Fractional Quantum Hall Effect of Lattice Bosons

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Support: NIST/NRC; Marie Curie IIF

Fractional Quantum Hall Effect in Lattice?

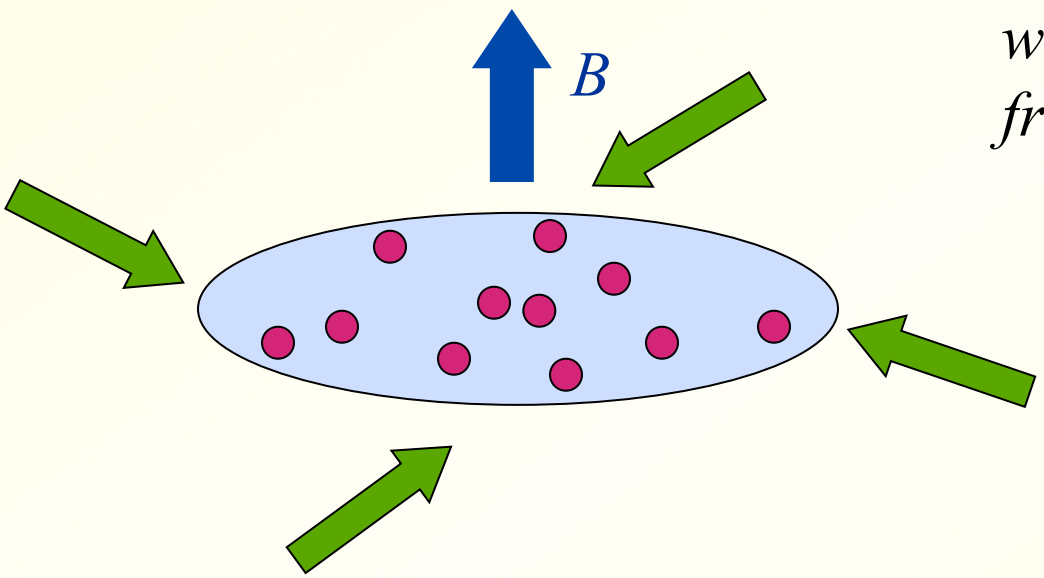
Ultracold atomic gasses (usually bosons)

Synthetic magnetic field (rotation, laser-induces fields)

Contact interaction →

At sufficiently small $\nu = N / N_\phi$ we expect Bosonic versions of fractional quantum Hall states.

Cooper et al., '01



$$\nu = \frac{1}{2} \qquad \psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 e^{-\sum_i |z_i|^2/4}$$

Laughlin State

$$\nu = 1 \qquad \psi_{MR}(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)\, e^{-\sum_i |z_i|^2/4}$$

Moore-Read State

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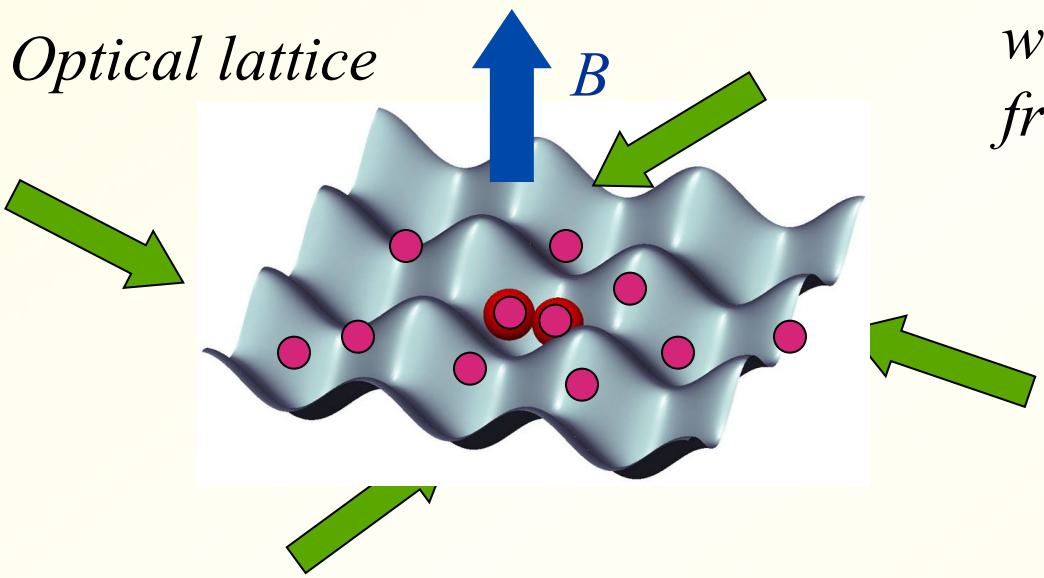
Optical lattice



At sufficiently small $\nu = N / N_\phi$ we expect Bosonic versions of fractional quantum Hall states.

Cooper et al., '01

How do the quantum Hall states modify in the presence of a lattice?

$$\nu = \frac{1}{2} \qquad \psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 e^{-\sum_i |z_i|^2/4}$$

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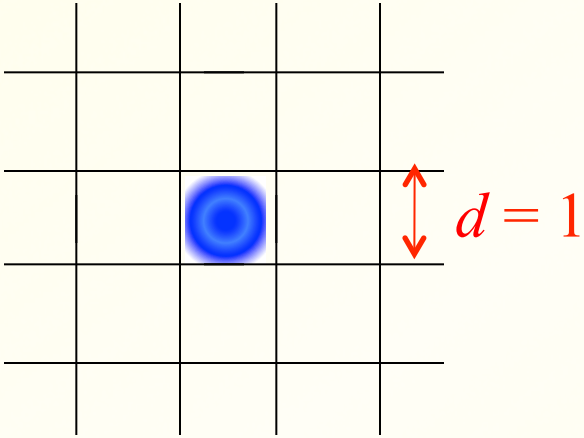
Moore-Read State

2D Bosons under Magnetic Field – Lattice



Filling Fraction $\nu = \dfrac{N}{N_\phi}$

Flux density $n_\phi = \dfrac{Bd^2}{h/q}$

$0 \le n_\varphi < 1$

$n_\phi << 1$ → *Effectively the continuum limit*

Sorensen et al., PRL '05

Hafezi et al., PRA '07

$n_\phi \sim p/q$ → *Map the lattice to a multi-layer model in the continuum limit.*

