Local Representations of the Loop Braid Group

Joost Slingerland

National University of Ireland Maynooth

In collaboration with

A. Kudryashov

R. Ainsworth

C. M. Lee

science foundation ireland
fondúireacht eolaíochta éireann

NUI MAYNOOTH
Ollscoil na hÉireann Má Nuad

From Braid Groups to Motion Groups

Particle exchange statistics

Described by the permutation group S_n (d>2) or the braid group B_n (d=2)

Comes about as
- Fundamental group of configuration space
- We can think of this as the π_1 of the space of "embeddings" of n points into space

Exchange behavior of generalized "particles"

Most generally, consider a "particle manifold" S inside a "space manifold" M.
D.M. Dahm,PhD thesis 1962, Princeton; D.L. Goldsmith, Michigan Math. J. 28, 1981

The Motion Group Mot(M;S) describes
"motions of/in M which bring N back to its original configuration in a nontrivial way"

We can think of Mot(M;S) (loosely) as the π_1 of the space of embeddings of S into M.

Formally one can define e.g. Mot(M;S) = $\pi_1(\mathfrak{Hom}_c(M); \mathfrak{Hom}_c(M;S))$

Usually, S is disconnected, e.g. if S={n points}, get S_n, B_n, Torus/Sphere braid groups etc. for different choices of M.

Here we will look at the case S={n unlinked unknotted circles} with $M=R^3$
We call this $\mathbf{Mot_n}$. It is also known as the Loop Braid Group.

Naming, similar work: J.C. Baez, D.K. Wise, A.S. Crans, Adv.Theor.Math.Phys.11,2007, also X.-S. Lin

Loops can perform some moves that particles can't...

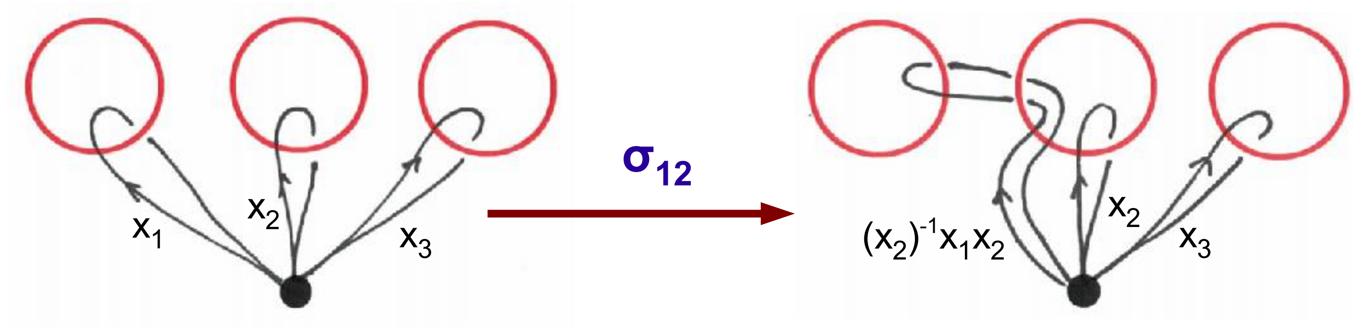
In the actual presentation, videos showing loops leapfrogging followed here....

Generators and their action on fundamental groups

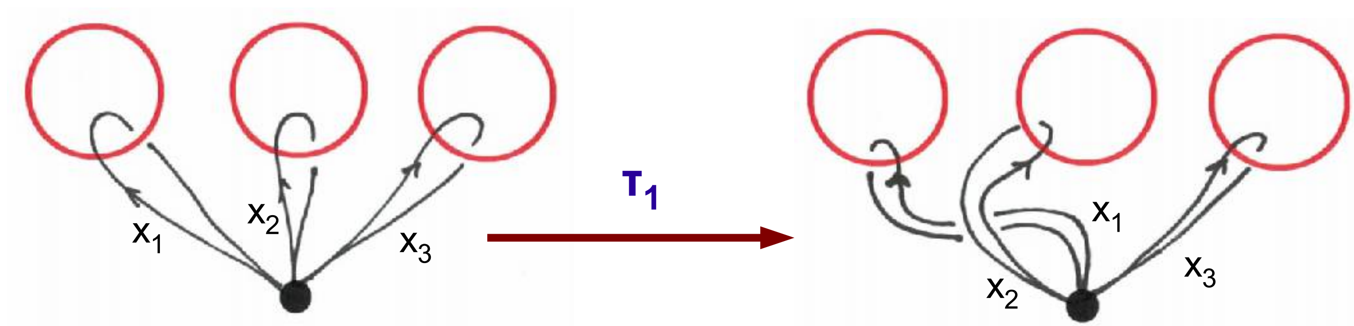
Mot_n is generated by three types of motions:
Slides (leapfrogging), Flips (flip ring over) and Exchanges (simply exchange rings like particles)

