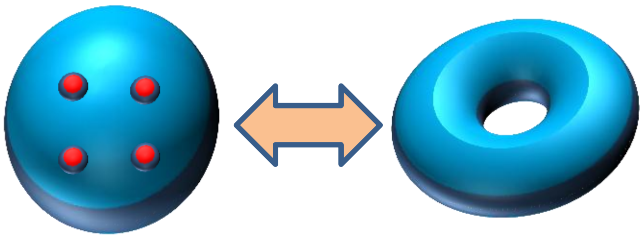

Topological nematic states, twist defects and genons

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NORDITA workshop, Aug. 8th, 2012

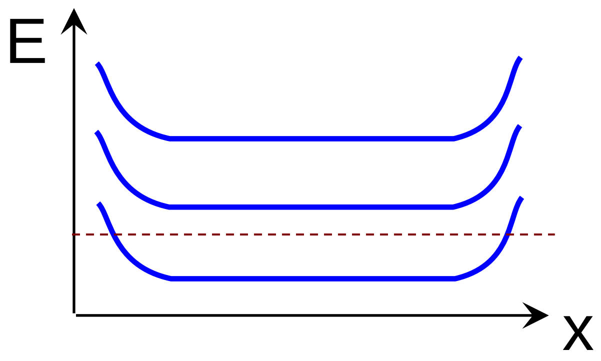
Outline

- A new class of topological states of matter: Fractional quantum anomalous Hall (FQAH) states
- 1D Wannier state description of FQAH states
- FQAH states with higher Chern number and the topological nematic states
- Lattice dislocations in topological nematic states as non-Abelian "genons" (generators of genus).
- Non-Abelian statistics of the genons.
- Relation to other twist defects
- Topological field theory description

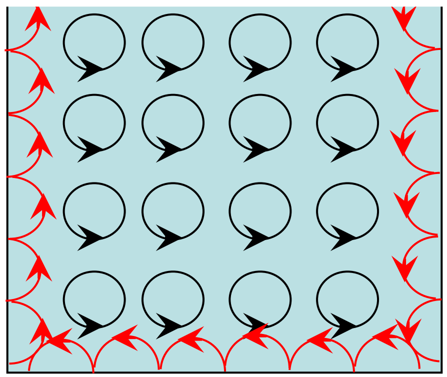
Maissam Barkeshli & XLQ, PRX, arXiv:1112.3311

Maissam Barkeshli, Chaoming Jian, XLQ, in preparation

Integer and fractional quantum Hall states



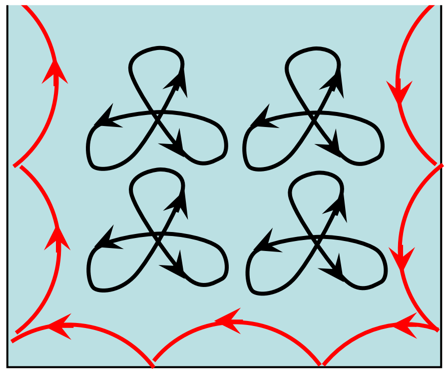
- Quantum Hall effect occurs in 2d electron system with strong perpendicular magnetic field



- Integer quantum Hall (IQH) state: Filling integer number of Landau levels. Momentum space Chern number (Thouless et al 1982)

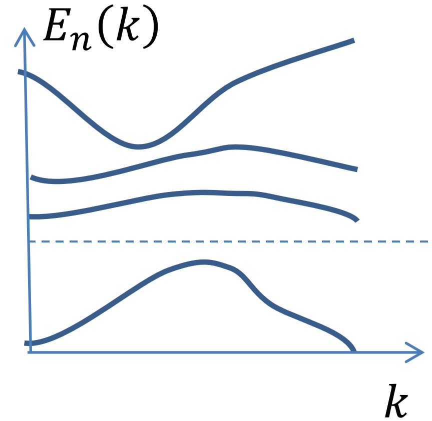
$$n = \frac{1}{2\pi}\int d^2k\left(\partial_x a_y - \partial_y a_x\right)$$

$$a_i(\boldsymbol{k}) = -i\langle\boldsymbol{k}|\partial_i|\boldsymbol{k}\rangle$$

