglement spectrum - Li and Haldane, PRL (2008)

example : system made of two spins 1/2

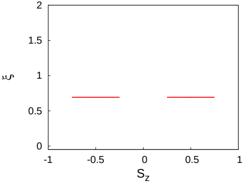


$\rho = |\Psi\rangle\langle\Psi|$, reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$
Entanglement spectrum : $\xi = -log(\lambda)$ (λ eigenvalues of ρ_A) as a function of S_z

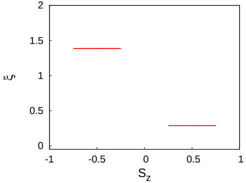
$$|\uparrow\uparrow\rangle$$



$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$



$$\frac{1}{\sqrt{4}}|\uparrow\downarrow\rangle + \sqrt{\frac{3}{4}}|\downarrow\uparrow\rangle$$



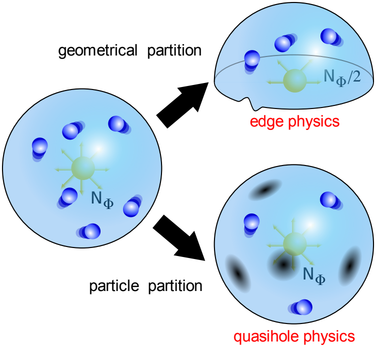
The counting (i.e the number of non zero eigenvalue) also provides informations about the entanglement

How to cut the system ?

The system can be cut in different ways :

- real space
- momentum space
- particle space

Each way may provide different information about the system (ex : trivial in momentum space but not in real space)



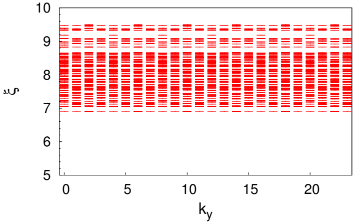
- **Orbital partitioning (OES)** : extracting the edge physics
- **Particle partitioning (PES)** : extracting the bulk physics

Particle entanglement spectrum

Particle cut : start with the ground state Ψ for N particles, remove $N - N_A$, keep N_A

$$\rho_A(x_1, \dots, x_{N_A}; x'_1, \dots, x'_{N_A})$$
$$= \int \dots \int dx_{N_A+1} \dots dx_N \quad \Psi^*(x_1, \dots, x_{N_A}, x_{N_A+1}, \dots, x_N)$$
$$\times \quad \Psi(x'_1, \dots, x'_{N_A}, x_{N_A+1}, \dots, x_N)$$

"Textbook expression" for the reduced density matrix.

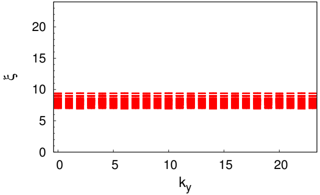


Laughlin $\nu = 1/3$ state $N = 8$, $N_A = 4$ on a torus

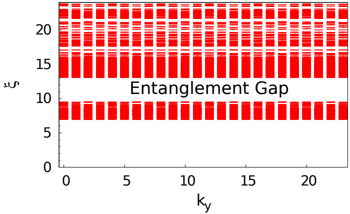
- Counting is the number of quasihole states for N_A particles on the same geometry
- the fingerprint of the phase.
- This information that comes from the bulk excitations is encoded within the groundstate !

Away from model states : Coulomb groundstate at $\nu = 1/3$

- Coulomb groundstate at $\nu = 1/3$ has the same universal properties than the Laughlin state
- The ES exhibits an entanglement gap.
- Depending on the geometry, this gap collapses after a few momenta away from the maximum one (the system "feels" the edge) or is along the full range of momenta (torus).
- The part below the gap has the same fingerprint than the Laughlin state : the entanglement gap protects the state statistical properties.



Laughlin on torus $\nu = 1/3$

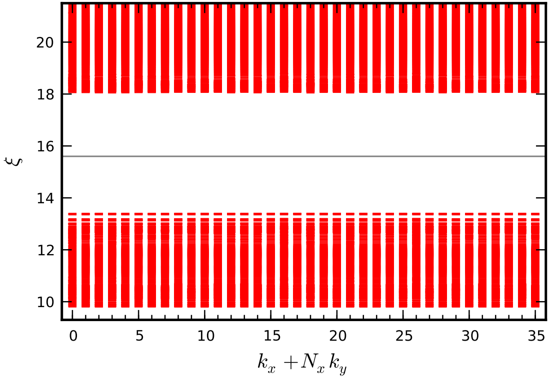


Coulomb on torus $\nu = 1/3$

Back to the FCI

Particle entanglement spectrum

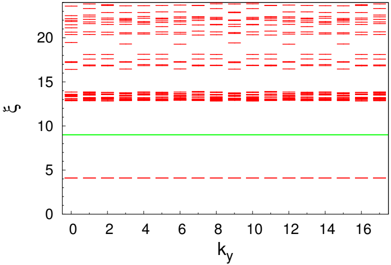
Back to the Fractional Chern Insulator



PES for $N = 12$, $N_A = 5$, 2530 states per momentum sector below the gap as expected for a Laughlin state

