Topological insulators in $D \geq 2$ dimensions : algebra of projected density operators

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Motivations

Topological phases of matter characterized by a **topological invariant** :
Chern numbers :

$$C_1 = \int F, \qquad C_2 = \int F \wedge F, \qquad \cdots$$

$\mathbb{Z}_2$ topological numbers :

$$P_1 = \int A, \qquad P_3 = \int F \wedge A + \frac{i}{3} A \wedge A \wedge A, \qquad \cdots$$

For 2D Chern insulators / Quantum Hall effect

Projected density operators probe the Berry curvature F

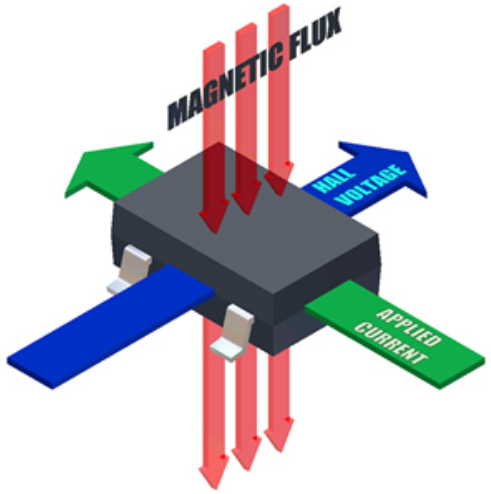
- Aharonov-Bohm effect
- Incompressibility (area preserving diffeomorphisms)
- Classification of edges (K matrices)

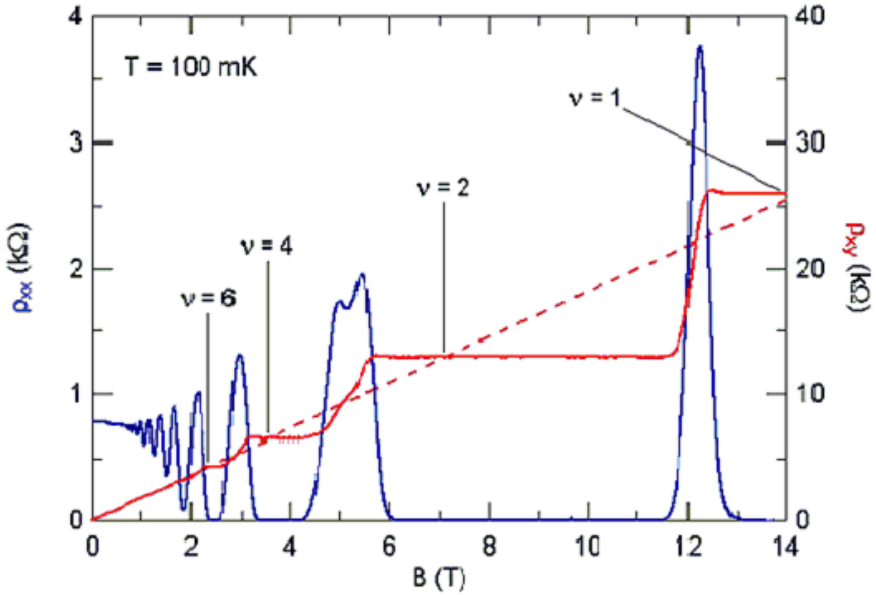
what about higher dimensional topological insulator ?

Integer Quantum Hall Effect Phenomenology and density algebra ($D = 2$)

Quantum Hall effect and quantized Hall conductance

Hall effect : a two-dimensional electron gas in a perpendicular magnetic field.
$\Rightarrow$ current $\perp$ voltage





IQHE : von Klitzing (1980)

Quantized Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

ν is an integer up to $O(10^{-9})$
Used in metrology

ν is the number of filled bands (Landau levels)

2D particle in a perpendicular $\vec{B} = B\vec{z}$: $H = \frac{1}{2m}(\vec{p} - q\vec{A})^2$

