Three-dimensional topological lattice models with surface anyons

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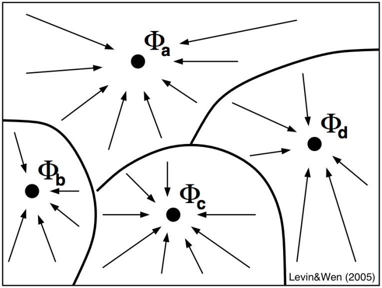
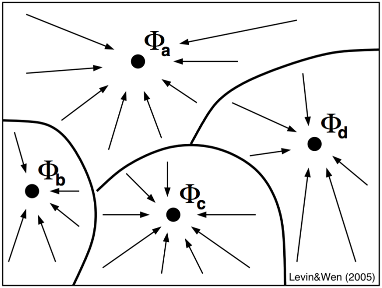
Exactly solvable spin models in 2D

- Toric code (Kitaev 2003), Levin-Wen models (Levin & Wen(2005)).
- Often contrived, so why study them?
- Toy models for topological order.
- Fixed point hamiltonians: These models capture 'low-energy long-wavelength' behaviour of 2D phases of matter – tell us about the topological order that might arise in 2D strongly interacting systems.



Exactly solvable spin models in 3D

- Here we study 3D cousins of 2D Levin-Wen models: Exactly solvable spin models proposed by Walker and Wang [arXiv:1104.2632v2].
- Class includes 3D toric code + many novel models.
- Models capture 'low-energy long-wavelength' behaviour of phases of matter.
- 'Confined models': Exactly solvable lattice realisation of a chiral Chern-Simons theory on boundary.

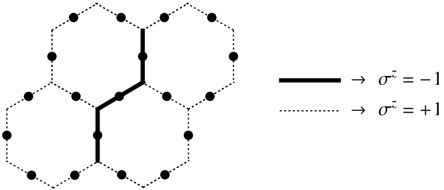


Outline

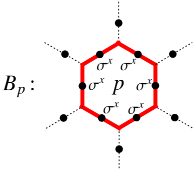
- Overview of Levin-Wen models, the 2D cousins of the 3D lattice models we study here.
- Review 3D toric code, the simplest Walker-Wang model.
- Introduce paradigm of *confined* WW models, the 3D Semion model. Understand its topological order, and the emergence of chiral anyons on its surface.
- Underlying field theories
- Generalisations
- Conclusion

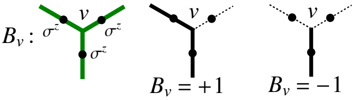
The 2D toric code

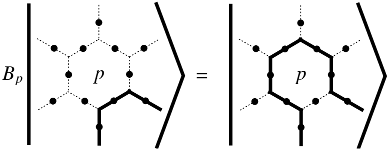
- Hilbert space: $\sigma^z = \pm 1$ on each edge of a trivalent lattice. Represent configurations by colouring in $\sigma^z = -1$ edges.



- $H_{TC} = -\sum_v \underbrace{\prod_{i \in s(v)} \sigma_i^z}_{B_v} - \sum_p \underbrace{\prod_{i \in \partial p} \sigma_i^x}_{B_p}$

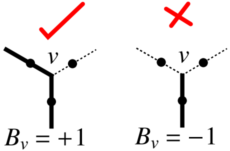


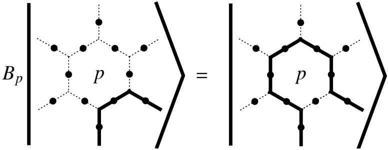




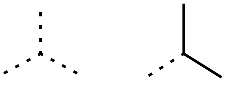
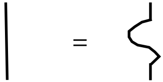
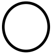
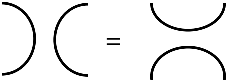
The 2D toric code: Ground states