They can be conveniently described by their action on $\pi_1(R^3\backslash\{n\ \text{rings}\})=F_n$:

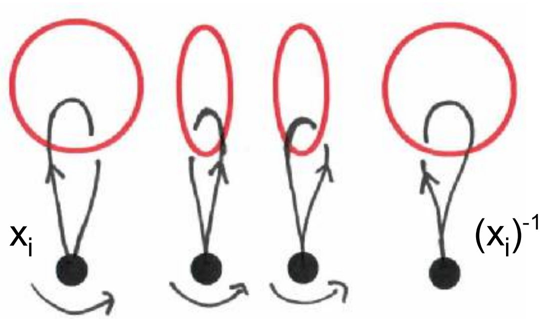
Slides σ_{ij}



Exchanges τ_i
(indistinguishable rings)



Flips f_i

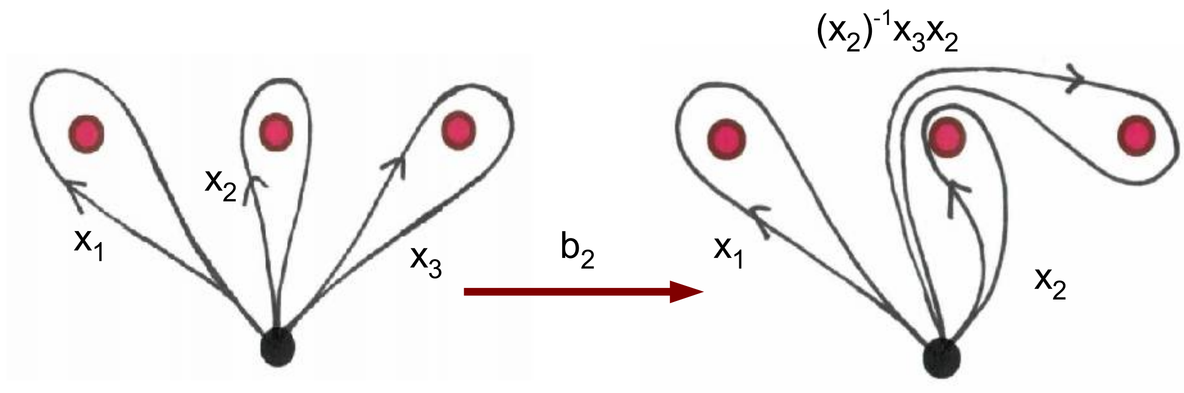


Actions of all generators of Mot_n on F_n

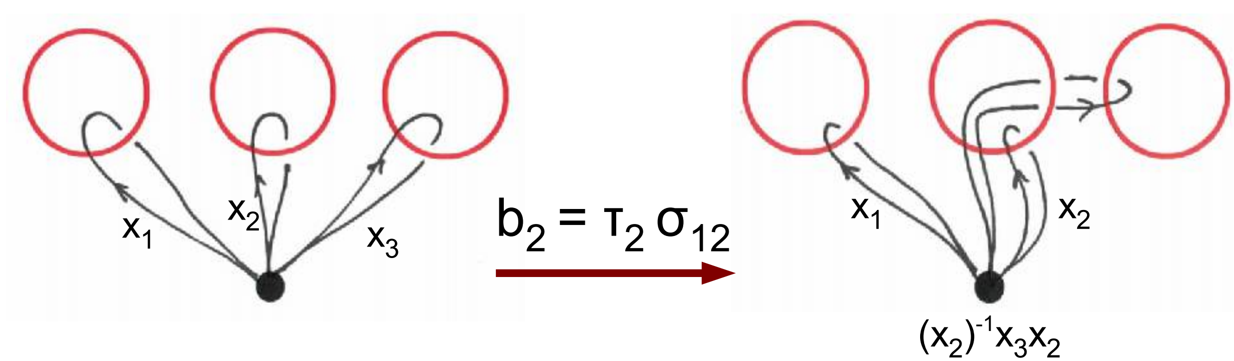
$$\begin{aligned}
\tau_i(x_1,\ldots,x_i,x_{i+1},\ldots,x_N) &= (x_1,\ldots,x_{i+1},x_i,\ldots,x_N)\\
f_i(x_1,\ldots,x_i,\ldots,x_N) &= (x_1,\ldots,x_i^{-1},\ldots,x_N)\\
\sigma_{ij}(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) &= (x_1,\ldots,x_j^{-1}x_ix_j,\ldots,x_j,\ldots,x_N)
\end{aligned}$$

Relation to the Braid Group

B_n can also be described by its action on $\pi_1(R^2\backslash\{n \text{ points}\})=F_n$:



The braid group lives inside the loop braid group!



General braid generators $b_i = \tau_i\,\sigma_{i,i+1,}$

Satisfy braid relations – same representation as above!