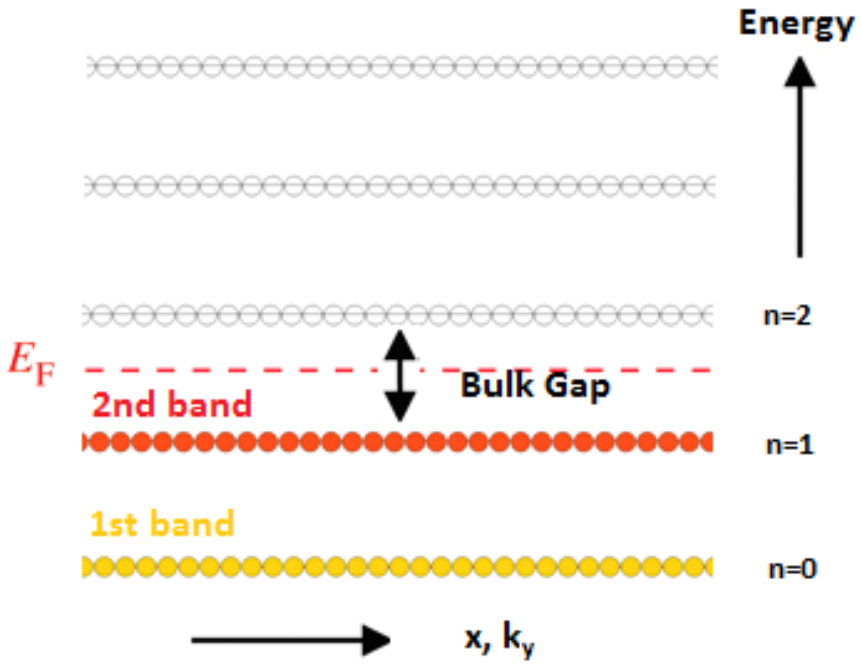
Discrete spectrum :

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega_c$$

Each Landau level n is highly degenerate.

$$\Psi_{n,k_y}(x,y) \sim e^{ik_y y} e^{-(x-k_y)^2/2}$$

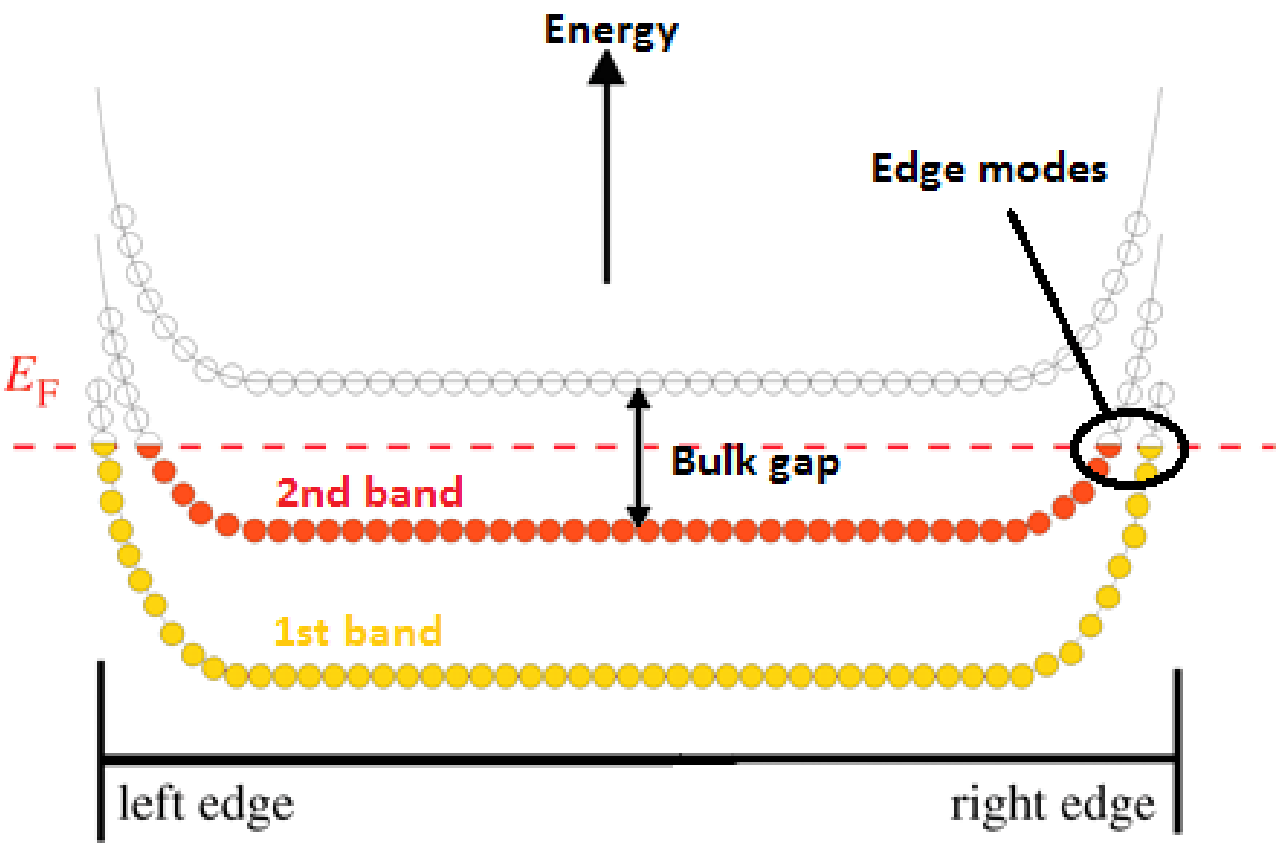
(Wannier type states)