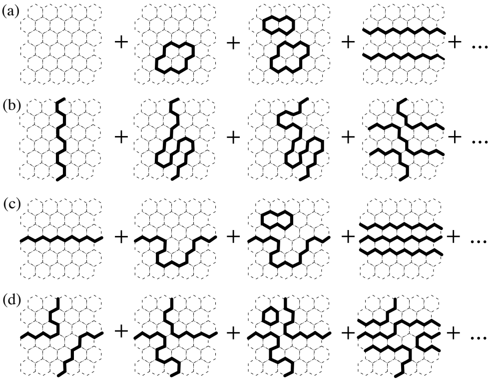
- $H_{TC} = -\sum_v \underbrace{\prod_{i \in s(v)} \sigma_i^z}_{B_v} - \sum_p \underbrace{\prod_{i \in \partial p} \sigma_i^x}_{B_p}$
- Lower bound on the energy:
  $B_v = 1 \,\forall\, v$ and $B_p = 1 \,\forall\, p$
- Exists because
  $[B_v, B_{v'}] = \left[B_p, B_{p'}\right] = [B_p, B_v] = 0.$
- $B_v = 1 \rightarrow$ Ground state a superposition of closed loop configurations ('Quantum loop gas').
- $B_p = 1 \rightarrow$ states related by plaquette flips have the same coefficient in the ground state.





Graphical rules

	Toric Code Rules
Allowed vertices	
Deformation	
Loop collapsing (Δ_1)	 $= +1$
Fusion	



Ground state degeneracy $= 2^2$ on torus.

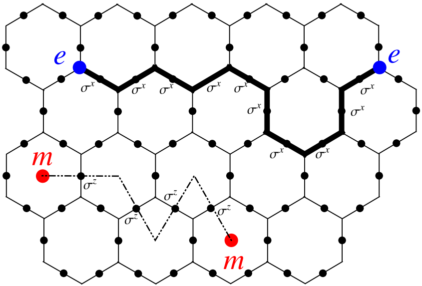
Topological order ✓.

Deconfined defects

- Vertex-type string operators create deconfined vertex defects (e) at their ends: $\hat{W}_V(\mathcal{C}) = \prod_{i\in\mathcal{C}} \sigma_i^x$.

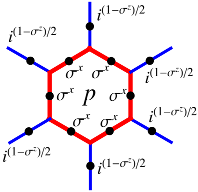
- Plaquette-type string operators create deconfined plaquette defects (m) at their ends: $\hat{W}_P(\mathcal{C}') = \prod_{i\in\mathcal{C}'} \sigma_i^z$.

- Berry phase of -1 on exchange. Topological order ✓.



The 2D doubled semion model (DSem)

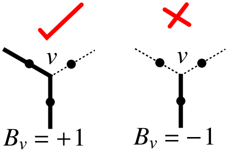
- DSem is another lattice model, which is similar to the toric code except its plaquette operators are different. (Freedman et al. (2004), Levin & Wen (2005)).
- Hilbert space: $\sigma^z = \pm 1$ living on each edge of a trivalent lattice.
- $H_{DSEM} = -\sum_v \underbrace{\prod_{s(v)} \sigma_i^z}_{B_v} + \sum_p \underbrace{(\prod_{i\in\partial p} \sigma_i^x) \prod_{j\in s(p)} i^{(1-\sigma_j^z)/2}}_{B_p}$

$B_p:$ 

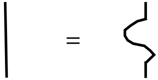
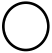
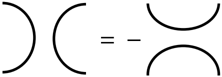
- **Different** B_p operator!