The full set of relations for Mot_n

$$\mathrm{Mot}_n = \mathrm{Slide}_n \rtimes (\mathbb{Z}_2)^n \rtimes S_n$$

Slides

$$\begin{aligned} \sigma_{ij}\sigma_{kl} &= \sigma_{kl}\sigma_{ij} \quad (i,j,k,l \text{ distinct}) \\ \sigma_{ik}\sigma_{jk} &= \sigma_{jk}\sigma_{ik} \quad (i \neq j) \\ \sigma_{ij}\sigma_{kj}\sigma_{ik} &= \sigma_{ik}\sigma_{kj}\sigma_{ij} \quad (i,j,k \text{ distinct}) \end{aligned}$$

"Yang-Baxter Equation"

Flips, $(\mathbb{Z}_2)^n$

$$f_i f_j = f_j f_i, \quad f_i^2 = e$$

Exchanges, S_n

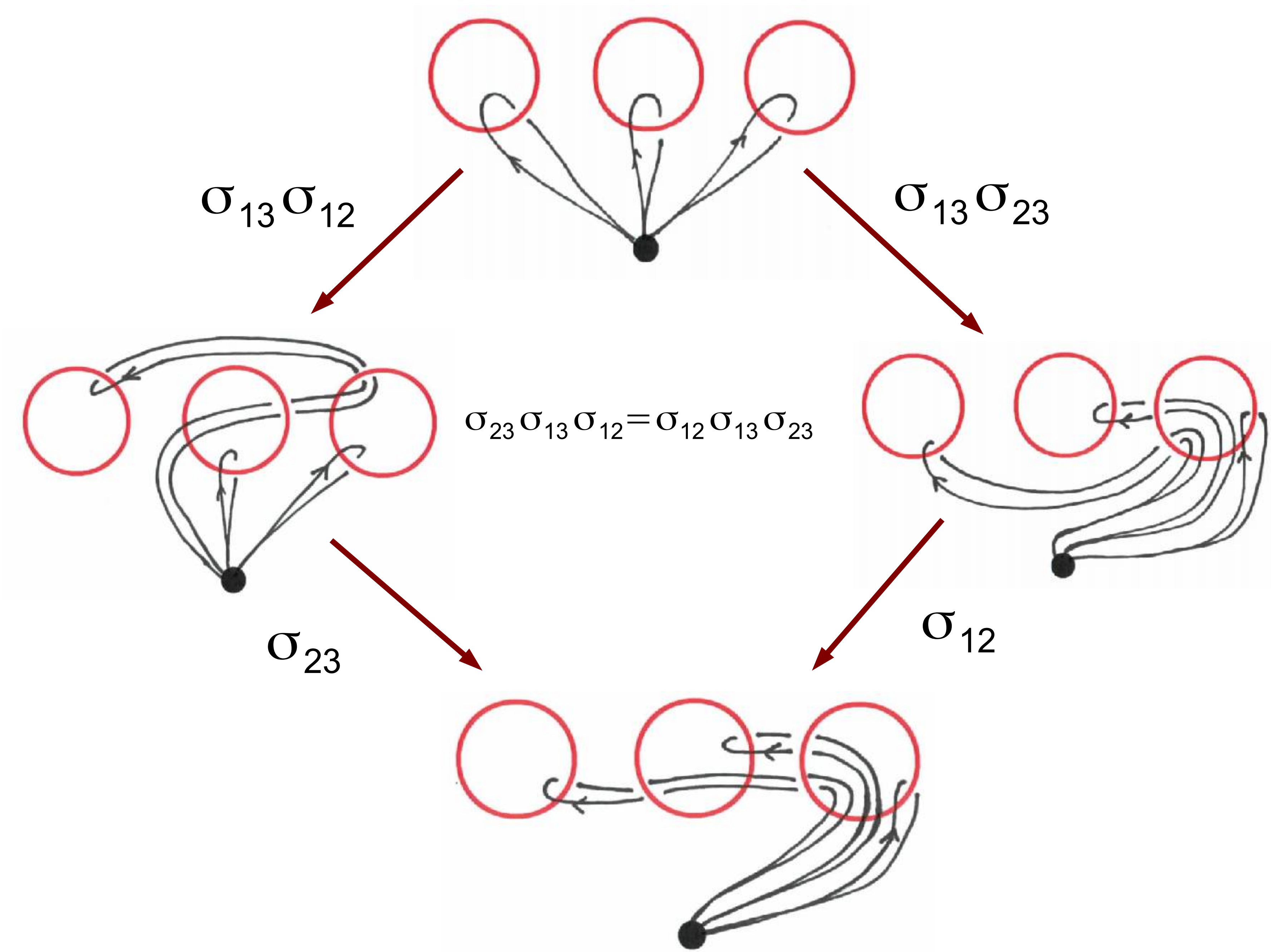
$$\begin{aligned} \tau_i\tau_j &= \tau_j\tau_i \quad (|i-j| \geq 2), \quad \tau_i^2 = e \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \end{aligned}$$

cross relations

$$\begin{aligned} \tau^{-1} f_i \tau &= f_{\tau(i)} \\ \tau^{-1}\sigma_{ij}\tau &= \sigma_{\tau(i)\tau(j)} \\ f_i\sigma_{jk}f_i &= \sigma_{jk} \quad (i \neq k) \\ f_i\sigma_{ji}f_i &= \sigma_{ji}^{-1}. \end{aligned}$$