IQHE : state obtained by filling ν Landau levels $\Rightarrow$ Bulk gap $\hbar\omega_c$.

ν is also the number of edge states

A state with momentum k_y is localized in real space around $x = k_y$



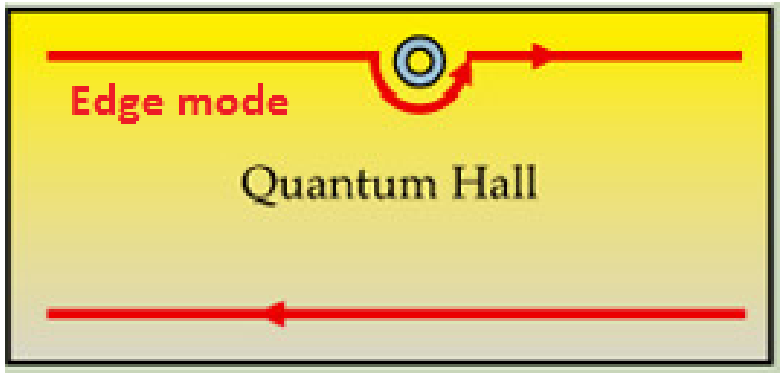
What is this integer ν ? TKNN and topology

- TKNN (Thouless *et al*, '82) :
  quantization insensitive to disorder or strong periodic potential.

  ν is a topological invariant, the first Chern number

- edge modes (Laughlin '81, Hatsugai '92) :
  edge states are robust, chiral

  number of edge modes = Chern number

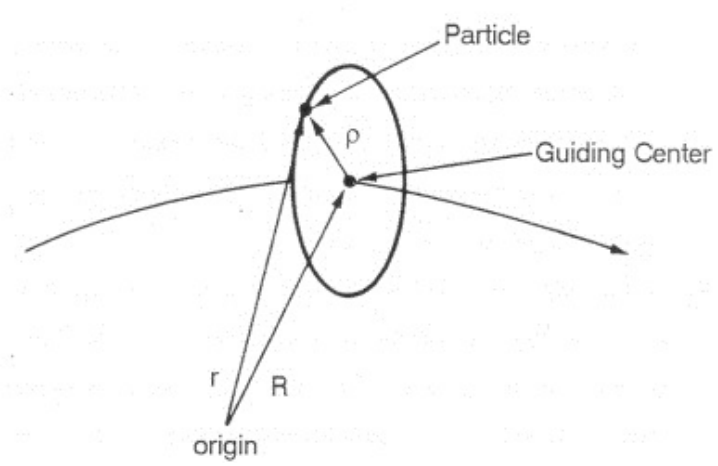


Each edge channel contributes e^2/h to the Hall conductance

$$\sigma_{xy} = \nu e^2/h = \nu/2\pi$$

Projection to the Lowest Landau Level

Decomposing the position $\mathbf{r} = \rho + \mathbf{R}$ where the guiding center $\mathbf{R}$ is



The Guiding center

$$R_i = r_i - \frac{1}{B}\epsilon_{ij}(p_j - A_j)$$

is a conserved quantity $[H, R_i] = 0$ but

$$[R_x, R_y] = [R_1, R_2] = i/B$$

Projection in the LLL and non-commutative space

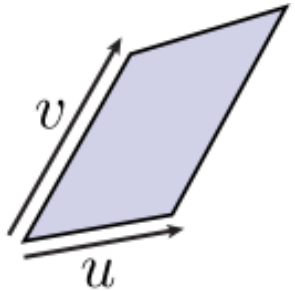
$$P\mathbf{r}P = \mathbf{R} \qquad\qquad [R_1, R_2] = i/B$$

The projected positions (R_1, R_2) are conjugate : non-commutative space ($+$ dimensional reduction : $4 \to 2$ dimensional phase-space).