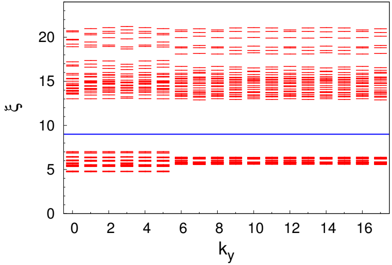
Particle entanglement spectrum : CDW

The PES for a CDW can be computed exactly and is not identical to the Laughlin PES



$\nu = 1/3,\ N_x = 1,\ N = 6,\ N_A = 3$

59 states below the gap $\longrightarrow$ CDW

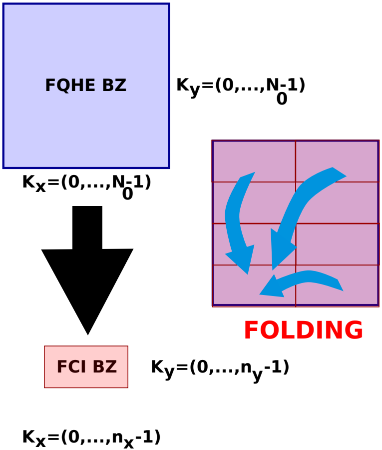


$\nu = 1/3,\ N_x = 6,\ N = 6,\ N_A = 3$

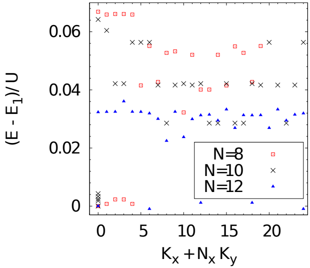
329 states below the gap $\longrightarrow$ Laughlin

Emergent Symmetries in the Chern Insulator

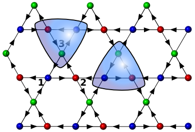
- in FQH, we have the magnetic translational algebra
- In FCIs, there is in principle no exact degeneracy (apart from the lattice symmetries).
- But both the low energy part of the energy and entanglement spectra exhibit an emergent translational symmetry.
- The momentum quantum numbers of the FCI can be deduced by folding the FQH Brillouin zone.
- FQH : $N_0 = GCD(N, N_\phi = N_x \times N_y)$
  FCI : $n_x = GCD(N, N_x)$,
  $n_y = GCD(N, N_y)$

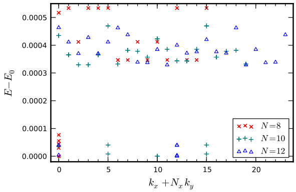


Beyond the Laughlin states



- A clear signature for composite fermion states at $\nu = 2/5$ and $\nu = 3/7$ (here Kagome at $\nu = 2/5$)
- Also observed for bosons at $\nu = 2/3$ and $\nu = 3/4$.

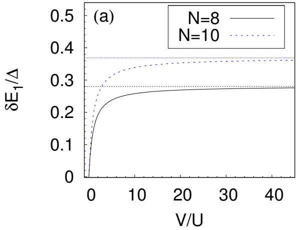




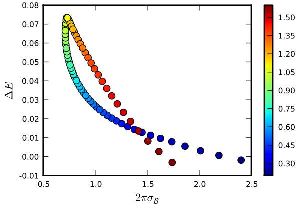
- Moore-Read state. Possible non-abelian candidate for $\nu = 5/2$ in the FQHE.
- MR state can be exactly produced using a three-body interaction.
- FCI require 3 body int.

FCI : a perfect world ?

- Not all models produce a Laughlin-like state
- Depends on the particle statistics : Haldane model fermions vs bosons
- Longer range interactions destabilize FCI
- Even more model dependent for the other states
- Does a flatter Berry curvature help ? Not really
- What are the key ingredients to get a robust FCI ?