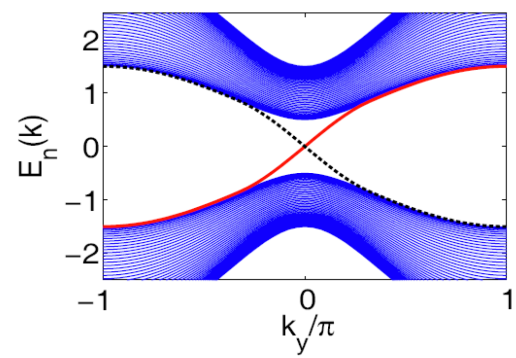


- Fractional quantum Hall state (FQH): Partially filled Landau level. Strongly correlated state with fractionalized excitations

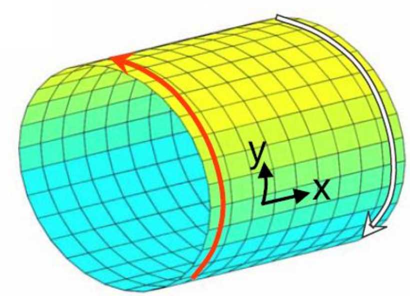
Integer Quantum Anomalous Hall States



- IQH effect can be realized in a lattice model without orbital magnetic field (Haldane 1988)
- A quantized version of the anomalous Hall effect
- General lattice Hamiltonian with translation symmetry $H = \sum_k c_k^+ h(k) c_k$



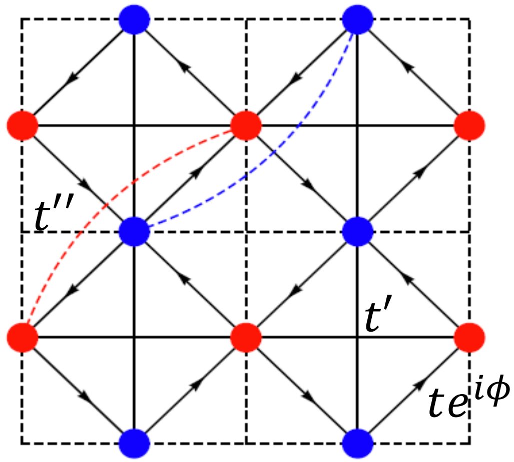
- There are n bands $|n, \boldsymbol{k}\rangle$
  Chern number is defined for each band
- Example: two-band models $H = \sum_a d_a(\boldsymbol{k})\sigma^a$ (Haldane 1988, Qi Wu Zhang 2005)

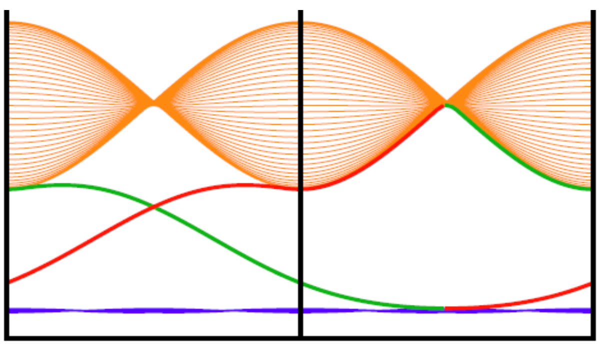


- Mateiral proposals: Hg(Mn)Te/CdTe (Liu et al PRL ‘08), Cr or Fe doped Bi2Se3 film (Yu et al ‘10)

Fractional quantum anomalous Hall (FQAH) states

- Can FQH state also be realized in a lattice system?
- Evidences have been observed (Sun *et al*, Neupert, *et al*, Tang *et al*, PRL 2011) (Sheng *et al* Nat. Comm. 2011, Regnault&Bernevig 1105.4867, Wu, Bernevig&Regnault arXiv:1111.1172)
- Supported by analytic results, e.g. Parameswaran, Roy, Sondhi,'11
- Example: Checkerboard model