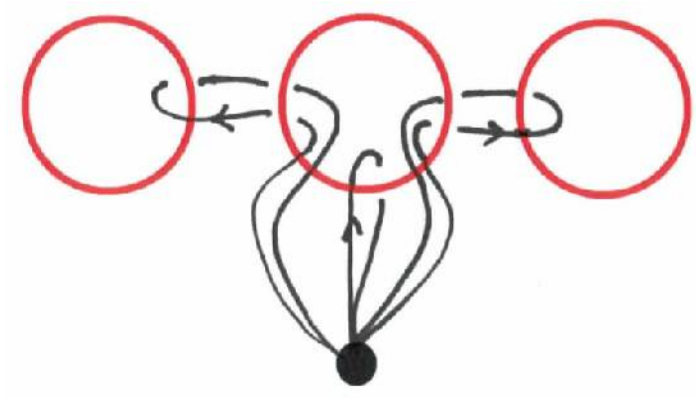
Note:
The slides generate a normal subgroup of finite index in Mot_n.
The braid group is not a normal subgroup and has infinite index.
This complicates the relation to anyon models (e.g. many braid groups).

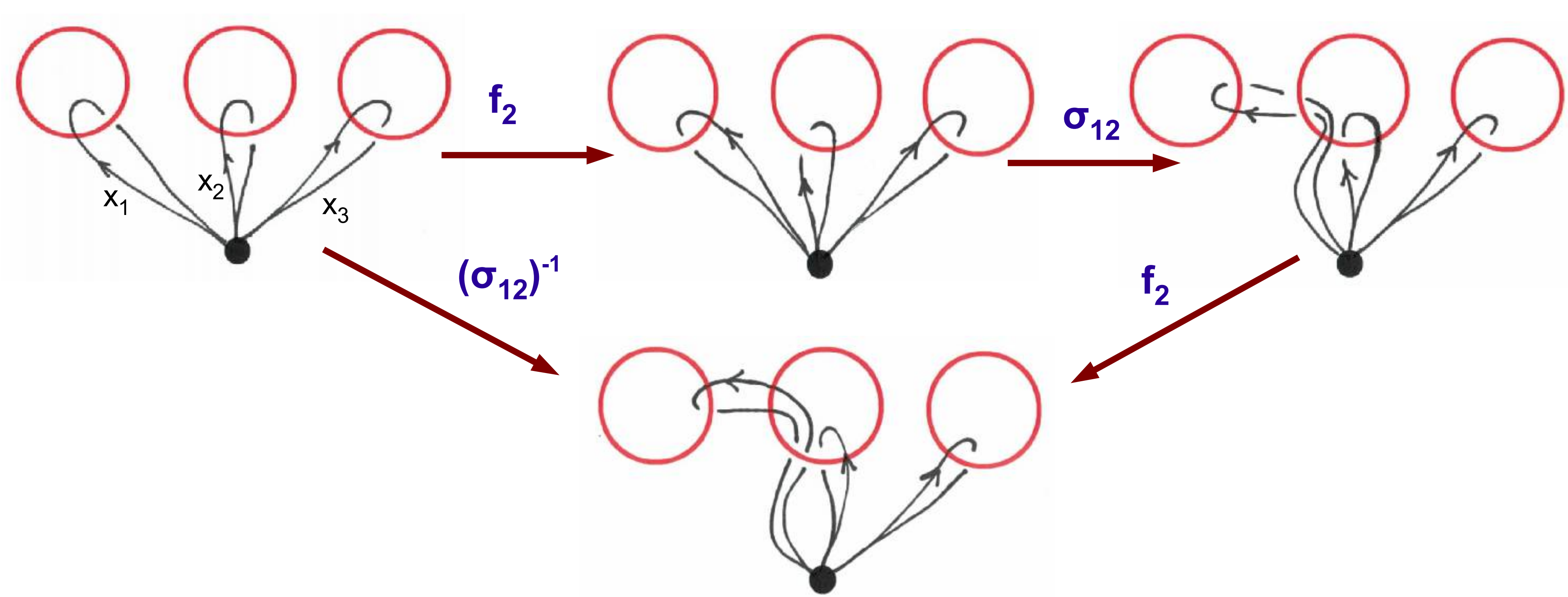
Some relations in pictures



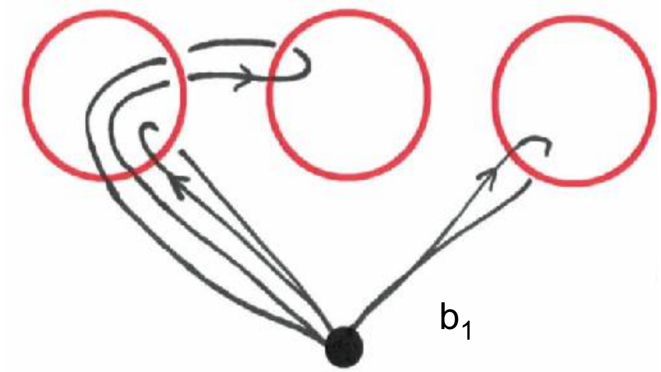
Some relations in pictures

$$\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}$$

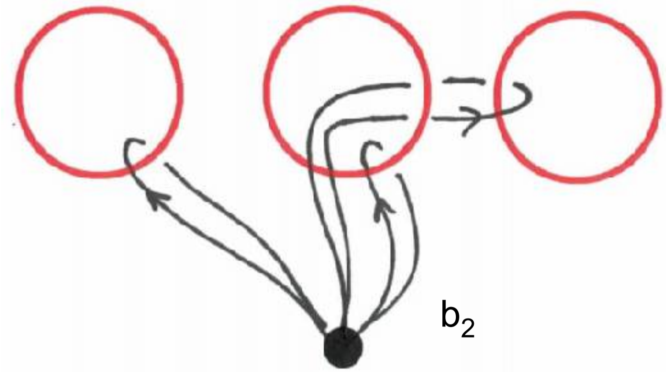




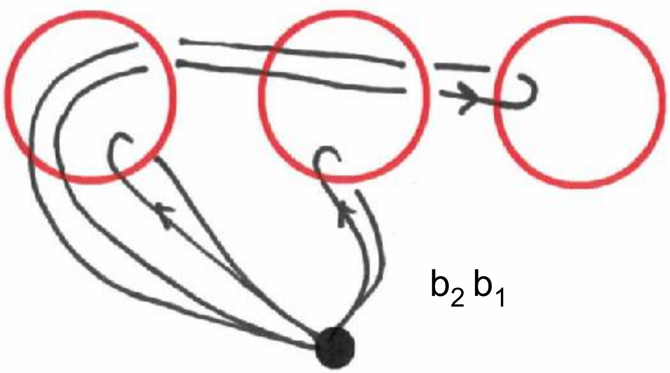
Braids inside Loop Braids; Yang-Baxter relation



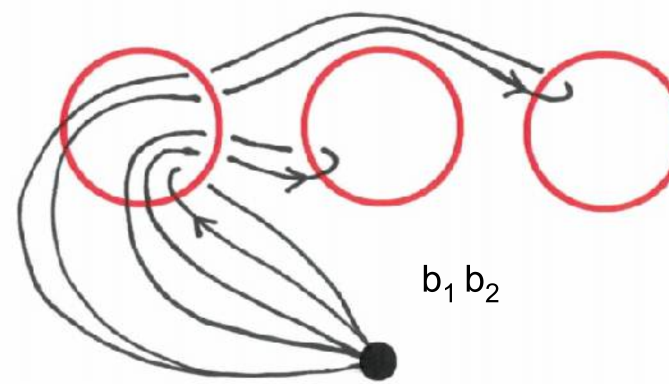
b_1



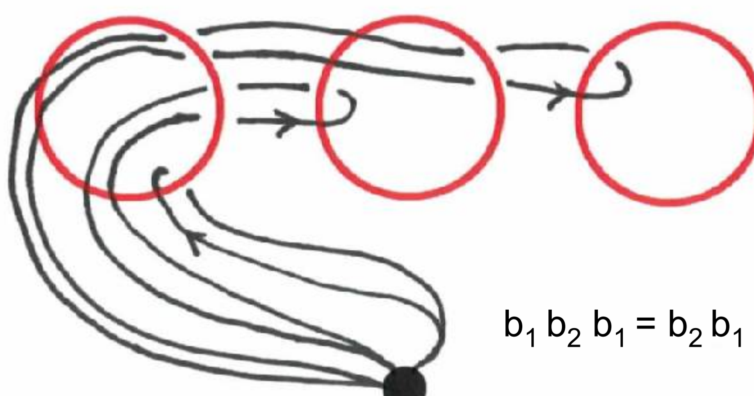
b_2



$b_2 b_1$



$b_1 b_2$



$b_1 b_2 b_1 = b_2 b_1 b_2$

1-dimensional representations

Slide group (oriented distinguishable rings)

The relations between the slides are all commutators $\longrightarrow$Trivial in a 1D representation.
If there are no further generators (distinguishable rings), get
1D unitary representations: $\sigma_{ij} \rightarrow \mathrm{Exp}(i\,\theta_{ij})$

Slides and Permutations (oriented indistinguishable rings)

Rings are bosons ($\tau_i \rightarrow 1$) or fermions ($\tau_i \rightarrow -1$) under exchange.
Invariance under permutations means all θ_{ij} must be equal, $\sigma_{ij} \rightarrow \mathrm{Exp}(i\,\theta)$
Such rings have been dubbed Bose-anyons and Fermi-anyons (Balachandran '89?)