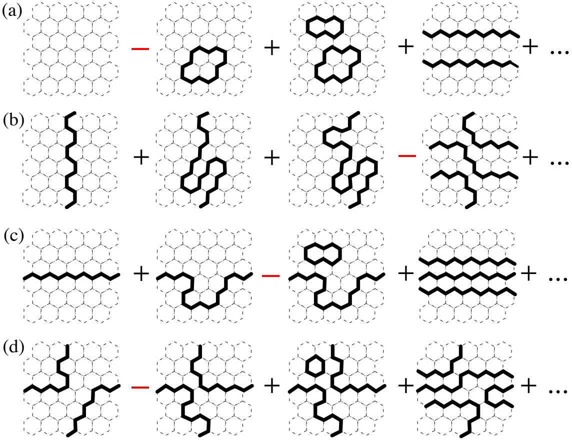
The 2D doubled semion model

- $H_{DSEM} = -\sum_v \underbrace{\prod_{s(v)} \sigma_i^z}_{B_v} + \sum_p \underbrace{(\prod_{i \in \partial p} \sigma_i^x) \prod_{j \in s(p)} i^{(1-\sigma_j^z)/2}}_{B_p}$
- Lower bound on the energy: $B_v = 1 \,\forall\, v$ and $B_p = -1 \,\forall\, p$
- $B_v = 1 \rightarrow$ Ground state is superposition of closed loops ('Quantum loop gas').
- $B_p = -1 \rightarrow$ New graphical rules.



Graphical rules

	Semion Rules
Allowed vertices	
Deformation	
Loop collapsing (Δ_1)	 $= -1$
Fusion	

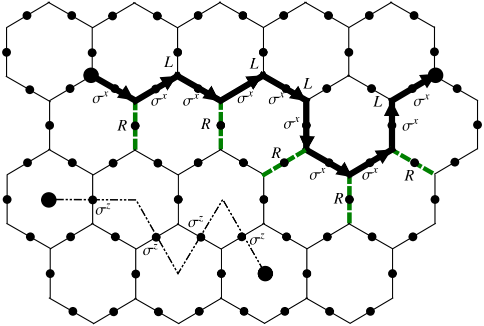


Ground state degeneracy $= 2^2$ on torus.

Topological order ✓.

String operators

- Two chiralities of vertex-type string operators:
$\hat{W}_V^{\pm}(\mathcal{C}) = \prod_{i\in\mathcal{C}} \sigma_i^x \prod_{k\in L \text{ vertices}} (-1)^{\frac{1}{4}(1-\sigma_i^z)(1+\sigma_j^z)} \prod_{l\in R \text{ legs}} (\pm i)^{(1-\sigma_l^z)/2}$.



- Plaquette-type string operator (achiral bound state): $\hat{W}_P(\mathcal{C}') = \prod_{i\in\mathcal{C}'} \sigma_i^z$.
- Vertex defects of same chirality are relative semions. Topological order ✓.

Topological order in the 2D models

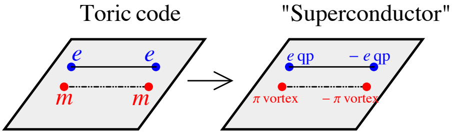
The 2D toric code

- GS degeneracy of 2^2 on torus ✓
- Relative statistics between point and vortex defects✓
- Topological entanglement entropy of $\log 2$✓
- Fixed point Hamiltonian✓
- $\rightarrow$ Topological order

The 2D doubled semion model

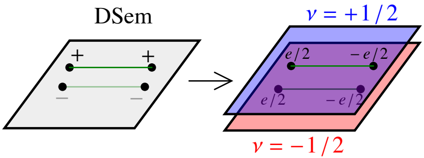
- GS degeneracy of 2^2 on torus ✓
- Semionic statistics between vertex defects of same chirality✓
- Topological entanglement entropy of $\log 2$✓
- Fixed point Hamiltonian✓
- $\rightarrow$ Different topological order

The 2D toric code



- Superconductors are topologically ordered (Hansson et al. (2004)).
- Vertex defects $\Leftrightarrow$ charge e quasi-particles.
- Plaquette defects $\Leftrightarrow$ π-flux vortices.

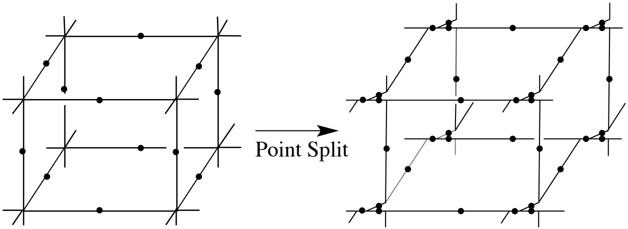
The 2D doubled semion model



- Two copies of a $\nu = \pm 1/2$ bosonic Laughlin.
- Vertex defects, chirality $\pm$ $\Leftrightarrow$ charge $e/2$ quasi-particles in $\nu = \pm 1/2$ layer.
- Plaquette defects $\Leftrightarrow$ Bound state of $\pm$ chirality quasi-particles.