Ruby+bosons $\nu = 2/3$

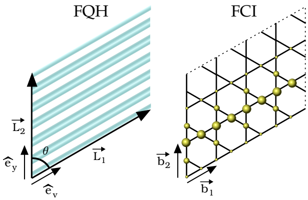


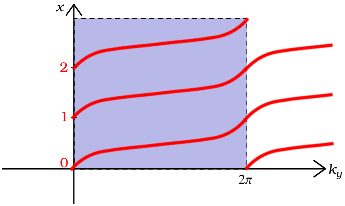
Kagome $N = 8$ and $N_x = 6, N_y = 4$

Overlaps and $C > 1$

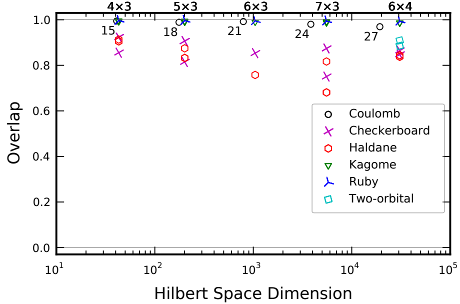
Wannier basis, computing overlaps

- Wannier states form a basis of maximally localized states in one direction $\hat{x}|W(n,k_y)\rangle = x_n|W(n,k_y)\rangle$
- flow into each other when $k_y \to k_y + 2\pi$: $|W(n,k_y+2\pi)\rangle \to |W(n+C,k_y)\rangle$
- mimic the Landau orbitals . Qi's prescription (PRL 2012) : take a FQH state and replace the Landau orbitals with the Wannier states.
- This prescription fails to produce good overlaps . Wannier states are not orthogonal in finite size (trade maximal localization to restore orthogonality). Need a correct gauge choice.



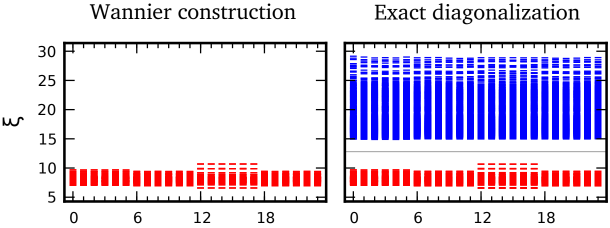


Overlaps at $\nu = 1/3$



- Large overlap with the Laughlin $\nu = 1/3$ (up to 99.9% depending on the model)
- Results are coherent with the PES observation (larger ent gap = better model)
- See also G. Moller's talk, Wu et al. arxiv :1207.4439

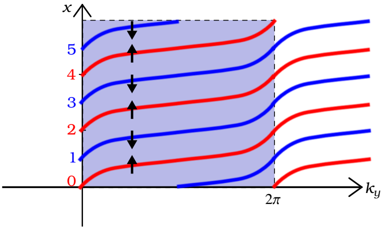
PES for the Kagome model $N = 8, N_A = 4$:



Beyond $C = 1$

Several almost flat band models with Chern number $C > 1$:

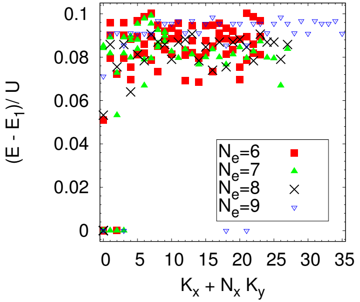
- Wang and Ran, PRB 2011 $C = 2$
- Wang et al, arxiv :1204.1697 $C = 2$
- Trescher and Bergholtz, arxiv :1205.2245 $C = N$
- Yang et al. arxiv :1205.5792 $C = N$



- Barkeshli and Qi, arXiv :1112.3311
- it decouples into spin up and down in the non-interacting case
- Do we get FQH bilayer physics for $C = 2$?

Example : $C = 3$

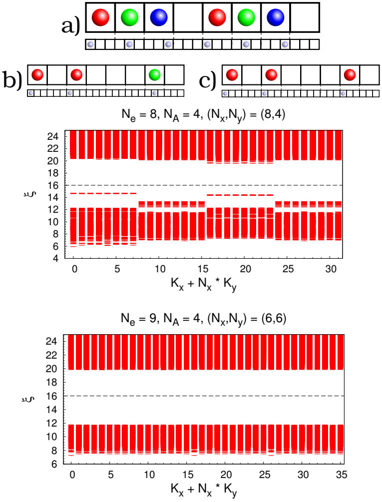
Several numerical evidences for topological states (arxiv :1204.1697 and arxiv :1206.3759), including states that might be related to the NASS (with $k+1$-body interaction)



- Example $C = 3$ on the 2−orbital model of Yang et al.
- bosons at $\nu = 1/4$, with on-site two body repulsion
- the (almost) degeneracy matches the 3-component Halperin (221) states ...
- ... but also the Laughlin $\nu = 1/4$ counting !
- even worse, the degeneracy appears were it should not be (only for N a multiple of C)
- **Are these really spinful (or colorful) states ?**

Example : $C = 3$ and PES

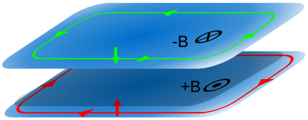
- **spinless FQH :** PES matches the qh counting for $N_A \leq \frac{N}{2}$.
- C**-color FQH :** PES gives the qh counting if $N_A \leq \lceil \frac{N}{C} \rceil$. For $N_A > \lceil \frac{N}{C} \rceil$, the counting is lower than the full quasihole counting.
- It can be obtained by removing some $SU(C)$ multiplets $\rightarrow$ clearly differs from the spinless counting
- A similar situation is observed for the $C > 2$ FCIs
- but the counting does not match the PES colorful FQH counting all the time.