Slides, Permutations and Flips (unoriented indistinguishable rings)

Conjugation by flips can invert slides. This forces $\theta=0$ or $\theta=\pi$.
In conclusion, there are 8 1-dimensional representations of Mot_n, given by

$$\tau_i \mapsto \pm 1, \quad f_i \mapsto \pm 1 \quad \sigma_{ij} \mapsto \pm 1.$$

Notes
Any number of rings may be (in)distinguishable or (un)oriented
$\longrightarrow$This gives rise to many intermediate motion groups.
With oriented rings, distinguishability depends on orientation!
$\longrightarrow$Makes sense to keep the flips even in this case...

Inducing representations of Slide_n

Suppose we have a representation (ρ, V_ρ) of $\mathrm{Slide}_{n.}$
We can produce corresponding representations of Mot_n by Induction.

Idea:
Can define an action of Mot_n on the vector space $\mathbb{C}((\mathbb{Z}_2)^n \rtimes S_n) \otimes V_\rho$

Elements of this space:
superpositions of $g \otimes v$ with $g \in (\mathbb{Z}_2)^n \rtimes S_n, \quad v \in V_\rho$

Action of Slide_n : $\sigma \cdot (g \otimes v) = g \otimes \rho(g^{-1}\sigma g)v$
Action of flips, permutations by left multiplication

Should find all representations of Mot_n as subrepresentations of these induced slide reps.

Inducing 1D reps of Slide_n (examples)

If all θ_{ij} different and not equal to 0 or π, get irreducible induced rep of dimension $2^n\, n!$
Get labels for all permutations and all orientations of the rings! Slides depend on these.

If all θ_{ij} equal but not equal to 0 or π, get irreducible induced reps of dimension 2^n
Get labels for the orientations of the rings only, acted on by flips.
Slide factors depend on the orientations. Natural for e.g. charge/flux composites.
Have Bose-/Fermi-anyons when all orientations equal.

Local representations

Local Representation of Mot_n (single type of ring)

Each ring has an internal vector space V

Total vector space is $V^{\otimes n}$

There are linear maps (matrices) $F : V \to V$

$$R : V \otimes V \to V \otimes V$$

The flips and slides are given by $f_i \mapsto F_i := \mathrm{id}^{\otimes(i-1)} \otimes F \otimes \mathrm{id}^{\otimes(n-i)}$

$\sigma_{ij} \mapsto R_{ij}$ (acts on tensor factors (i,j))

The permutations act by permutation of the tensor factors (as for particles)

Remaining nontrivial relations

$$R_{13}R_{23} = R_{23}R_{13}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad \text{(YBE)}$$

$$F_2RF_2 = R^{-1}$$

$$F^2 = 1$$

Note: this works for any n (same R, F)

Loop Braid representations of local dimension 2

Aim: classify all local loop braid reps with d_V=2 up to equivalence.

Equivalence: $\{R, F\} \sim \{(U \otimes U)R(U^{-1} \otimes U^{-1}),\ UFU^{-1}\}$

Start with R.

All solutions to the d=2 YBE are known (unitary and non-unitary)
Hietarinta, Phys Lett. A 165, 1992, Dye, Quantum Information Processing 2, 2003.
We classified these up to the equivalence above.

Imposing $R_{13}R_{23}=R_{23}R_{13}$
We find that the surviving solutions have R diagonal up to equivalence
(all diagonal R make slides commute and solve the relations)

Loop Braid representations of local dimension 2

Aim: classify all local loop braid reps with d_V=2 up to equivalence.

Equivalence: $\{R, F\} \sim \{(U \otimes U)R(U^{-1} \otimes U^{-1}),\ UFU^{-1}\}$

Start with R.

All solutions to the d=2 YBE are known (unitary and non-unitary)
Hietarinta, Phys Lett. A 165, 1992, Dye, Quantum Information Processing 2, 2003.
We classified these up to the equivalence above.

Imposing $R_{13}R_{23}=R_{23}R_{13}$
We find that the surviving solutions have R diagonal up to equivalence
(all diagonal R make slides commute and solve the relations)

Loop Braid representations of local dimension 2

Aim: classify all local loop braid reps with d_V=2 up to equivalence.

Equivalence: $\{R, F\} \sim \{(U \otimes U)R(U^{-1} \otimes U^{-1}),\ UFU^{-1}\}$

Start with R.

All solutions to the d=2 YBE are known (unitary and non-unitary)
Hietarinta, Phys Lett. A 165, 1992, Dye, Quantum Information Processing 2, 2003.
We classified these up to the equivalence above.