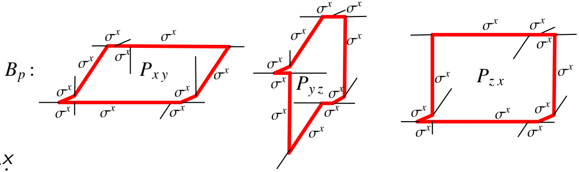
Generalising the 2D models to 3D

- $H = -\sum_v B_v - \sum_p B_p$
- The first of these will be the familiar 3D toric code (Hamma et al. 2005).
- The second will the the '3D semion model'. It is qualitatively very different from the 3D toric code.

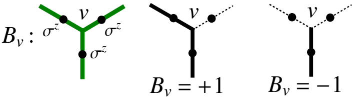


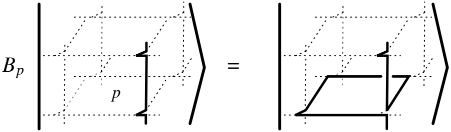
The 3D toric code

- Hilbert space: $\sigma^z = \pm 1$ on each edge.



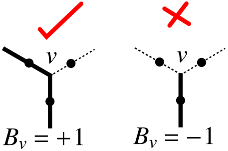
- $H_{TC} = -\sum_v \underbrace{\prod_{i \in s(v)} \sigma_i^z}_{B_v} - \sum_p \underbrace{\prod_{i \in \partial p} \sigma_i^x}_{B_p}$

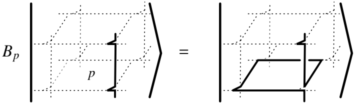




The 3D toric code

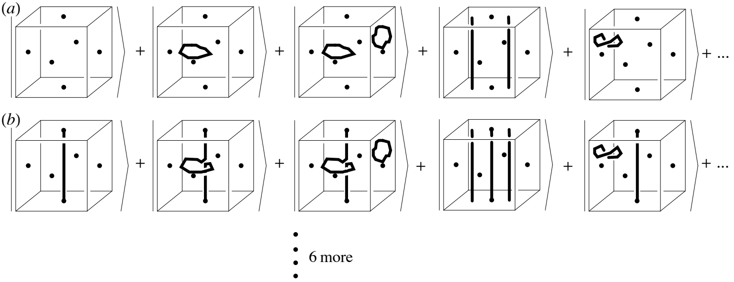
- $H_{TC} = -\sum_v \underbrace{\prod_{i \in s(v)} \sigma_i^z}_{B_v} - \sum_p \underbrace{\prod_{i \in \partial p} \sigma_i^x}_{B_p}$
- $[B_v, B_{v'}] = [B_p, B_{p'}] = [B_p, B_v] = 0.$
- $B_v = 1 \rightarrow$ Ground state is superposition of closed loops.
- $B_p = 1 \rightarrow$ states that are related by plaquette flips have the same coefficient in the ground state.

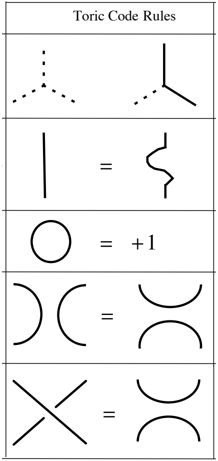




Graphical rules

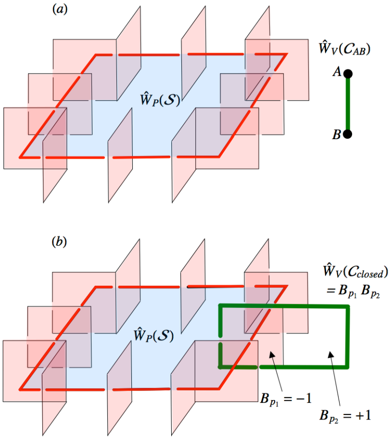
- 2^3 ground states on the 3-torus.
- Topological order ✓.