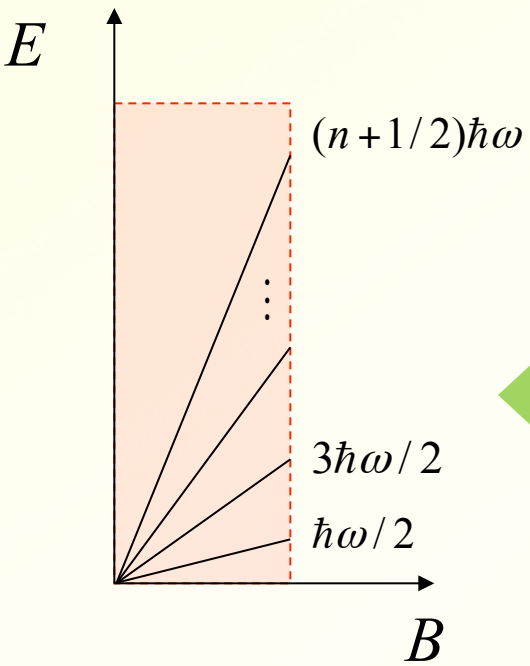
Palmer and Jacksch, PRL '06

Moller and Cooper, PRL '09

Powell et al., PRL '10

Charged Particles in a Magnetic Field – Single Particle Picture

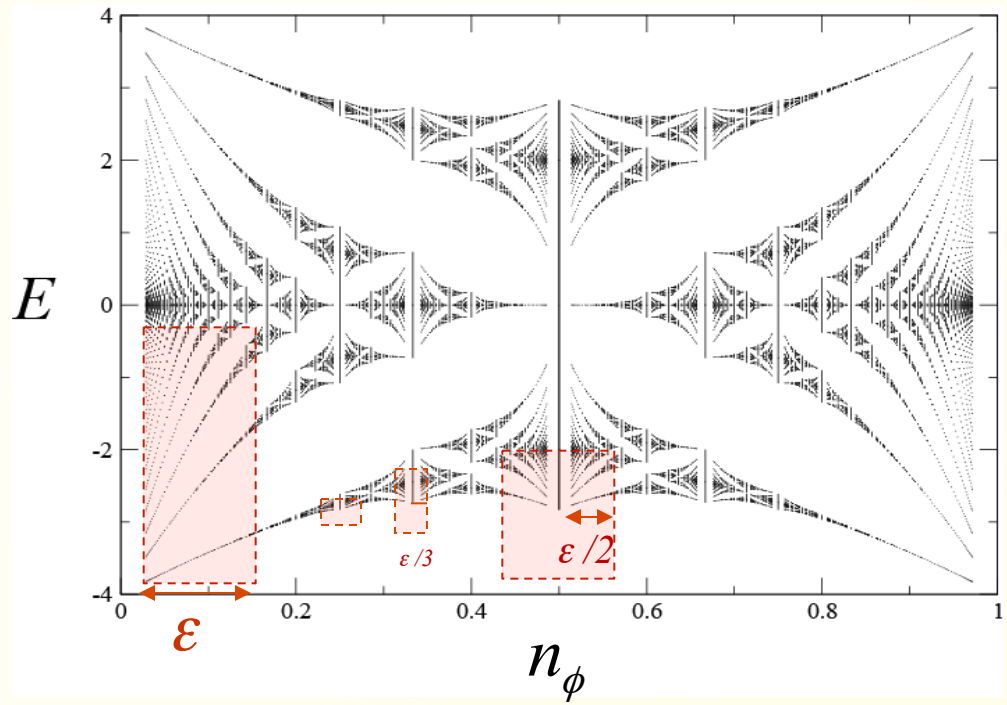
Continuum



Landau Levels

$E_n = (n+1/2)\hbar\omega$ $\quad$ $\omega = qB/m$

Lattice



Hofstadter Butterfly

Hofstadter, PRB '76

Map the lattice near rational n_ϕ to a model in the continuum!

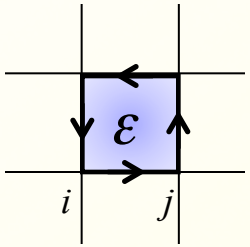
Non-interacting System: $0 < n_\varphi = \varepsilon << 1$

Palmer and Jacksch, PRL '06

$$H = -J \sum_{\langle ij \rangle} (e^{i\theta_{ij}} c_i^\dagger c_j + h.c.)$$

$$\theta_{ij} = \int_i^j \vec{A} \cdot d\vec{l}$$

$$\sum_{plaquette} \theta_{ij} = 2\pi\varepsilon$$

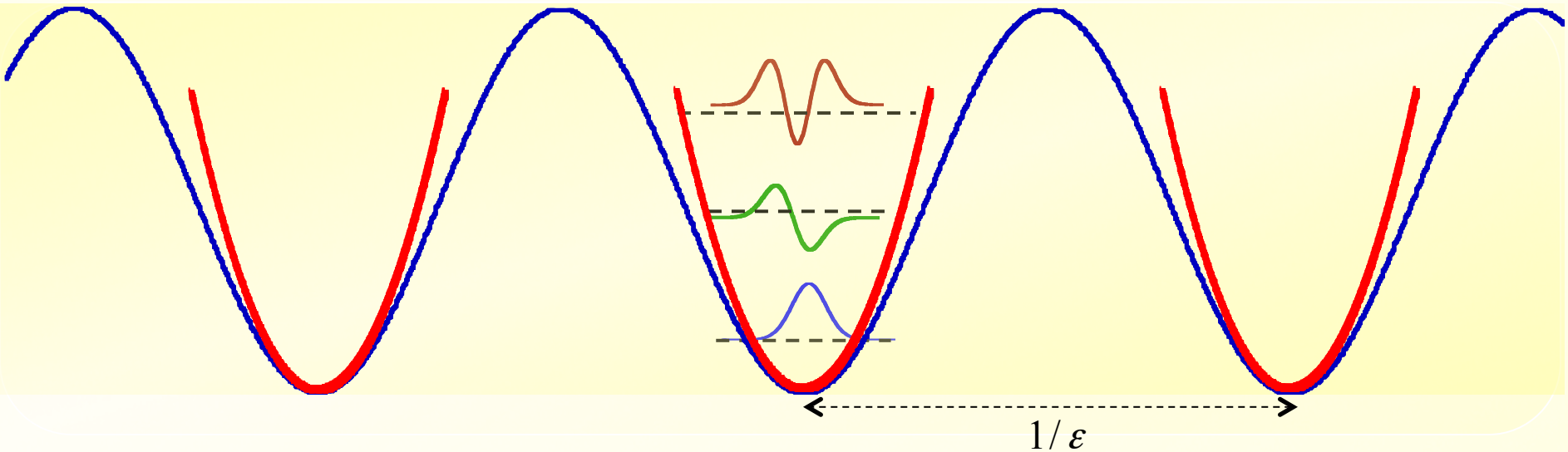


Landau Gauge
$\vec{A} = (0, -Bx, 0)$ → $\psi_k(x,y) \sim \phi(x)\, e^{iky}$

$$\phi(x+1) + \phi(x-1) + 2Cos\ (2\pi\varepsilon\, x - k)\phi(x) = -E/J\, \phi(x)$$

$\varepsilon << 1$ → $$-\partial_x^2 \phi(x) + 2\pi\varepsilon (x - x_k)^2 \phi(x) = \tilde{E}\phi(x)$$

$$x_k = \frac{k}{2\pi\varepsilon}$$



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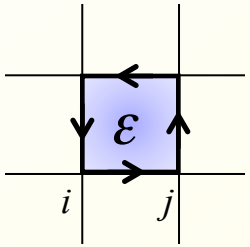


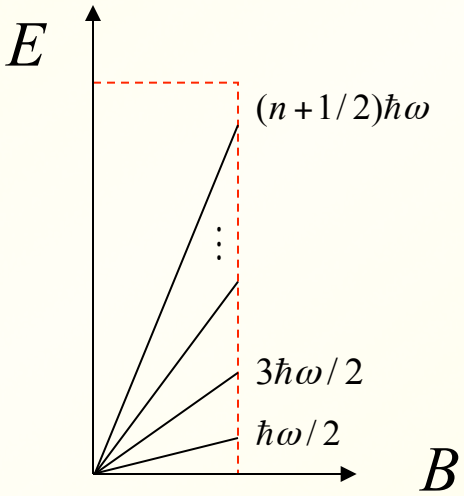
Landau Gauge
$\vec{A} = (0,-Bx,0)$

$$H = -\partial_x^2 + 2\pi\varepsilon(x - x_k)^2$$

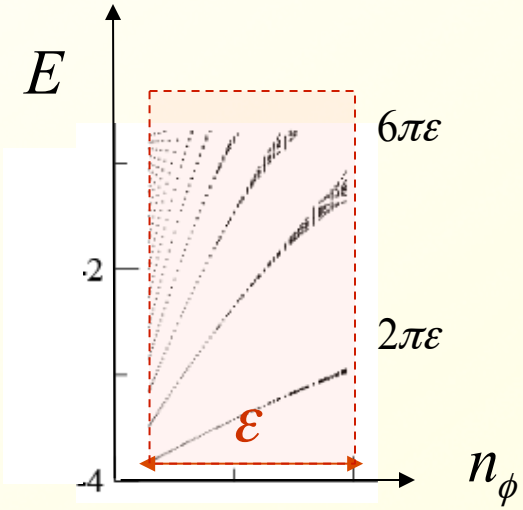
$$x_k = \frac{k}{2\pi\varepsilon}$$

Ground State → $\psi_k(x,y) \sim e^{-\pi\varepsilon(x-x_k)^2}e^{iky}$ → *Lowest Landau Level*

Energy Spectrum → $E_n = 4J\pi\varepsilon(n+1/2)$



$$\hbar\omega \equiv 4J\pi\varepsilon$$



Non-interacting System: $n_\phi = 1/2 + \varepsilon \quad 0 < \varepsilon << 1$

Palmer and Jacksch, PRL '06

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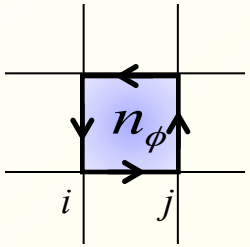