Girvin-MacDonald-Platzmann algebra (or W_∞)

Projected density operators $\rho_{\mathbf{u}} = Pe^{i\mathbf{u}\cdot\mathbf{r}}P \propto e^{i\mathbf{u}\cdot\mathbf{R}}$ obey

$$[\rho_{\mathbf{u}}, \rho_{\mathbf{v}}] = 2i \sin\left(\frac{\mathbf{u}\wedge\mathbf{v}}{2B}\right)\rho_{\mathbf{u}+\mathbf{v}}$$



long wavelength ($\mathbf{u}, \mathbf{v} \ll 1$) : algebra of area-preserving diffeomorphisms.

Projected density operators also act as magnetic translations :

$$T_{\mathbf{a}} = e^{i\mathbf{u}\cdot\mathbf{D}} \qquad\qquad D_i = B\epsilon_{ij}R_j$$

whose algebra describes the Aharonov-Bohm effect in a uniform B

$$T_{\mathbf{u}}T_{\mathbf{v}} = e^{iB(\mathbf{u}\wedge\mathbf{v})}T_{\mathbf{v}}T_{\mathbf{u}}$$

$\Rightarrow$ $\rho_{\mathbf{q}}$ **implements parallel transport w.r.t. the Berry curvature** B

This algebra predicts the center-of-mass degeneracy : a state at filling p/q has q-fold degeneracy on the torus [Haldane, '85].

Density algebra for topological insulators ($D \geq 2$)

TI in D space dimensions : notations

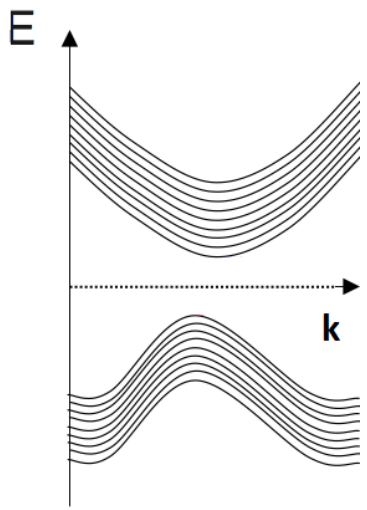
One-body tight biding model Hamiltonian on the infinite lattice $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^D$.

$$H = \sum_{\mathbf{i},\mathbf{j}} c^\dagger_{\mathbf{i}\alpha} h^{\alpha\beta}(\mathbf{i}-\mathbf{j}) c_{\mathbf{j}\beta}$$

Momenta are restricted to the Brillouin zone (BZ) $\mathbf{k} \in \mathbb{T}_D$ $(k_i \equiv k_i + 2\pi)$

$$H = \int_{\mathrm{BZ}} \mathrm{d}^D\mathbf{k}\ c^\dagger_{\mathbf{k}\alpha} h^{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\beta}$$

Diagonalizing the the Bloch Hamiltonian $\sum_\beta h_{\alpha\beta}(\mathbf{k}) u^n_{\mathbf{k},\beta} = E_n(\mathbf{k}) u^n_{\mathbf{k},\alpha}$:



$$H = \sum_n \int_{\mathrm{BZ}} \mathrm{d}^D\mathbf{k}\ E_n(\mathbf{k}) |\mathbf{k}, n\rangle\langle\mathbf{k}, n|$$

with states $|\mathbf{k}, n\rangle = \sum_\beta u^n_{\mathbf{k},\beta} c^\dagger_{\mathbf{k},\beta} |0\rangle$

TI in D space dimensions : electromagnetic response

In two dimensions, the response to an external A_μ^{ext} is

$$j^\mu = \frac{C_1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho^{\text{ext}} \qquad C_1 = \frac{1}{2\pi}\int_{\text{BZ}} d^2k \,\text{Tr}\left(F_{xy}(\mathbf{k})\right) \in \mathbb{Z}$$

i.e. the winding number from the mapping of $A_\mu(\mathbf{k}) : \mathbb{T}_2 \to U(N)$

$$C_1 \neq 0 \Rightarrow \text{2D Chern insulator.}$$

The Berry connection in $\mathbf{k}$ space

$$A_\mu^{nm}(\mathbf{k}) = i\langle \mathbf{k}, n|\partial_{k_\mu}|\mathbf{k}, m\rangle = i\sum_\alpha u_{\mathbf{k},\alpha}^{n\star}\partial_{k_\mu} u_{\mathbf{k},\alpha}^m$$

defines a non-Abelian U(N) Berry field strength :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