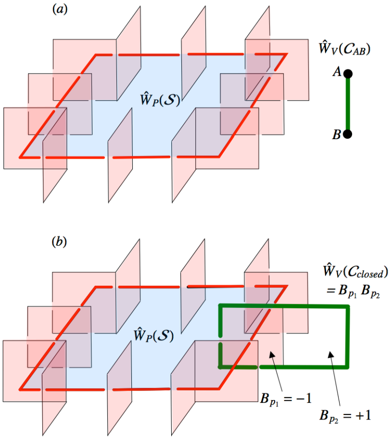
Defects

- Vertex-type string operators: $\hat{W}_V(\mathcal{C}) = \prod_{i\in\mathcal{C}} \sigma_i^x$.
- *Lines* of plaquette defects ('vortex rings'): $\hat{W}_P(\mathcal{S}) = \prod_{i\in\mathcal{S}} \sigma_i^z$.
- Berry phase of -1 on braiding. Topological order ✓.



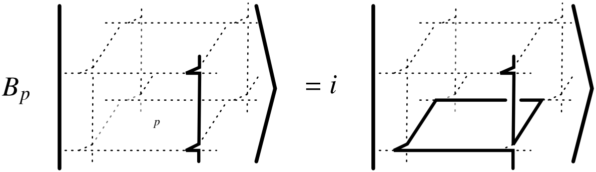
3D Toric code ⇔ '3D superconductor'

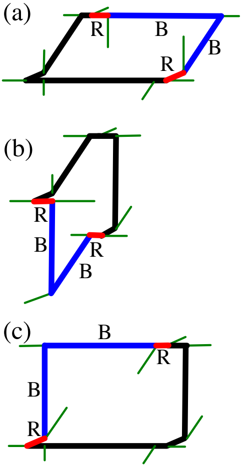
- Superconductors are topologically ordered (Hansson et al. (2004)).
- Vertex defects
  ⇔ charge e quasi-particles.
- *Lines* of plaquette defects
  ⇔ π-flux vortex rings.



The 3D semion model

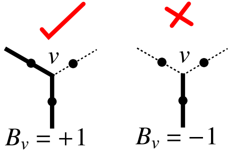
- Hilbert space: $\sigma^z = \pm 1$ on each edge. Again represent configurations by colouring in $\sigma^z = -1$ edges.
- $H_{3DSem} = -\sum_v B_v$
  $+ \sum_p \underbrace{\Big(\prod_{i\in\partial p} \sigma_i^x\Big)\Big(\prod_{j\in s(p)} i^{n_j}\Big)\, i^{\sum_{j\,\mathrm{red}} n_j - \sum_{j\,blue} n_j}}_{B_p}$

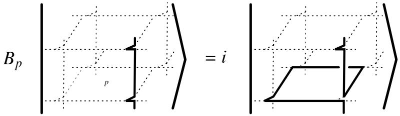




The 3D semion model

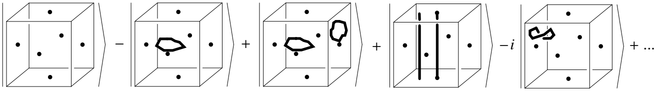
- $H_{3DSem} = -\sum_v B_v + \sum_p B_p$
- $B_v = 1 \rightarrow$ Ground state is a superposition of closed loops.
- $B_p = -1 \rightarrow$ Semion graphical rules determine relative amplitudes of loop configurations.



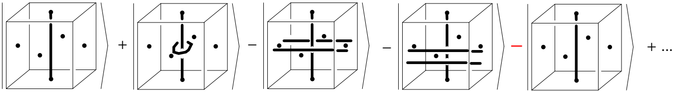


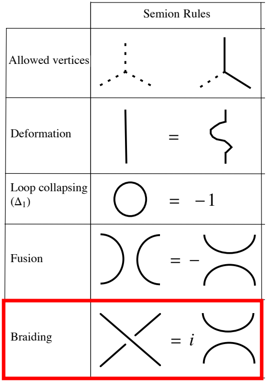
Graphical rules

- Unique ground state on the 3-torus.