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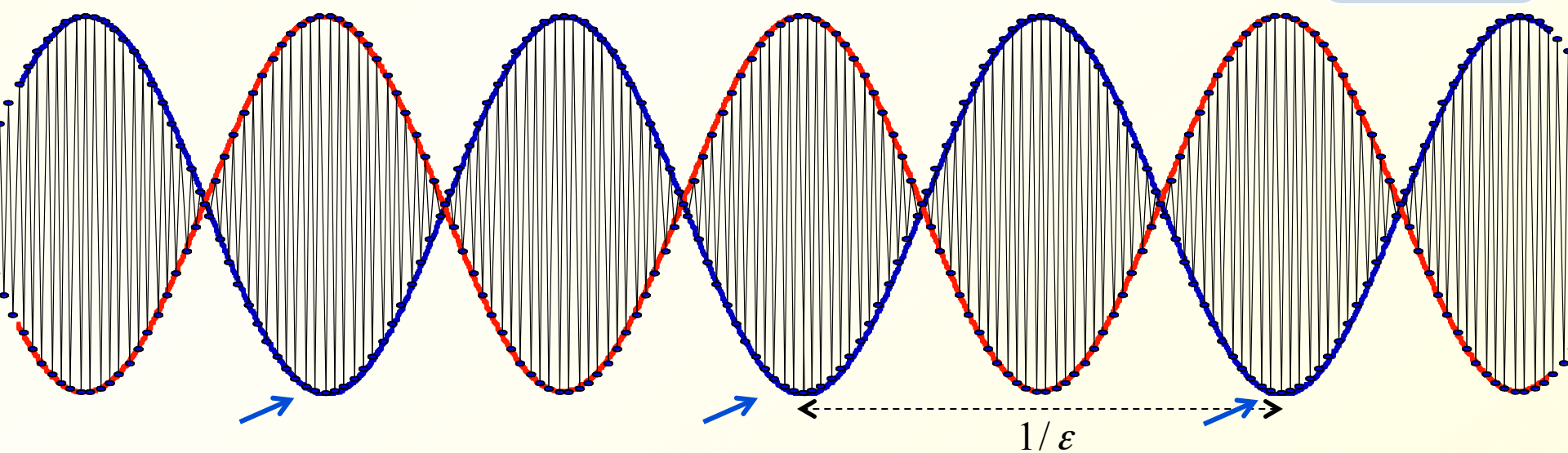

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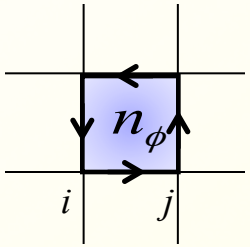

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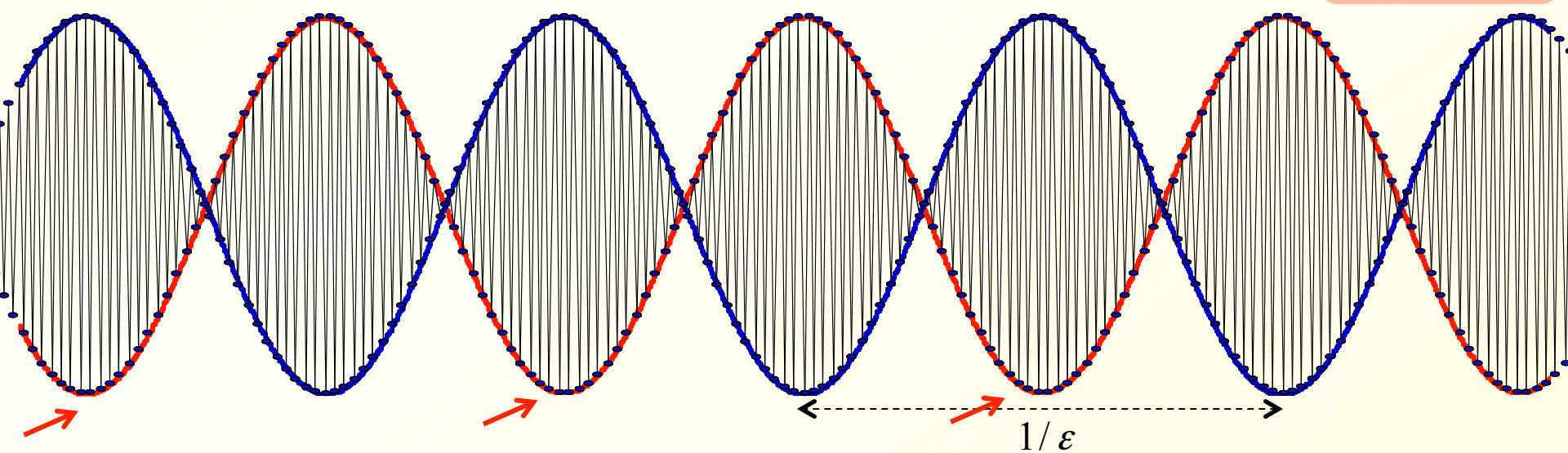
$\varepsilon << 1$ → $$-\partial_x^2 \phi(x) - 2\pi\varepsilon(-1)^x (x - x_{k-\pi})^2 \phi(x) = \tilde{E}\phi(x)$$

$$x_{k-\pi} = \frac{k-\pi}{2\pi\varepsilon}$$



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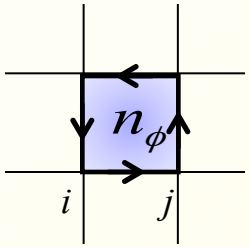

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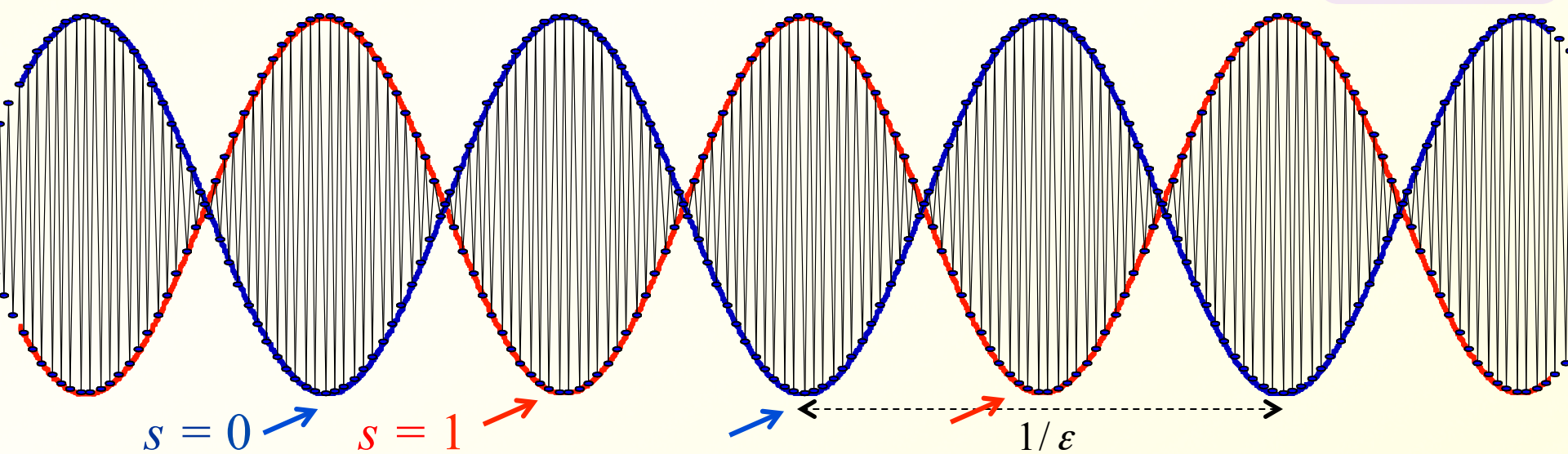
$\varepsilon << 1$ → $$-\partial_x^2 \phi(x) + 2\pi\varepsilon(-1)^{x+s}(x - x_{k-s\pi})^2 \phi(x) = \tilde{E}\phi(x)$$