Imposing $R_{13}R_{23}=R_{23}R_{13}$
We find that the surviving solutions have R diagonal up to equivalence
(all diagonal R make slides commute and solve the relations)

Becomes a bit more interesting with F.

We have either $F = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $F \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In the second case, the rings have clear orientation states,
but R is not always diagonal

Loop Braid representations of local dimension 2

Aim: classify all local loop braid reps with d_V=2 up to equivalence.

Equivalence: $\{R, F\} \sim \{(U \otimes U)R(U^{-1} \otimes U^{-1}),\ UFU^{-1}\}$

Start with R.

All solutions to the d=2 YBE are known (unitary and non-unitary)
Hietarinta, Phys Lett. A 165, 1992, Dye, Quantum Information Processing 2, 2003.
We classified these up to the equivalence above.

Imposing $R_{13}R_{23}=R_{23}R_{13}$
We find that the surviving solutions have R diagonal up to equivalence
(all diagonal R make slides commute and solve the relations)

Becomes a bit more interesting with F.

We have either $F = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $F \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In the second case, the rings have clear orientation states,
but R is not always diagonal

All representations with local dimension 2 where F flips the two states

$R = \begin{bmatrix} e^{i\phi_1} & 0 & 0 & 0 \\ 0 & e^{-i\phi_1} & 0 & 0 \\ 0 & 0 & e^{i\phi_2} & 0 \\ 0 & 0 & 0 & e^{-i\phi_2} \end{bmatrix}$	Generalization of Bose-/Fermi-Anyons. Note slide factors reverse when ring 2 is flipped. Sliding does not change orientation. Only this solution has eigenvalues other than +/-1
$R = \begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Sliding leaves ring 2 unaffected. Ring 1 experiences a reflection in orientation space
$R = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ $R = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ $R = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$ $R = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$	Sliding maximally mixes the orientation states (Both rings' sliding eigenstates have mixed orientation)
$R = F \otimes 1, R = 1 \otimes F, R = F \otimes F$	Sliding is Flipping

Local representations from monodromy/gauge theory 1

Nonabelian representations of the slide group are possible.
Many of them arise naturally from gauge theories.

Consider the holonomies g_i of the gauge connection around the loops x_i through the rings

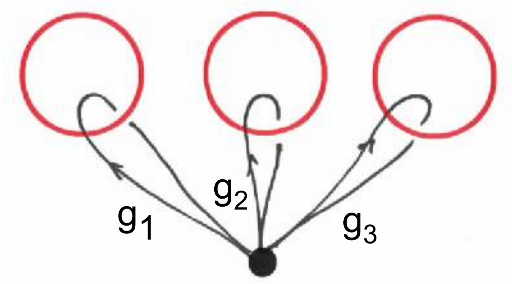
$$g_i = \mathcal{P} \oint_{x_i} \exp(iA) \cdot d\ell$$

(aka generalized fluxes carried by the rings)

These are topological if the connection is flat,
e.g. in BF-theories, in Toric code like models (discrete gauge theories)

Mot$_n$ acts on the holonomies in exactly the same way as on the loops themselves,

$$\begin{aligned}
\tau_i(g_1,\ldots,g_i,g_{i+1},\ldots,g_N) &= (g_1,\ldots,g_{i+1},g_i,\ldots,g_N)\\
f_i(g_1,\ldots,g_i,\ldots,g_N) &= (g_1,\ldots,g_i^{-1},\ldots,g_N)\\
\sigma_{ij}(g_1,\ldots,g_i,\ldots,g_j,\ldots,g_N) &= (g_1,\ldots,g_j^{-1}g_ig_j,\ldots,g_j,\ldots,g_N)
\end{aligned}$$

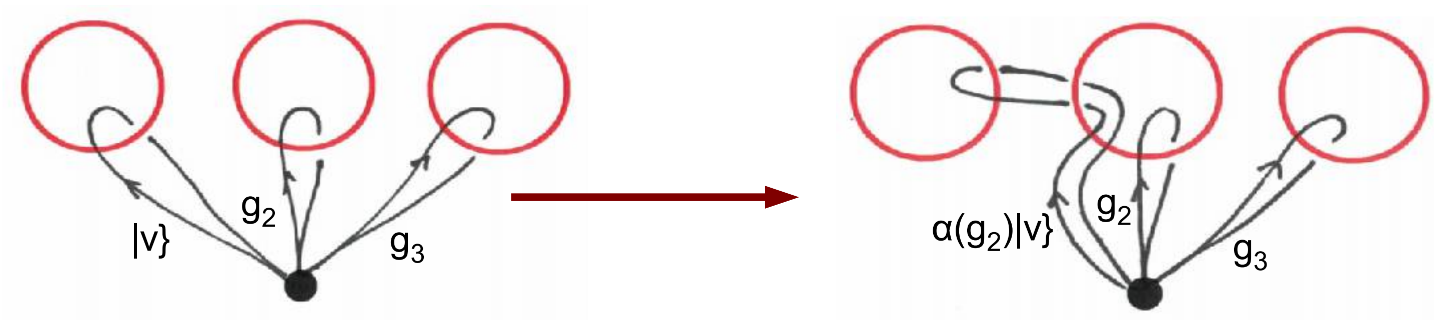


Mot$_n$ also acts on the space of holonomy states – superpositions of flux states.
For identical rings, each ring carries a superposition of fluxes g_i from a conjugacy class A of the gauge group. On sliding,these act on each other by conjugation.