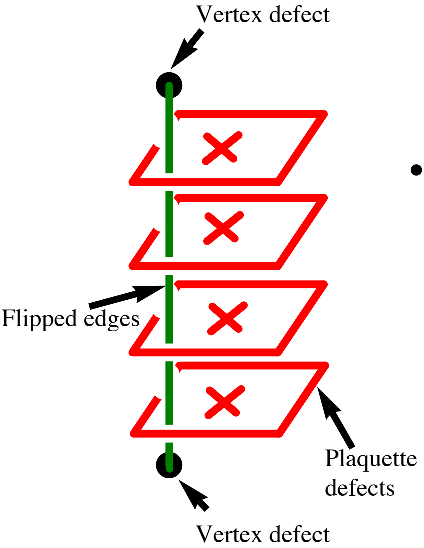


- NOT a ground state:





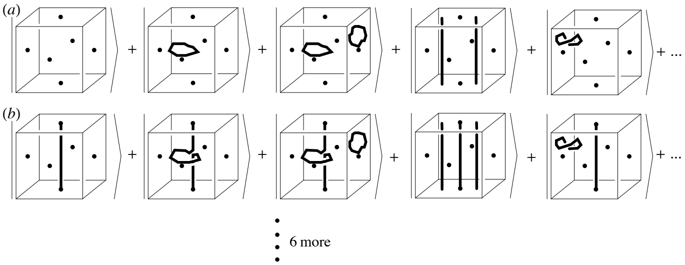
Defects



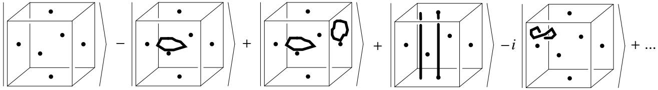
- Vertex-type string operators are confined. Any operator $\hat{W}_V(\mathcal{C}) = \prod_{i\in\mathcal{C}} \sigma_i^x \times \textit{phases}$ necessarily produces plaquette defects along its length.

Unique ground state ⇔ Confined excitations

- $\hat{W}_V(\mathcal{C}) = \prod_{i\in\mathcal{C}} \sigma_i^x$ toggles between two ground state sectors.



- In 3D semion model, applying such an operator to GS leaves an energetic string behind.



Is the 3D semion model even topologically ordered?

3D toric code:

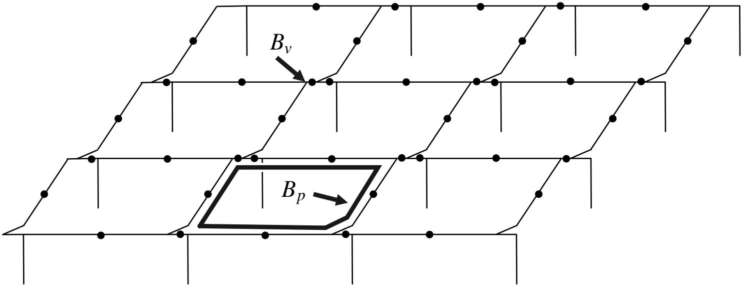
- GS degeneracy of 2^3 ✓
- Universal statistics between point defects and vortex lines✓
- Topological entanglement entropy of $\log 2$✓
- Fixed point Hamiltonian✓
- $\rightarrow$ Topological order

3D semion model:

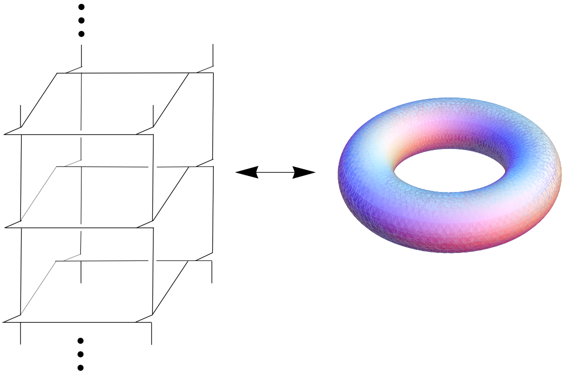
- Unique ground state ✗
- All excitations confined in the bulk ✗
- Topological entanglement entropy of 0 ✗
- Fixed point Hamiltonian✓
- $\rightarrow$ Topological order ??? Not in the traditional sense . . . **at least in the bulk!**

3D semion model with boundary

- Cut off the lattice in a 'smooth' manner.