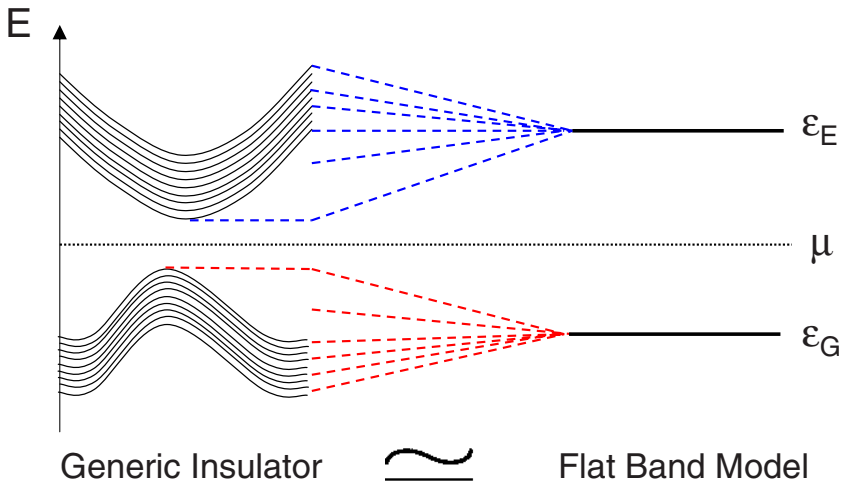
Flat-Band limit and symmetries

The flat-band limit is obtained by collapsing all occupied bands

$$H = \sum_{\mathbf{k},n} E_n(\mathbf{k}) |\mathbf{k}, n\rangle\langle n, \mathbf{k}|$$

$$\downarrow$$

$$H_{\text{FB}} = \epsilon_G P + \epsilon_E (1 - P)$$



where $P = \sum_{n,\mathbf{k}} |\mathbf{k}, n\rangle\langle n, \mathbf{k}|$ is the projector to the occupied bands.

Any projected operator is a symmetry of H_{FB}

$$[H_{\text{FB}}, POP] = 0$$

In particular the projected position $\mathbf{R} = P\mathbf{r}P$ implements parallel transport

$$R_\mu = -i\left(\frac{\partial}{\partial k_\mu} - iA_\mu(k)\right) \qquad [R_\mu, R_\nu] = iF_{\mu\nu}(\mathbf{k})$$

Projected density operators : the fundamental relation

The projected density operators are

$$\rho_{\mathbf{q}} = P \sum_{\mathbf{j},\alpha} e^{i\mathbf{q}\cdot\mathbf{j}} c^{\dagger}_{\mathbf{j}\alpha} c_{\mathbf{j}\alpha} P$$

- They commute with the one-body Hamiltonian.
- They annihilate the many-body ground state (filled band).

Commutation relation [Parameswaran et. al . '11] :

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}]|\mathbf{k}, n\rangle = -iq_1^{\mu} q_2^{\nu} (F_{\mu\nu}(\mathbf{k}))_{nm} \, |\mathbf{k} + \mathbf{q}_1 + \mathbf{q}_2, m\rangle \qquad (\mathbf{q}_1, \mathbf{q}_2 \ll 1)$$

Parallel transport $\rho_{\mathbf{q}} = 1 + iq^{\mu} R_{\mu} + O(\mathbf{q}^2)$

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}] = -iq_1^{\mu} q_2^{\nu} F_{\mu\nu}(\mathbf{k})$$

Density algebra in 2d and first Chern number

Density algebra in 2d and first Chern number

In two dimensions the Berry curvature is $F_{\mu\nu}(\mathbf{k}) = B(\mathbf{k})\epsilon_{\mu\nu}$,
Hall type response $j^i = \frac{C_1}{2\pi}\epsilon^{ij}E_j$ where C_1 is the first Chern number

$$C_1 = \frac{1}{4\pi}\int_{\mathrm{BZ}} d^2k\epsilon^{\mu\nu}\mathrm{Tr}\left(F_{\mu\nu}(\mathbf{k})\right)$$

i.e. the winding number from the mapping of $A_\mu(\mathbf{k}) : \mathbb{T}_2 \to U(N)$

First Chern number as an obstruction to commutativity

$$\mathrm{Tr}\left([\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}]\rho_{-\mathbf{q}_1-\mathbf{q}_2}\right) = \frac{L^2}{2\pi i}\left(\mathbf{q}_1 \wedge \mathbf{q}_2\right) C_1$$

if $F_{xy}(\mathbf{k}) = B = \mathrm{constant}$ we recover the $\mathbf{q} \ll 1$ limit of the GMP algebra