$$x_{k-s\pi} = \frac{k - s\pi}{2\pi\varepsilon}$$



Non-interacting System: $n_\phi = 1/2 + \varepsilon$

Palmer and Jacksch, PRL '06

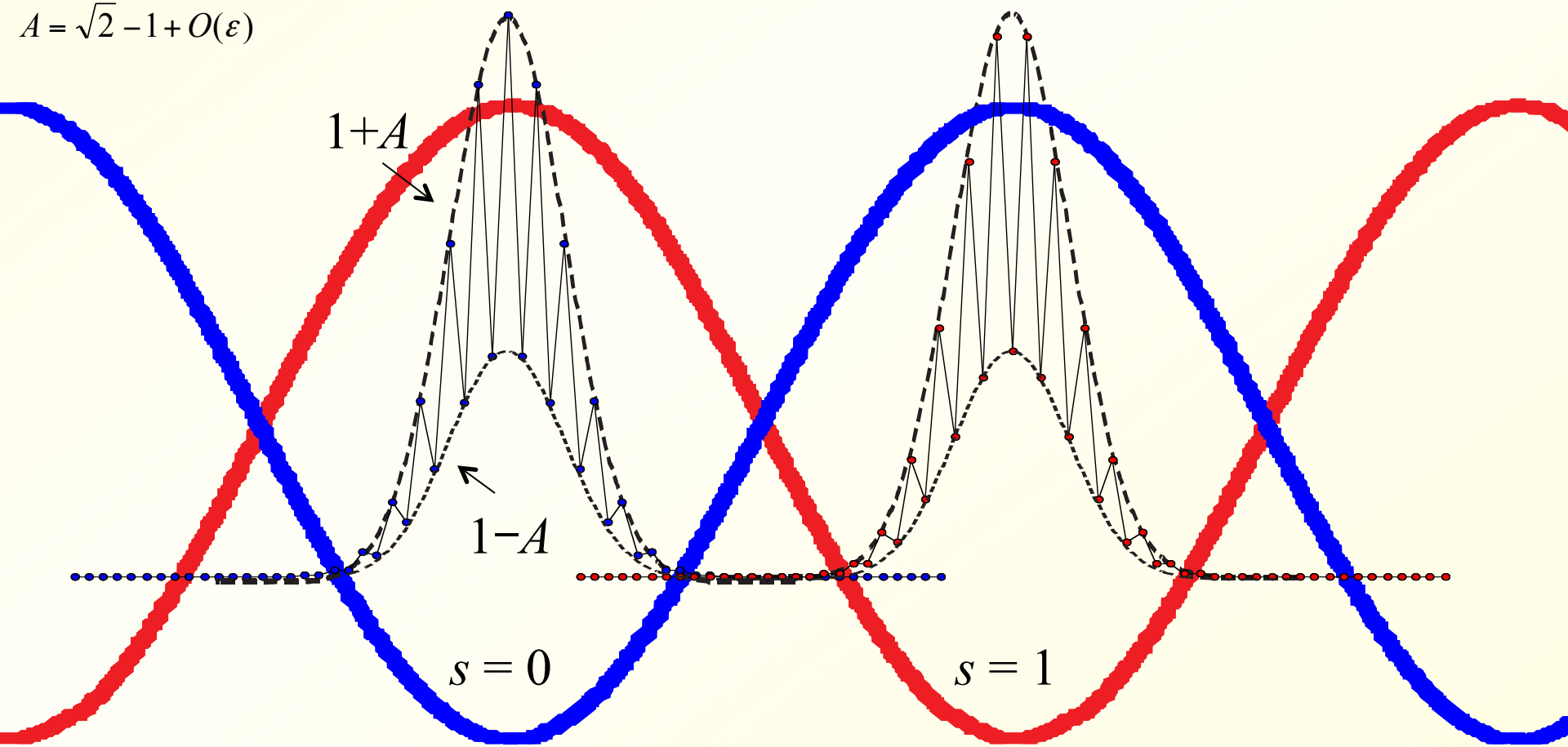
Ground State → $\psi_{k'}(x,y) \sim F_s(x)\, e^{-\pi\varepsilon(x-x_{k-s\pi})^2}\, e^{iky}$

$$x_{k-s\pi} = \frac{k - s\pi}{2\pi\varepsilon}$$

$F_s(x) = (1 + (-1)^{s+x} A)$

$s = 0,\ 1$ → ***band index***

$A = \sqrt{2} - 1 + O(\varepsilon)$



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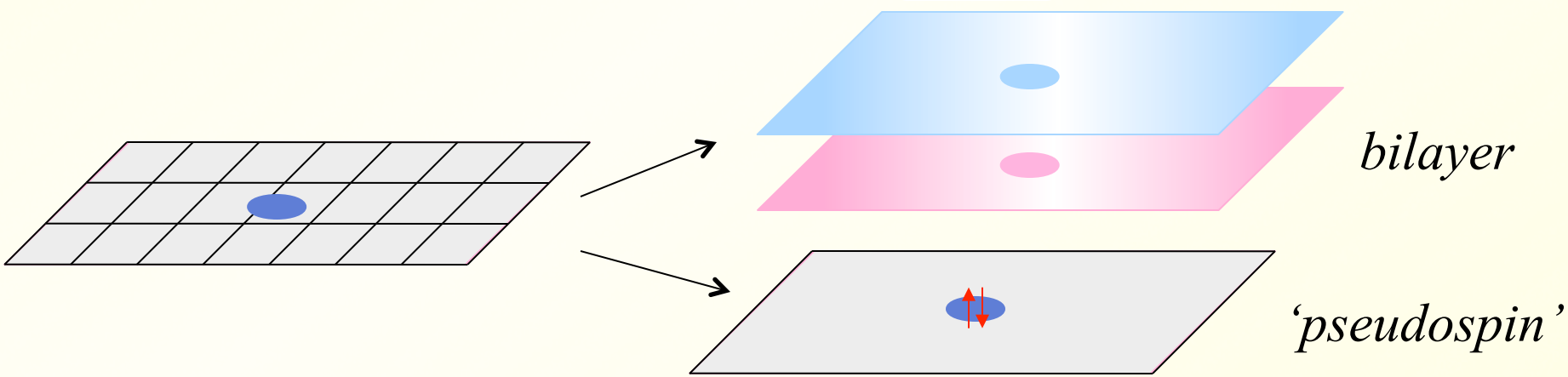
$$x_{k-s\pi} = \frac{k - s\pi}{2\pi\varepsilon}$$

$F_s(x) = (1 + (-1)^{s+x} A)$ $\quad s = 0, 1$ → *band index*

Similar to the continuum case but now with a form factor that depends on the band index.

Energy Spectrum → $E_n = 2\sqrt{2} J\pi\varepsilon(n + 1/2)$



By introducing the band index we can map the energy spectrum near n_ϕ = ½ to Landau levels where each state is two-fold degenerate.

Interacting System

Interaction $\hat{U} = \frac{1}{2}\sum_{k_1k_2k_3k_4} U_{k_1k_2k_3k_4} c_{k_1} c_{k2} c^{\dagger}_{k_3} c^{\dagger}_{k_4}$

Haldane, '83
Cooper, '08

$$U_{k_1k_2k_3k_4} = \int dr_1 dr_2 U(r_1 - r_2)\psi^*_{k_1}(r_1)\psi^*_{k_2}(r_2)\psi_{k_3}(r_2)\psi_{k_4}(r_1)$$

$\psi_{k_i}(r_\alpha) \leftarrow$ *single-particle basis states at the lowest Landau level*

We carry out exact diagonalization of the potential for finite size systems.

Contact Interaction $\hat{U} = U\sum_{i<j}\delta(r_i - r_j)$

Continuum Limit

$n_\phi = \varepsilon << 1$

Single Particle Ground State $\psi_k(x,y) \sim e^{-\pi\varepsilon(x-x_k)^2} e^{iky}$

$\rightarrow$ $U_{k_1k_2k_3k_4} = U\sqrt{\varepsilon}\, e^{-\sum_{i<j}(k_i-k_j)^2/(16\pi\varepsilon)} \delta_{k_1+k_2,k_3+k_4}$ $\rightarrow$ $\forall i,j \quad k_i = k_j$

Interacting Case Near $n_\phi = 1/2$

Single Particle Ground State

$$\psi_{k'}(x,y) \sim F_s(x) e^{-\pi\varepsilon(x-x_{k'})^2} e^{iky}$$

$$k_i' = k_i - s_i\pi$$

$$U_{k_1'k_2'k_3'k_4'} = U\, G_{s_1s_2s_3s_4} \sqrt{\varepsilon}\; e^{-\sum_{i<j}(k_i'-k_j')^2/(16\pi\varepsilon)}\; \delta_{k_1+k_2,k_3+k_4}$$

$$G_{s_1s_2s_3s_4} = \frac{1}{2} \sum_{s_1,s_2,s_3,s_4=0}^{1} F_{s_1}F_{s_2}F_{s_3}F_{s_4}$$

$$\forall i,j \quad k_i' = k_j'$$

$$k_i = k_j, \quad s_i = s_j$$

$$k_i = k_j - \pi, \quad s_i \neq s_j$$

Conserve the band index

k_1	k_2	k_3	k_4	s_1	s_2	s_3	s_4	$G_{s_1s_2s_3s_4}$
k	k	k	k	1	1	1	1	$(3-\pi\varepsilon)$
k	$k+\pi$	k	$k+\pi$	1	0	1	0	$(1+\pi\varepsilon)$
k	$k+\pi$	$k+\pi$	k	1	0	0	1	$(1+\pi\varepsilon)$
k	k	$k+\pi$	$k+\pi$	1	1	0	0	$(1+\pi\varepsilon)$

$$k_1 + k_2 = k_3 + k_4 + 2\pi$$

umklapp scattering

no band index conservation

anomalous term

Rotating the Basis

Pseudospin

Band Index

$$G = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 3-\pi\varepsilon & & & 1+\pi\varepsilon \\ & 1+\pi\varepsilon & 1+\pi\varepsilon & \\ & 1+\pi\varepsilon & 1+\pi\varepsilon & \\ 1+\pi\varepsilon & & & 3-\pi\varepsilon \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{pmatrix} 1 & & & \pi\varepsilon \\ & 1 & 1 & \\ & 1 & 1 & \\ \pi\varepsilon & & & 1 \end{pmatrix} \end{array}$$