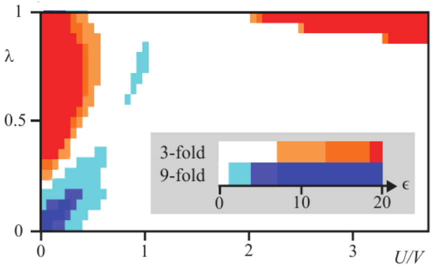


FTI with time reversal symmetry

From FCI to FQSH

- QSH can be built from two CI copies
- One can do the same for FQSH
- How stable if the FQSH wrt when coupling the two layers ? Coupling via interaction or the band structure
- Neupert et al., PRB 84, 165107 (2011) using the checkerboard lattice
- Not really conclusive for the FQSH : does it survive beyond the single layer gap ?

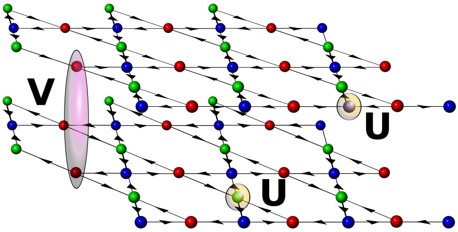


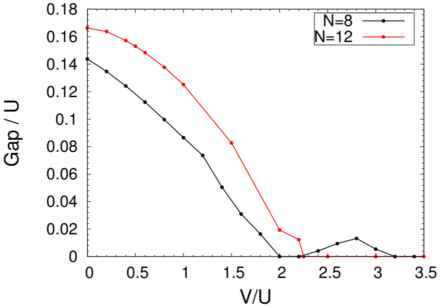


$N = 16$ at $\nu = 2/3$
V intralayer, U on-site interlayer, λ NN interlayer

From FCI to FQSH

- Two copies of the Kagome model with bosons.
- Hubbard model with two parameters for the interaction : U on-site same layer, V on-site interlayer





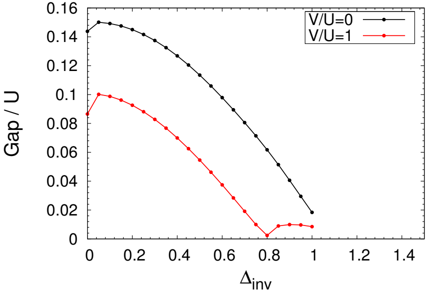
From FCI to FQSH

We can also couple the two layers through the band structure by adding an inversion symmetry breaking term.

$$H(\mathbf{k}) = \begin{bmatrix} h_{\mathrm{CI}}(\mathbf{k}) & \Delta_{inv} C \\ \Delta_{inv} C^{+} & h^{*}_{\mathrm{CI}}(-\mathbf{k}) \end{bmatrix}$$

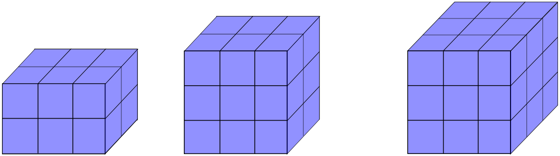
with $C = -C^t$, here

$$C_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$



3D FTI : a technical challenge

- an unknown territory : nature of the excitations (strings ?), effective theory for the surface modes (beyond Luttinger ?), algebraic structure (GMP algebra in 3D ?)
- is there a microscopic model ?
- example : Fu-Kane-Mele model with interaction for N electrons at filling $\nu = 1/3$



3x2x2	3x3x2	3x3x3
dim=61,413	dim=69,538,908	dim=3,589,864,780,047
960kb	1Gb	52Tb

Conclusion

- Fractional topological insulator at zero magnetic field exists as a proof of principle.
- A clear signature for several states : Laughlin, CF and MR (using many body interactions)
- There is a counting principle that relates the low energy physics of the FQHE and the FCI
- Entanglement spectrum powerful tool to understand strongly interacting phases of matter.
- PES results are now backed up by overlap calculations
- Interesting physics beyond $C = 1$
- What are the good ingredients for an FCI ? Does the knowledge of the one body problem is enough ?
- Do we observe spinful FQH for $C > 1$? A counting principle is needed...