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}] = -iB\,\mathbf{q}_1 \wedge \mathbf{q}_2\,\rho_{\mathbf{q}_1+\mathbf{q}_2} + O(\mathbf{q}^3)$$

Unitarity and Parallel transport

Projected density operators enjoy the small correct q behavior

$$\rho_q = 1 + iq_\mu R^\mu + O(q^2)$$

but they are not unitary

$$\langle \mathbf{k}, n | \rho_{\mathbf{q}}^\dagger \rho_{\mathbf{q}} | \mathbf{k}, n \rangle = \sum_m |\langle u^m_{\mathbf{k}-\mathbf{q}} | u^n_{\mathbf{k}} \rangle|^2 = 1 - O(\mathbf{q}^2)$$

In 2D we know the cure : parallel transport w.r.t. A_μ

$$\tilde{\rho}_{\mathbf{q}} | \mathbf{k}, n \rangle = \sum_m \left(\mathcal{P} e^{-i \int_{\mathbf{k}}^{\mathbf{k}+\mathbf{q}} dp^\mu A_\mu(p)} \right)_{nm} | \mathbf{k} + \mathbf{q}, m \rangle$$

(i.e. $\tilde{\rho}_{\mathbf{q}} = e^{iq^\mu R_\mu}$) and we recover the full GMP algebra

$$[\tilde{\rho}_{\mathbf{q}_1}, \tilde{\rho}_{\mathbf{q}_2}] = -2i \sin\left(B \, \frac{\mathbf{q}_1 \wedge \mathbf{q}_2}{2} \right) \tilde{\rho}_{\mathbf{q}_1 + \mathbf{q}_2}$$

Chern insulator in 2D = same phenomenology and algebra as the IQHE

Chern insulators/QHE in higher dimensions

Density algebra in even D dimensions

A D dimensional TI is characterized by the topological number

$$C_{D/2} = \frac{1}{(D/2)!(2\pi)^{D/2}} \int d^D k \mathrm{Tr}\left(F(\mathbf{k}) \wedge \cdots \wedge F(\mathbf{k})\right)$$

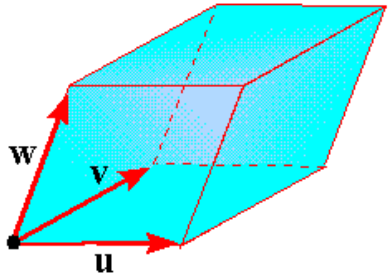
We want to probe $F \wedge F \wedge \cdots \wedge F$!

We need a "D-commutator" :

$$[A_1, A_2, \cdots, A_D] = \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_D} A_{\alpha_1} A_{\alpha_2} \cdots A_{\alpha_D}$$

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \cdots, \rho_{\mathbf{q}_D}] \left|\mathbf{k}\right\rangle \simeq (\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_D) \left[F(\mathbf{k}) \wedge \cdots \wedge F(\mathbf{k})\right] \left|\mathbf{k}\right\rangle$$

$(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})$ is the volume delimited by



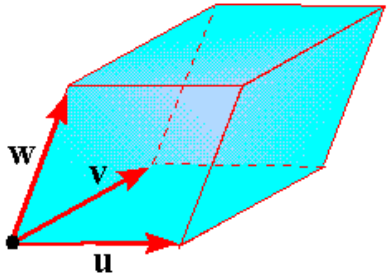
D-algebra

The D-algebra closes for $\mathbf{q}_i \ll 1$

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \cdots, \rho_{\mathbf{q}_D}] \propto (\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_D)\, C_{\frac{D}{2}} \rho_{\mathbf{q}_1 + \ldots + \mathbf{q}_D}$$

for a uniform Chern density $[F \wedge \cdots \wedge F]_{nm} \propto C_{\frac{D}{2}} \delta_{nm}$

Flux of the D-form $F \wedge \cdots \wedge F = \epsilon^{\mu_1 \mu_2 \cdots \mu_D} F_{\mu_1 \mu_2} \cdots F_{\mu_{D-1} \mu_D}$ through the volume $(\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_D)$. **parallel transport ?**