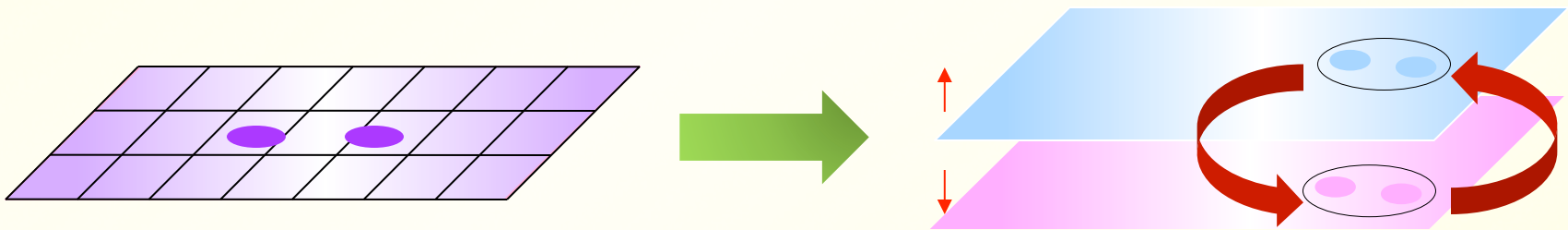
$$\Psi_{\uparrow\downarrow}(x,y) = \frac{\psi_0(x,y) \pm i\psi_1(x,y)}{\sqrt{2}}$$

This change of basis isolates the effect of ε to two matrix elements.

Interacting Case at $n_\phi = 1/2 + \varepsilon$

$$U_{k'_1 k'_2 k'_3 k'_4} = 2U\sqrt{\varepsilon}\begin{array}{c} \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{pmatrix} 1 & & & \pi\varepsilon \\ & 1 & 1 & \\ & 1 & 1 & \\ \pi\varepsilon & & & 1 \end{pmatrix} \end{array} e^{-\sum_{i<j}(k'_i - k'_j)^2/(16\pi\varepsilon)} \delta_{k_1+k_2,k_3+k_4}$$

Similar to the continuum case but now with matrix elements that depend on pseudospin.



The anomalous terms do not conserve pseudospin → *BCS Pairing?*

Can a pairing process be observed in an incompressible state?

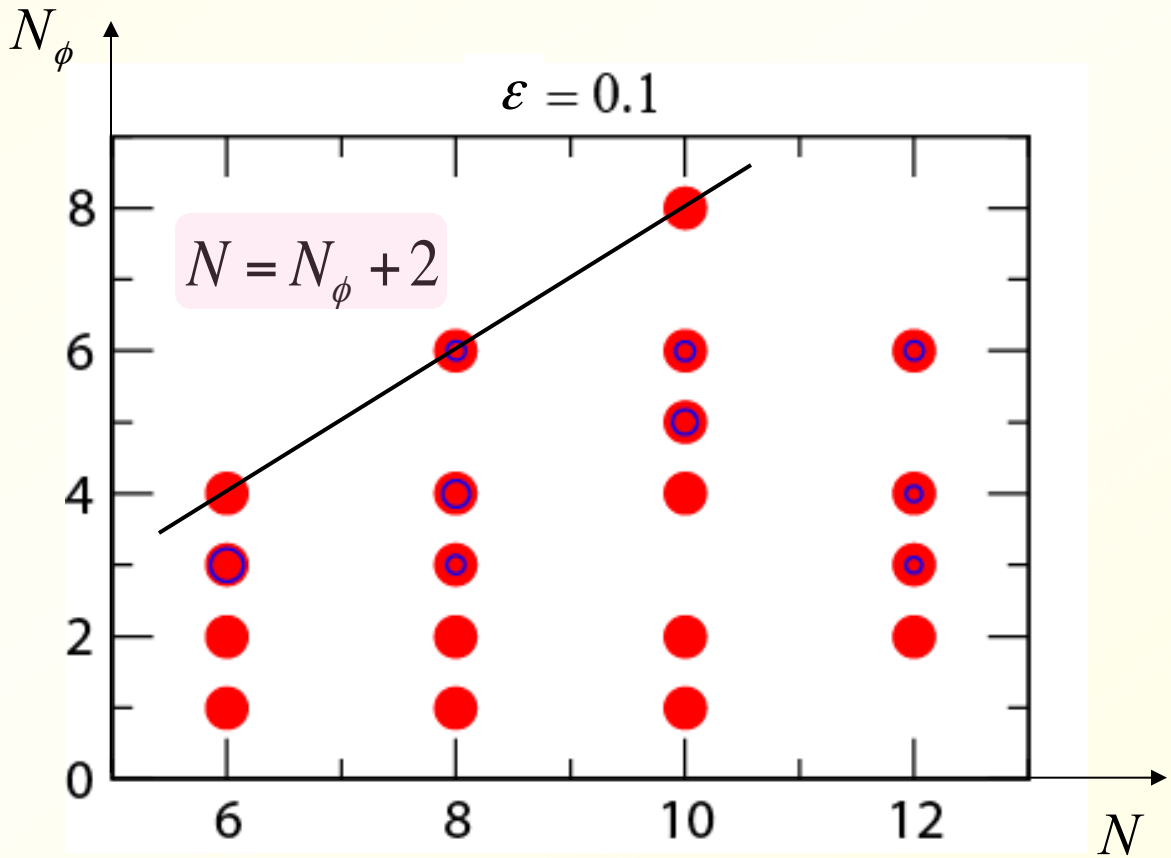
Incompressible States

We carry out exact diagonalization of the potential for finite size systems and identify incompressible states.