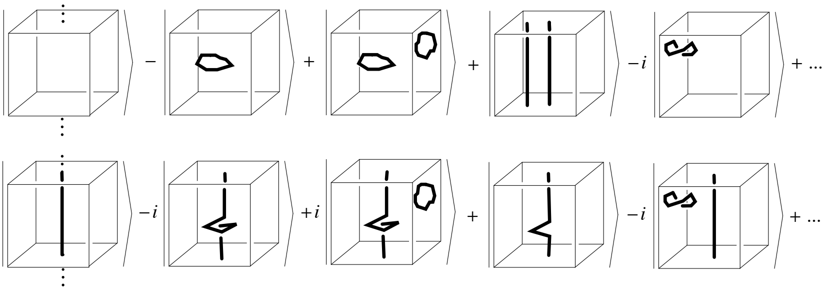


3D semion model on the solid donut

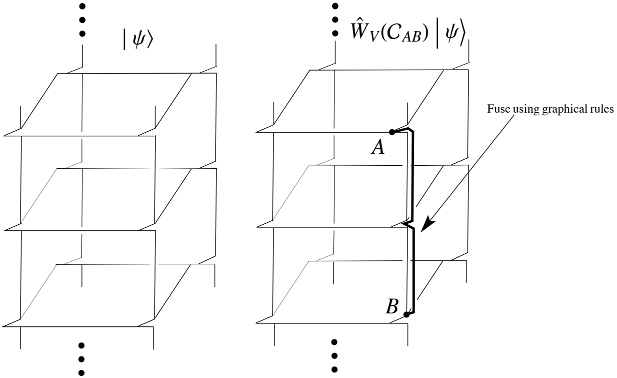


Ground states on the solid donut

- Ground state degeneracy of 3D semion model on solid donut is 2. Topological order ✓.

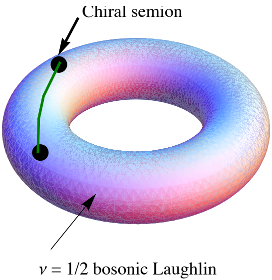


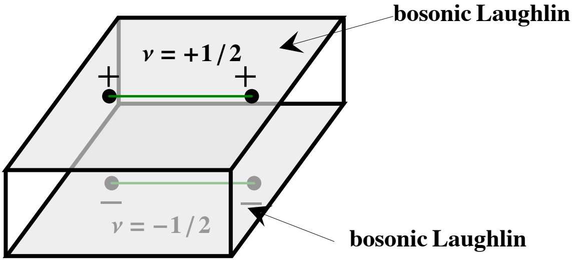
Deconfined surface anyons



What is the topological order of the 3D semion model?

- Not topologically ordered in the bulk, at least according to Wen's criteria.
- However, a sample with a surface has topological order.
- Topological order: Bosonic $\nu = 1/2$ laughlin state

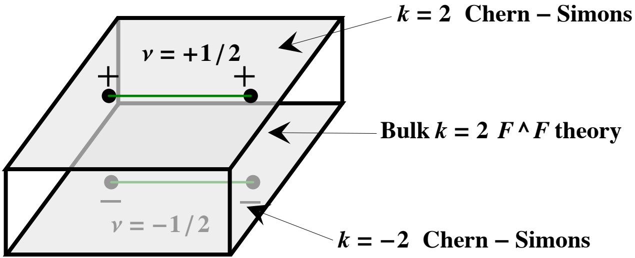




Field theory

- $\nu = 1/2$ bosonic hall effect $\rightarrow$ k=2 Chern-Simons theory.
- Ground-space of 3D semion model on solid donut $\Leftrightarrow$ Hilbert space of compact $k = 2$ Chern-Simons theory on surface. Using $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \left(A_{\nu} \partial_{\rho} A_{\sigma} \right)$ suggests:

$$S_{FF}\left[A\right] = \int d^4x \left(\frac{k}{16\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + A_{\mu} J^{\mu} \right) ,$$