Projection to the lower bands $\Rightarrow$ non-commutative D-dimensional phase-space

$$[R_1, R_2, \cdots, R_D] \propto i^{D/2} C_{\frac{D}{2}}$$

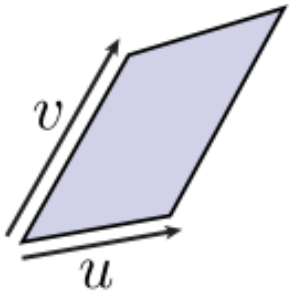
Recap

In two dimensions :

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}] \propto C_1 \left(\mathbf{q}_1 \wedge \mathbf{q}_2\right) \rho_{\mathbf{q}_1+\mathbf{q}_2}$$

Non-commutative plane $[R_1, R_2] \propto iC_1$ (*i.e.* uncertainty $\Delta R_1 \Delta R_2 \geq C_1$)

Parallel transport $\tilde{\rho}_{\mathbf{q}} = e^{iq^\mu R_\mu}$, Aharonov-Bohm effect



$$\tilde{\rho}_{\mathbf{u}} \tilde{\rho}_{\mathbf{u}} = e^{iB\mathbf{u}\wedge\mathbf{v}} \tilde{\rho}_{\mathbf{v}} \tilde{\rho}_{\mathbf{u}}$$

In D dimensions (D even)

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \cdots, \rho_{\mathbf{q}_D}] \propto C_{\frac{D}{2}} \left(\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_D\right) \rho_{\mathbf{q}_1+\ldots+\mathbf{q}_D}$$

Non-commutative D-dimensional phase-space $[R_1, R_2, \cdots, R_D] \propto i^{D/2} C_{\frac{D}{2}}$

Parallel transport w.r.t the D-form $F \wedge F \wedge \cdots \wedge F$?

Topological insulators in odd dimensions

$\mathbb{Z}_2$ topological number in odd dimensions

In odd space dimensions the topological number is the integrand of the Chern-Simons form

$$P_1 = \frac{1}{2\pi} \int dk \, \mathrm{Tr}\,[A]$$
$$P_3 = \frac{1}{8\pi^2} \int d^3k \, \mathrm{Tr}\left[F \wedge A + \frac{i}{3} A \wedge A \wedge A\right]$$
$$\ldots$$

and are only defined modulo an integer (large gauge transformations).

We've got a problem : projected density operators are gauge invariant

$\Rightarrow$ It is not possible to repeat the contruction obtained for D even

Density algebra in three dimensions

$$[A, B, C] = [A, B]C + [B, C]A + [C, A]B$$

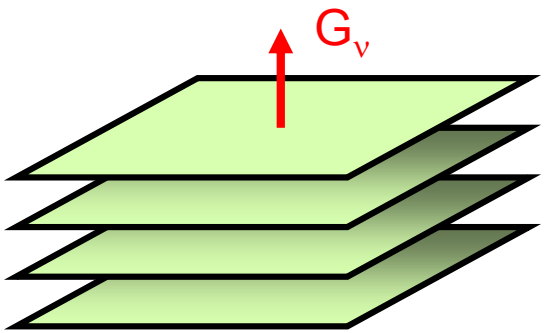
In odd dimensions the D-commutator is annoying :

$$[A, B, 1] = [A, B] \neq 0$$

Upon expanding $\rho_{\mathbf{q}} = 1 + i\mathbf{q} \cdot \mathbf{R}$ we get and anisotropic term

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \rho_{\mathbf{q}_3}] = -i(q_1^\mu q_2^\nu + q_3^\mu q_1^\nu + q_2^\mu q_3^\nu) F_{\mu\nu} \rho_{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3}$$

which is sensitive to a weak 3D TI
(layers of 2D TI) instead of a strong one.



Check : subdominant term in the 3-commutator

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \rho_{\mathbf{q}_3}] = \left[-i(q_1^\mu q_2^\nu + q_3^\mu q_1^\nu + q_2^\mu q_3^\nu)F_{\mu\nu} + O(\mathbf{q}^3)\right] \rho_{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3}$$

Is the $O(\mathbf{q}^3)$ term the isotropic $(\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \mathbf{q}_3)\left[F \wedge A + \frac{i}{3} A \wedge A \wedge A\right]$?