$L = 0$ ← *incompressibility*

$$\nu = \frac{N}{N_\phi + \chi}$$

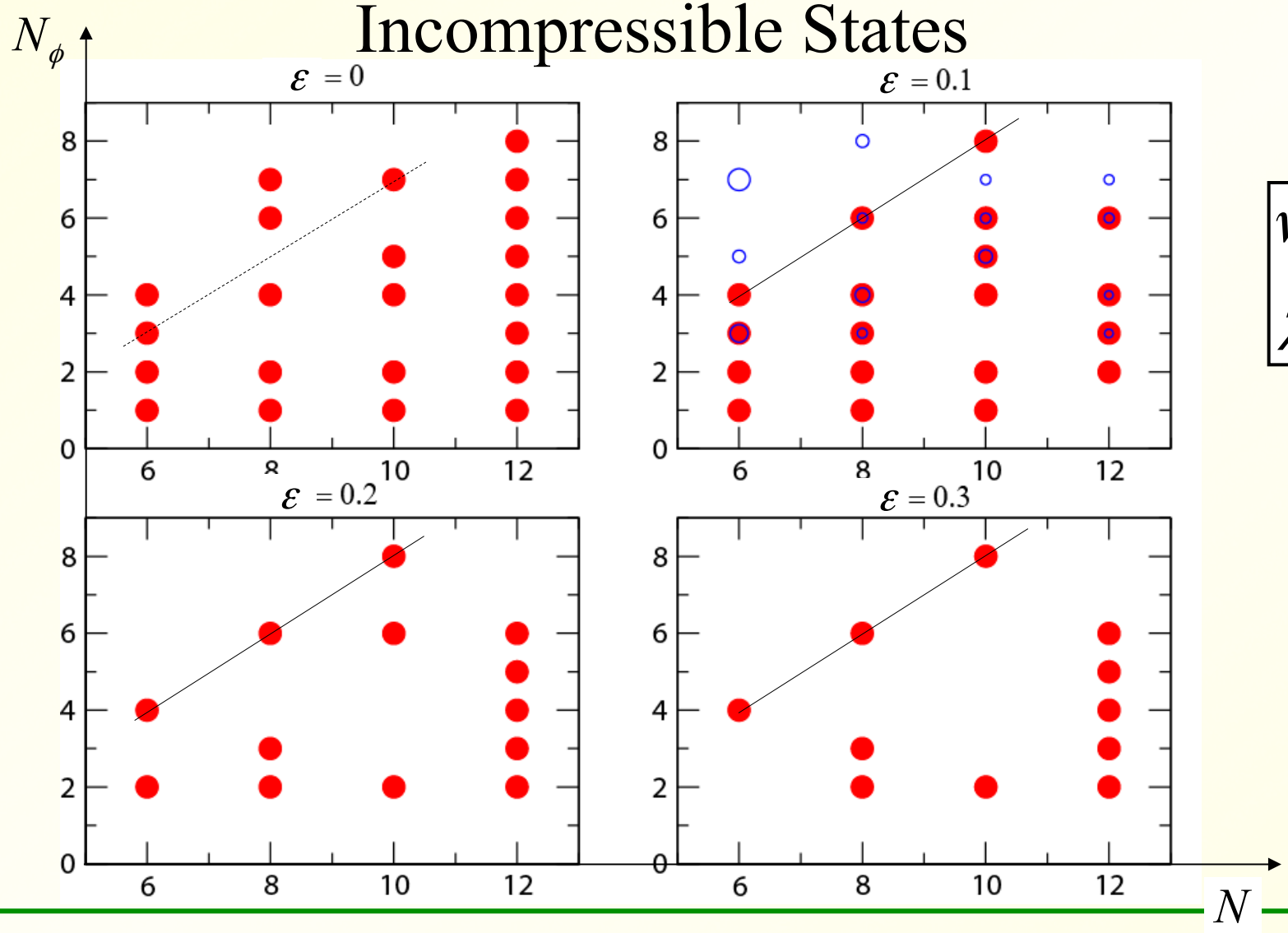
← *Shift*



$$\nu = 1$$

$$\chi = 2$$

Incompressible States

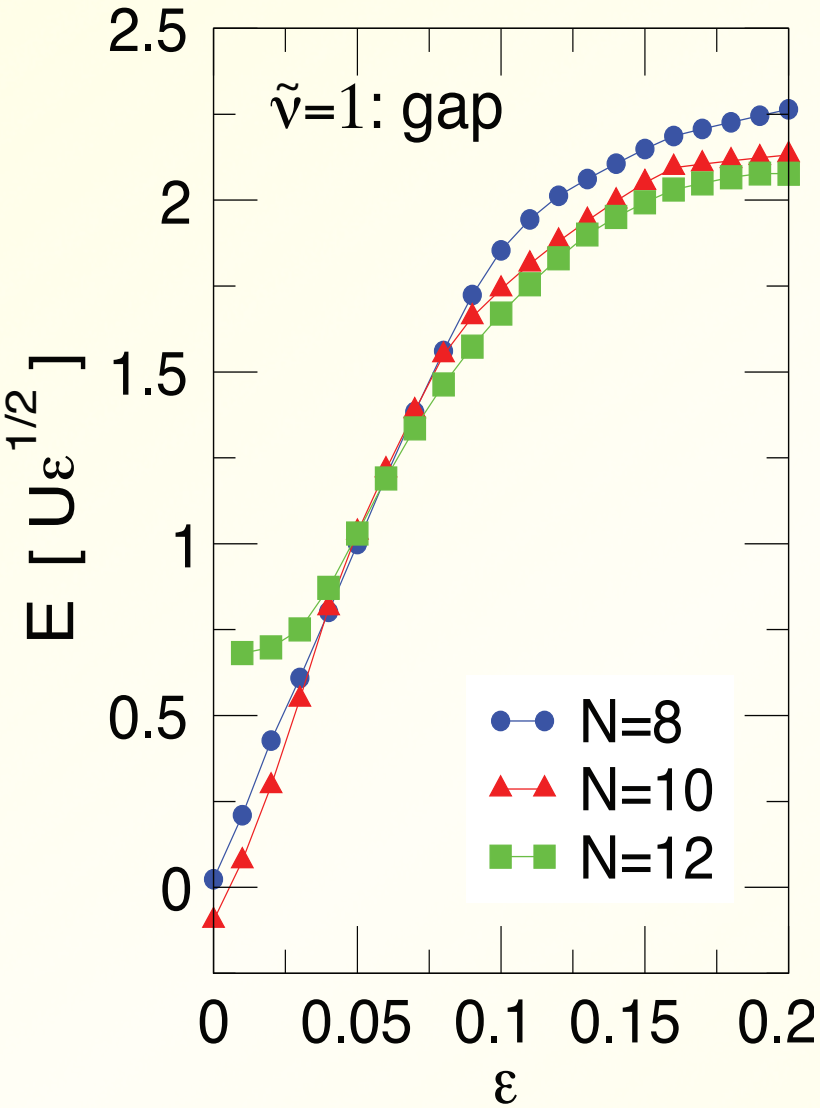


$\nu = 1$

$\chi = 2$

The incompressible state at ν = 1 seems to be persistent over a range of flux densities.

Incompressible States



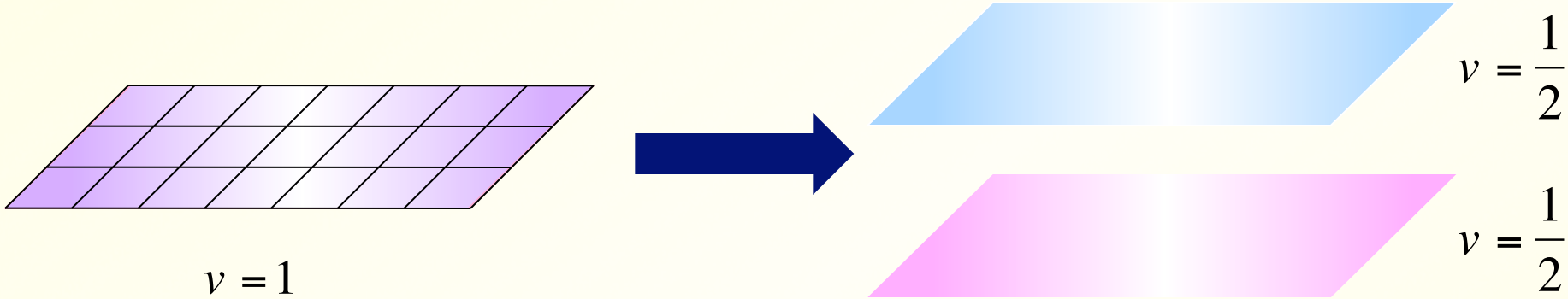
The incompressible state at $\nu = 1$ seems to get more robust as ε increases.

Gap closes as ε vanishes!

→ *The anomalous terms seem to stabilize the incompressible state.*

Trial Wavefunction

Near flux density n_ϕ = ½: Effectively a bilayer system $\quad \nu = \frac{1}{2} + \frac{1}{2}$



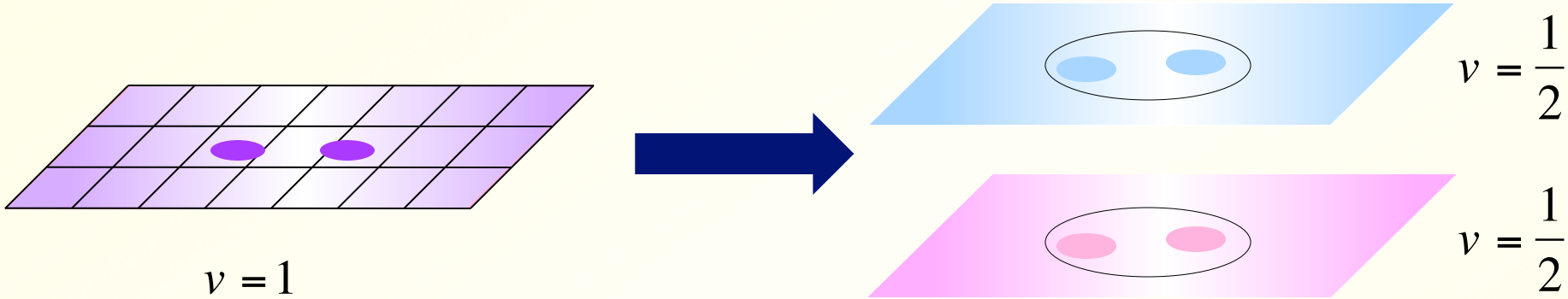
Continuum $\quad \nu = \frac{1}{2} \quad \Psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2$

$n_\phi = \frac{1}{2} + \varepsilon \quad \cancel{\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^2 \prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})^2}$

- *Does not prevent particles of opposite pseudospin from approaching one another --- not energetically favorable.*
- *The overlap is not good either.*

Trial Wavefunction

Near flux density n_ϕ = ½: Effectively a bilayer system $\quad \nu = \frac{1}{2} + \frac{1}{2}$



Continuum $\quad \Psi_L(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j) \qquad \nu = 1$

$n_\phi = \frac{1}{2} + \varepsilon$

$$\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)Pf\left(\frac{1}{z_i^\uparrow - z_j^\uparrow}\right)\prod_{i<j}(z_i^\downarrow - z_j^\downarrow)Pf\left(\frac{1}{z_i^\downarrow - z_j^\downarrow}\right)\prod_{i\neq j}(z_i^\uparrow - z_j^\downarrow)$$

- *Symmetric under exchange of interspecies particles*
- *Produces the right filling fraction and the right shift*
- *Pfaffian factor is the real space form of the BCS pairing*