Field theory

$$S_{FF}[A] = \int d^4x \ \left(\frac{k}{16\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + A_\mu J^\mu \right) ,$$

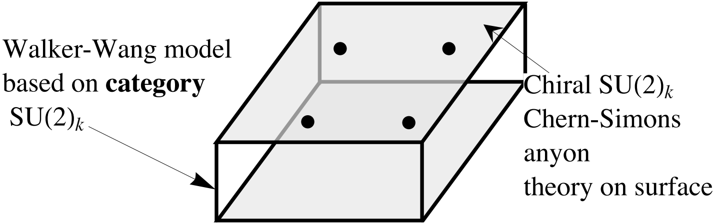
- Correct GS degeneracy, and anyonic statistics between surface particles (J).
- No deconfined point particles in the bulk: If J is non-vanishing in the bulk, the partition function becomes:

$$\int DA_S \, e^{\frac{ik}{4\pi} S_{CS}[A_S] + i \int J_S \cdot A_S} \delta \left[J^0 = -\frac{k}{4\pi} \epsilon^{ijk} \partial_i \partial_j A_k \text{ in bulk} \right]$$

- Pairs of bulk point particles connected by a line defect in A (corresponds to the presence of a energetic line defect in lattice model).

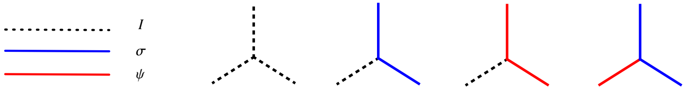
Generalisations

- 3D semion model
  $\Leftrightarrow k = 2$ $F \wedge F$ theory $\Leftrightarrow k = 2$ surface CS theory.
- Can realise many compact chiral (non-abelian) Chern-Simons theory on the surface of a Walker-Wang model.

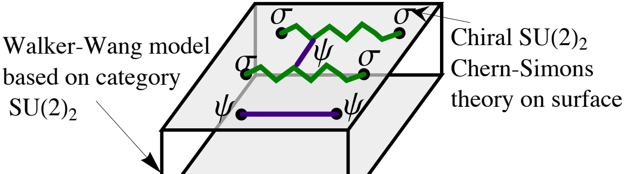


Generalisations

- e.g. To realise an $SU(2)_2$ Chern-Simons theory (Ising anyon theory), construct a Walker-Wang model based on the Ising category.
- Hilbert space: 3-state system on each edge of the lattice. Labels $\{0, \frac{1}{2}, 1\}$ (or $\{I, \sigma, \psi, \}$)
- Allowed vertices:



- ...more complicated graphical rules determining ground states.
- Leads to non-abelian anyons on the surface of the model.



Conclusion

- We described the 'surface topological order' of the 3D semion model.
- Explicitly constructed the chiral surface semions.
- Made more concrete the correspondence:
  Lattice model $\Leftrightarrow$ FF-theory.
- Works for any Chern-Simons theory!
- Known that
  3D toric code $\Leftrightarrow$ 3D superconductor $\Leftrightarrow$ bF-theory
  (Hansson et al. (2004))
- Speculate:
  3D semion model $\Leftrightarrow$ *Novel phase?!* $\Leftrightarrow$ $F \wedge F$-theory
- Putative novel phase: Gapped bulk with confined excitations, deconfined anyons on boundary. P and T odd low-energy Hamiltonian.

Thanks for listening!

Three-dimensional topological lattice models with surface anyons

Curt von Keyserlingk

With Fiona J. Burnell and Steven H. Simon.

Nordita, August 9, 2012

A useful graphical mnemonic

- The vertex type string operators can be drawn as off-lattice strings.
  $+$ chirality $\rightarrow$ lay string **above** lattice.
  $-$ chirality $\rightarrow$ lay string **below** lattice.