For finite gauge groups this gives a finite dimensional representation ($d=|A|^n$)

Local representations from monodromy/gauge theory 2

Rings can also carry charge and experience a generalized Aharonov-Bohm effect on sliding.
The flux g acts in the representation α of the gauge group characterizing the charge



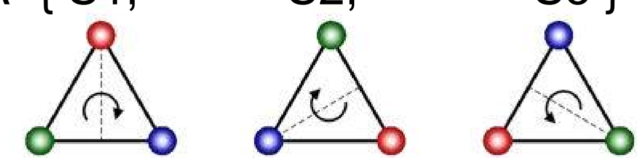
One may also have rings which carry flux A and charge α.
In this case α is representation of the centralizer N_A of an element of A

The braiding of these representations is well known and is described using D(G), the quantum double of the gauge group G (this is a quantum group).
Here, the R-matrix of D(G) describes the slides.

For any representation labeled (A,α) of D(G) we have a Representation ρ^A_α of the slides and permutations

Simplest example: G=D_3 (symmetries of a triangle),α=1, A={ S1, S2, S3 }

Here $S_i\, S_j\, S_i = S_k$ (i,j,k distinct) with $S_i=(S_i)^{-1}$



General Structure of Local Representations 1

Conjecture/Theorem (can prove a version with extra assumptions)

For every local irreducible representation ρ of Slide_n, there is
- a group G,
- a conjugacy class A in G
- a representation α of the centralizer N_A of an element of A

Such that ρ is lies inside the representation ρ^A_α that comes from D(G)

Note:
These representations are not universal for Quantum Computation by braiding
They can often be made universal with addition of topological measurement.

Proof Sketch

Start from $R_{13}R_{23} = R_{23}R_{13}$

With $R = \sum_k r_k^1 \otimes r_k^2$ get $\sum_{k,l} r_k^1 \otimes r_l^1 \otimes r_k^2 r_l^2 = \sum_{k,l} r_k^1 \otimes r_l^1 \otimes r_l^2 r_k^2$

And from there $r_k^2 r_l^2 = r_l^2 r_k^2 \; (\forall k, l)$

General Structure of Local Representations 2

Proof Sketch (continued)

Now assuming the r^2_k can be diagonalized,

$$r_k^2 = \sum_i \lambda_{k,i} P_i \quad \text{with} \quad P_i P_j = \delta_{ij} P_i \quad \sum_i P_i = \mathrm{id}_V$$

Then

$$R = \sum_k r_k^1 \otimes \sum_i \lambda_{k,i} P_i = \sum_i \sum_k \lambda_{k,i} r_k^1 \otimes P_i = \sum_i \tilde{r}_i \otimes P_i$$

Hence

$$R^{-1} = \sum_i (\tilde{r}_i)^{-1} \otimes P_i$$

And the $\tilde{r}_i$ are invertible – generate a group G!

Note:

R "looks" at ring 2 and acts on ring 1 accordingly. It is a "controlled gate".

General Structure of Local Representations 3

Proof Sketch (continued, sketchier)

Now use the YBE and irreducibility to show that

- The r_i close under conjugation (→ conjugacy class A of G)
- The action of R on the images of the P_i is essentially to permute them according to conjugation of the corresponding r_i
- The remaining freedom is captured by a centralizer representation

Discussion/Outlook

- Local Representations with multiple types of rings.
- Projective Representations, Ribbon Loop Braid Groups?
- Relaxed notions of locality (no tensor product structure)
- Induction of B_n representations (instead of induction on slides)
- Loop Braid Hecke/BMW algebras?
- Loop Fusion! Notion of a Loop Braid quantum group...
- No go theorem for braiding from modular anyon models??
- (More) Physical Models!
- Potential for experimental observations?
 (Slide interferometers, Loop scattering acmplitudes, Collective effects?)

Discussion/Outlook

- Local Representations with multiple types of rings.
- Projective Representations, Ribbon Loop Braid Groups?
- Relaxed notions of locality (no tensor product structure)
- Induction of B_n representations (instead of induction on slides)
- Loop Braid Hecke/BMW algebras?
- Loop Fusion! Notion of a Loop Braid quantum group...
- (More) Physical Models!
- Potential for experimental observations?
  (Slide interferometers, Loop scattering acmplitudes, Collective effects?)

Thank You!