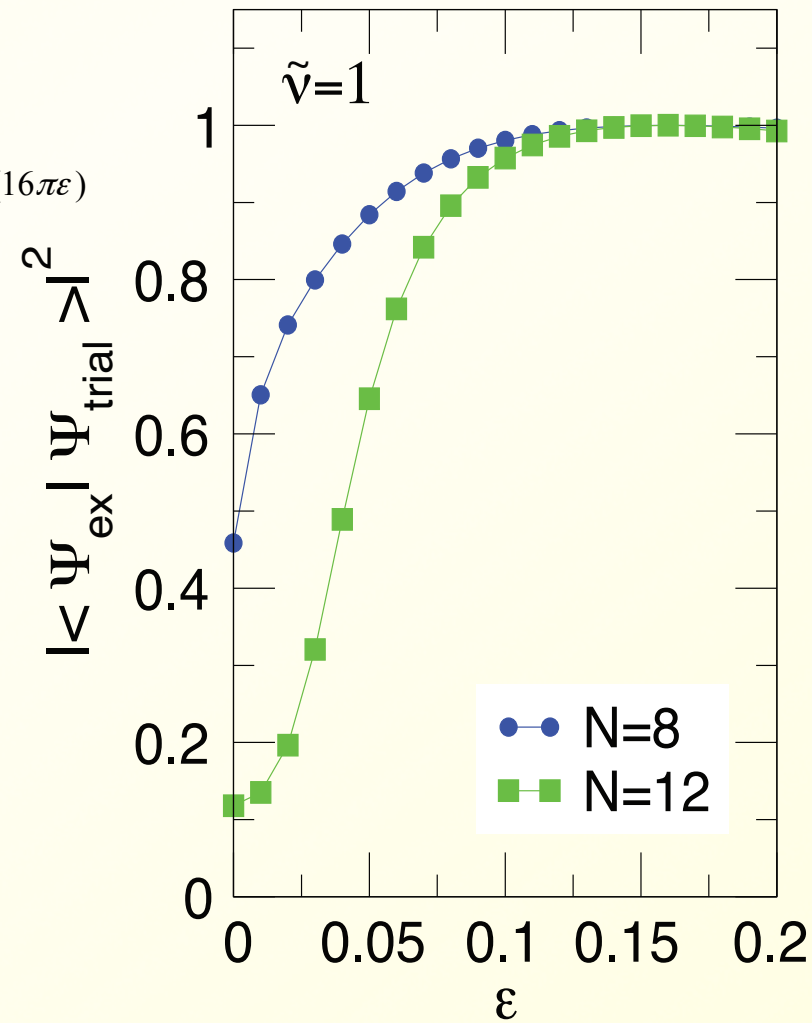
$\nu = 1$

$\chi = 2$

How Good is the Trial Wave function?

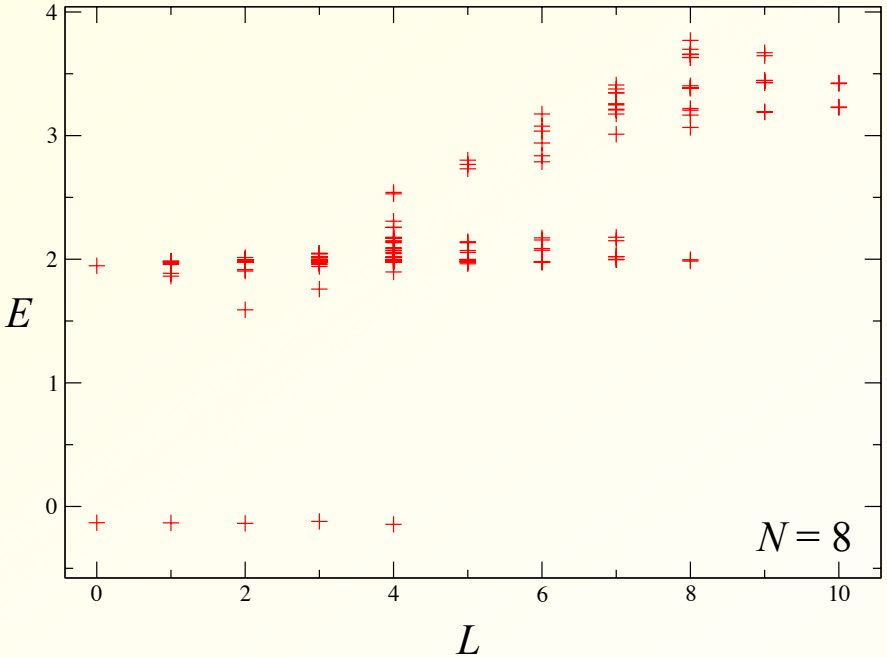
$$\Psi(\{z_i\})_{trial} = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow}) Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right) \prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow}) Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right) \prod_{i\neq j}(z_i^{\uparrow} - z_j^{\downarrow})$$

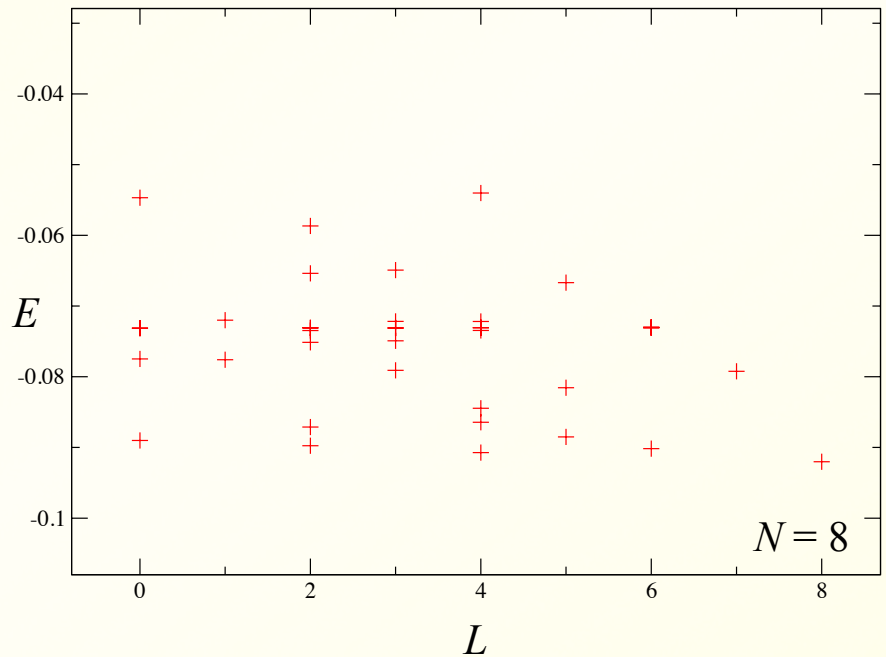
$$U_{k'_1 k'_2 k'_3 k'_4} = U\sqrt{\varepsilon}\begin{pmatrix} 1 & & & \pi\varepsilon \\ & 1 & 1 & \\ & 1 & 1 & \\ \pi\varepsilon & & & 1 \end{pmatrix} e^{-\sum_{i<j}(k'_i - k'_j)^2/(16\pi\varepsilon)}$$



For ε = 0.16, the overlap is 0.99999!

Excitations





Lattice Hamiltonian + 1 flux

$L = 0 + 1 + 2 + 3 + 4$

Lattice Hamiltonian + 2 fluxes

$L = 0^5 + 1^2 + 2^5 + 3^6 + 4^8 + 5^3 + 6^5 + 7 + 8$

MR x MR + 1 flux

$L = 0^2 + 1 + 2^3 + 3 + 4$

MR x MR + 2 fluxes

$L = 0^6 + 1^5 + 2^{13} + 3^9 + 4^{11} + 5^5 + 6^5 + 7 + 8$

Excitations between the two layers are coupled.

Abelian or Not?

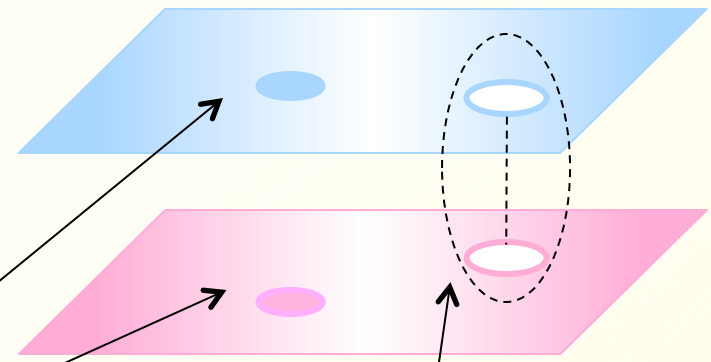
Moore-Read State

$\psi_b = \psi \; e^{i\phi}$

$\psi_{qh} = \sigma \; e^{i\phi/2}$

Non-Abelian excitation

*Two **coupled** Moore-Read States*



There are two kinds of bosons but only one type of quasihole with two sigma fields that are locked.