No ! It's anisotropic

$$O(\mathbf{q}^3) = \epsilon_{\alpha_1\alpha_2\alpha_3} q^\mu_{\alpha_1} q^\nu_{\alpha_1} q^\sigma_{\alpha_2} C_{\mu\nu\sigma}$$

$$C_{\mu\nu\sigma} = iD_\sigma B_{\mu\nu} - i\partial_\mu\partial_\nu A_\sigma - (A_\mu\partial_\nu + A_\nu\partial_\mu)A_\sigma + F_{\mu\sigma}A_\nu + F_{\nu\sigma}A_\mu$$

where $B_{\mu\nu}$ is the subleading term in $\rho_{\mathbf{q}} = \left(1 - iq^\mu A_\mu - \frac{i}{2} q^\mu q^\nu B_{\mu\nu}\right)$

Moreover $[R_1, R_2, R_3] = F \wedge (\partial - iA)$ and its trace is not well defined.

Contrary to Neupert *et al*, arXiv :1202.5188

the CS density cannot and does not appear !

Classical limit of the D-commutator : Nambu bracket

Nambu bracket and volume perserving diffeomorphisms

The classical limit a of D-commutator $[R_1, \cdots, R_D] = (i\hbar)^{D/2}$ is a multisimplectic structure describing a D-dimensional phase-space

$$\{x_1, \cdots, x_D\} = 1$$

with the Nambu bracket :

$$\{A_1, \cdots, A_D\} = \epsilon_{\alpha_1\alpha_2\cdots\alpha_D} \frac{\partial A_1}{\partial x_{\alpha_1}} \frac{\partial A_2}{\partial x_{\alpha_2}} \cdots \frac{\partial A_D}{\partial x_{\alpha_D}}$$

Invariant under volume-preserving diffeomorphisms (VPD)

$$x_i \to y_i(x) \qquad \det \frac{\partial y_i}{\partial x_j} = 1 \qquad \{y_1, \cdots, y_D\} = 1$$

Liouville theorem :

$$\frac{dx_i}{dt} = \{x_i, H_1, \cdots, H_{D-1}\}$$

Hamiltonian(s) evolutions generates all VPDs.

Nambu bracket and extended objects

Nambu formalism associates to a classical string $x_i(t,\sigma)$ in N dimensions $N(N-1)/2$ "momenta" p_{ij}.

$$\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial \sigma} - \frac{\partial x_i}{\partial \sigma}\frac{\partial x_j}{\partial t} = \partial H/\partial p_{ij}$$

$$\sum_j \left(\frac{\partial p_{ij}}{\partial t}\frac{\partial x_j}{\partial \sigma} - \frac{\partial p_{ij}}{\partial \sigma}\frac{\partial x_j}{\partial t} \right) = -\partial H/\partial x_i$$

For a string in 2D this becomes, writing $x_3 = p_{12}$.

$$\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial \sigma} - \frac{\partial x_i}{\partial \sigma}\frac{\partial x_j}{\partial t} = \{x_i, x_j, H\}$$

A Nambu bracket appears, with a single Hamiltonian !

Can be extended to $D-1$ membranes.

Extended objects (cf Joost's talk)

3D TI and BF theory (Cho and Moore, '11)

$$\mathcal{L}_{BF} = \frac{k}{4\pi} b \wedge F = \frac{k}{4\pi} \epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} \partial_\rho a_\sigma$$

2-form gauge field couple to string-like objects

$b_{\mu\nu}$ is a 2-form gauge field ($b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu \beta_\nu - \partial_\nu \beta_\mu$)

higher gauge theories, and parallel transport of strings/loops.

4D QHE and higher CS theory (Bernevig *et al*, '02)

$$\mathcal{L} = A \wedge dA \wedge dA - \frac{3i}{2} A \wedge A \wedge A \wedge A \wedge dA - \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A$$

membranes excitations, fractional statistics

Conclusion

Algebraic structure of projected densisty operators in D dimensions :

D even

Isotropic D-algebra that probes the Hall conductance in D dimensions

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \cdots, \rho_{\mathbf{q}_D}] \propto C_{\frac{D}{2}} \left(\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \cdots \wedge \mathbf{q}_D\right) \rho_{\mathbf{q}_1 + \ldots + \mathbf{q}_D}$$

Non-commutative D-dimensional phase-space $[R_1, R_2, \cdots, R_N] \propto i^{D/2} C_{\frac{D}{2}}$

D odd

anisotropic, probes the Hall conductance in $D-1$ dimensions

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}, \rho_{\mathbf{q}_3}] = -i(q_1^\mu q_2^\nu + q_3^\mu q_1^\nu + q_2^\mu q_3^\nu) F_{\mu\nu} \rho_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3}$$

Consequences of D-algebra poorly understood at this point :

- volume preserving differomorphisms and incompressibility
- extended excitations (strings or membranes)