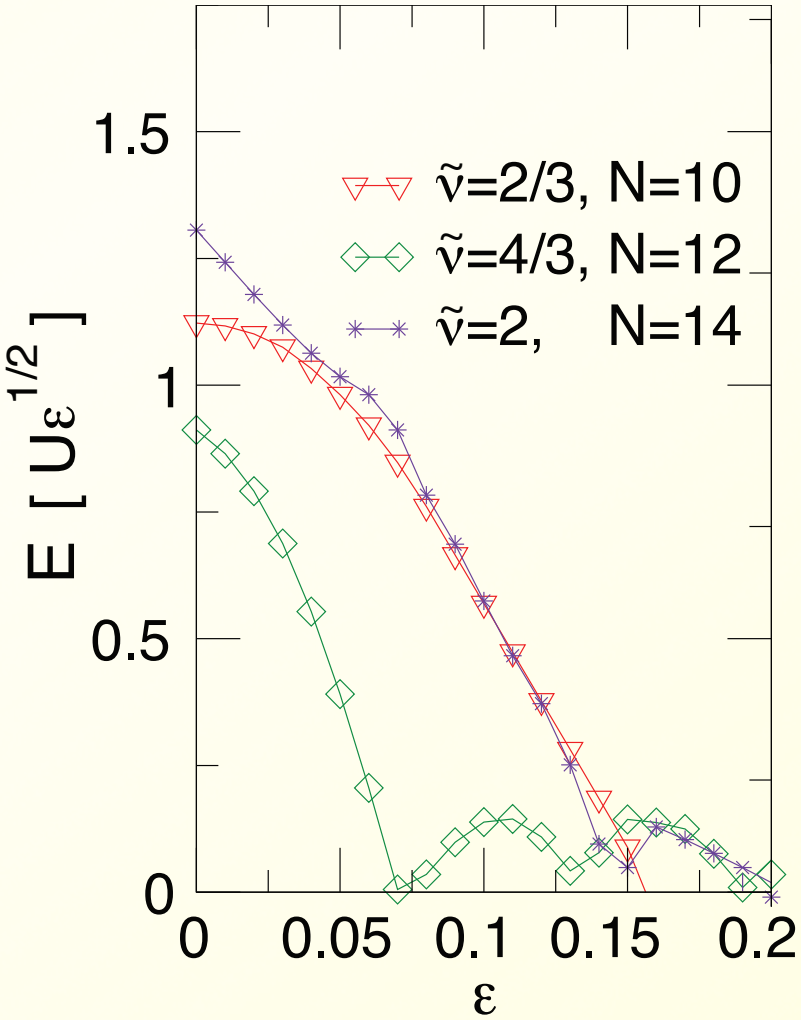
The combined excitations are effectively Abelian.

Other Filling Fractions?

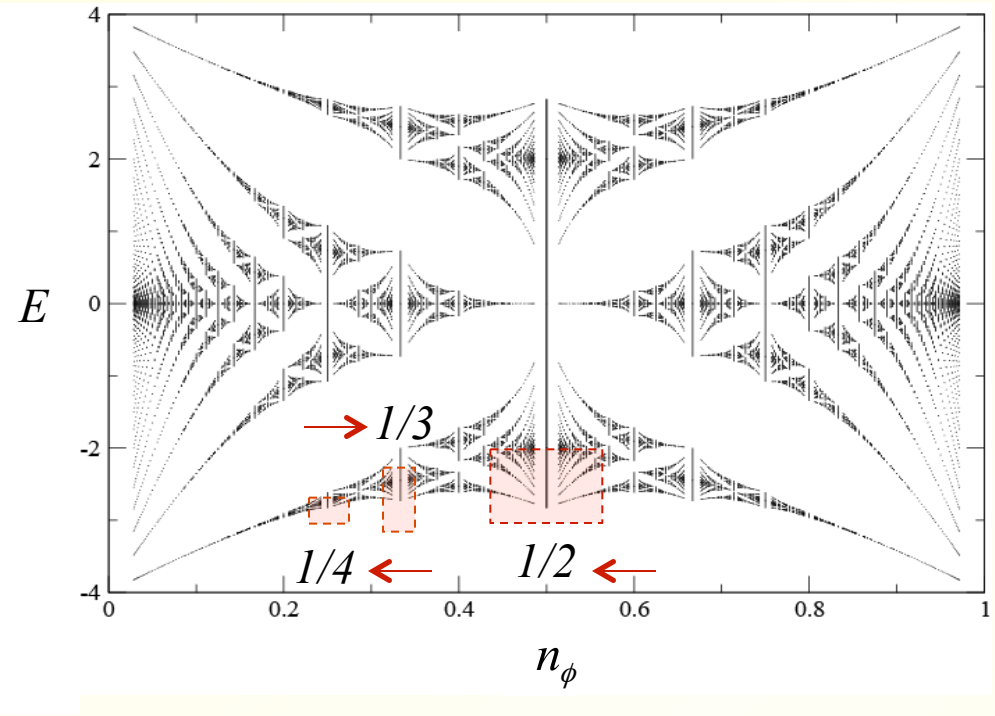
$\nu = 4/3 \longrightarrow$ *NASS State*

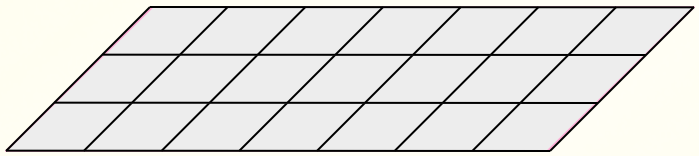
$\nu = 2/3 \longrightarrow$ 221 *State*

$\nu = 2 \longrightarrow$ 2 *Copies of CF liquid*

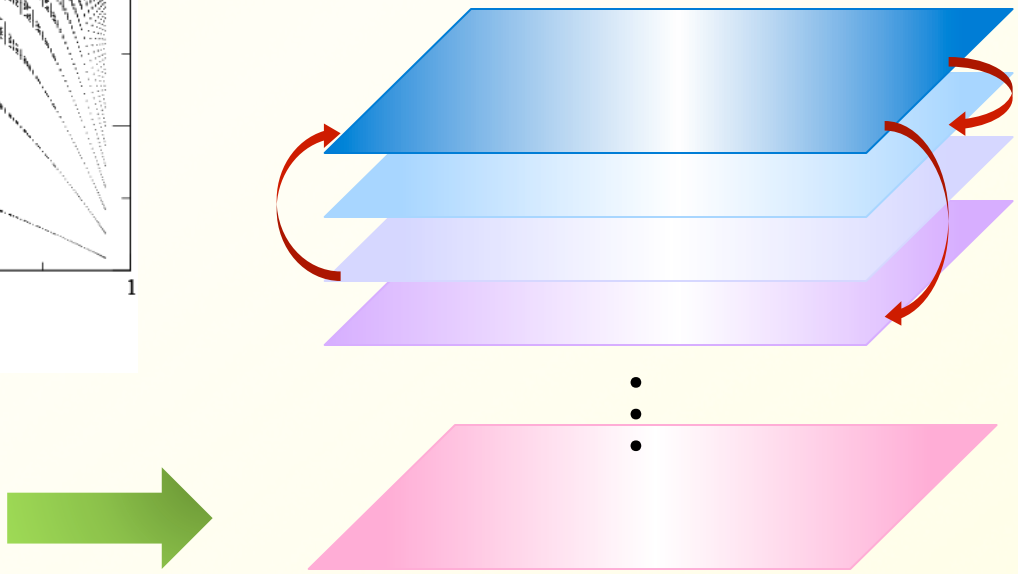


Generalization to $n_\phi = p/q + \varepsilon$





Lattice near $n_\phi = p/q + \varepsilon$



q-layer continuum system

Potentially more interesting states but probably harder to realize…

Summary

- *Near $n_\phi = ½$ ⟶ two-fold degeneracy due to pseudospin.*
- *Interaction potential suggests pairing of particles with the same pseudospin.*
- *At $\nu = 1$ pairing terms stabilize the groundstate.*
- *Trial wave function for the groundstate of $\nu = 1$ has excellent overlap with ED result:*

$$\Psi(\{z_i\}) = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})\prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})\prod_{i<j}(z_i^{\uparrow} - z_j^{\downarrow})Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right)Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right)$$

- *Pairing terms might be important for other filling fractions, flux densities, other types of interactions, fermions, etc.*

LH, G. Moller and S. H. Simon, *Phys. Rev. Lett.* **108**, 256809 (2012)