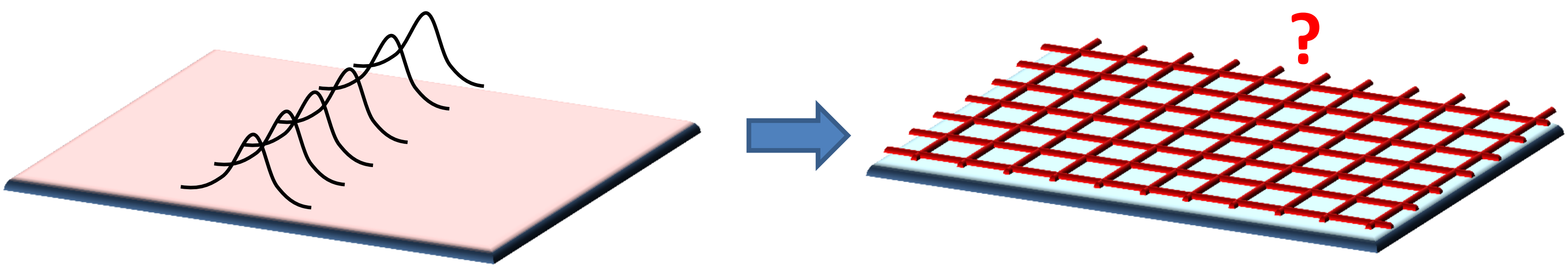




Flat band for $t = 1, t' = \frac{1}{2+\sqrt{2}}, t'' = \frac{1}{2+2\sqrt{2}}, \phi = \frac{\pi}{4}$
(Sun *et al 2011*)

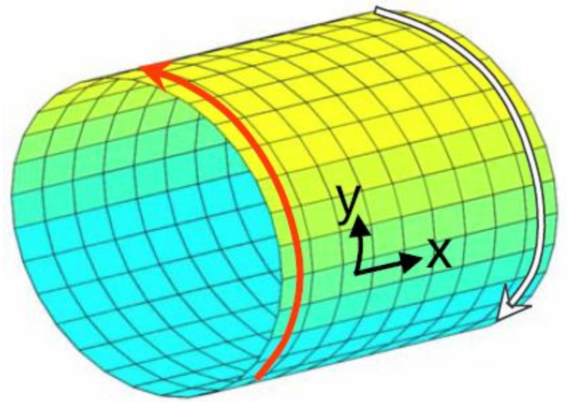
Wave-function description of FQAH states

- FQH states can be described by many-body wavefunctions such as the Laughlin wavefunction (Laughlin 1983)
- $\Psi_{\frac{1}{m}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m \exp(-\sum_i |z_i|^2/2l_B^2)$
- What are the many-body wavefunctions describing FQAH states?
- What FQH states can be realized on the lattice?
- Idea: Finding the single-particle basis corresponding to the Landau level wavefunctions in the ordinary QH states.



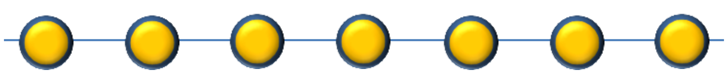
FQH FQAH

Wave-function description of FQAH states: 1D Wannier functions

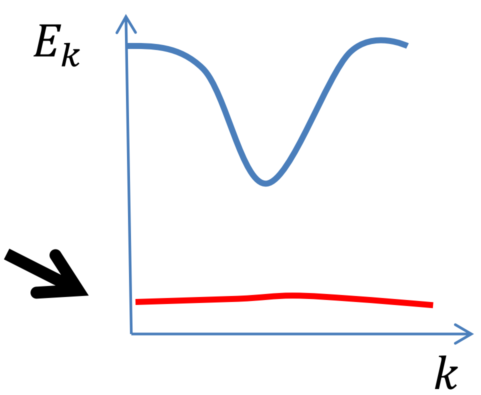


- The proper basis can be found by using 1D Wannier functions
- Consider FQAH state on a cylinder
- The states for each fixed k_y forms a 1D chain.
- 1D Wannier functions:
  a local basis for the 1D system.
  Fourier transform of Bloch states
- $|W_{nk_y}\rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_x n} e^{i\varphi(\boldsymbol{k})} |k_x, k_y\rangle$

k_y eigenstates



$h_{nm}(k_y)$

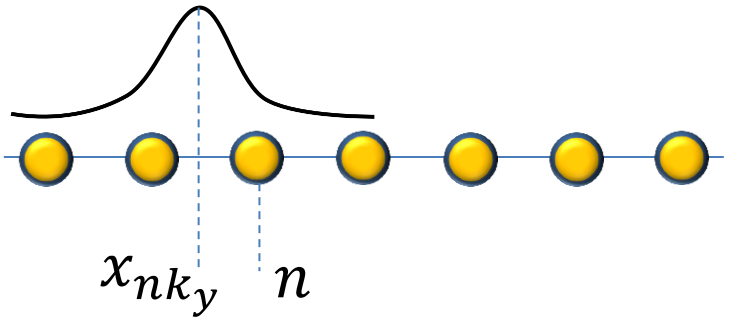


1D Wannier functions

- $|W_{nk_y}\rangle = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{ik_x n} e^{i\varphi(\boldsymbol{k})} |k_x, k_y\rangle$
- The ambiguity of $\varphi(\boldsymbol{k})$ can be fixed by requiring the Wannier functions to be maximally localized, i.e., by minimizing

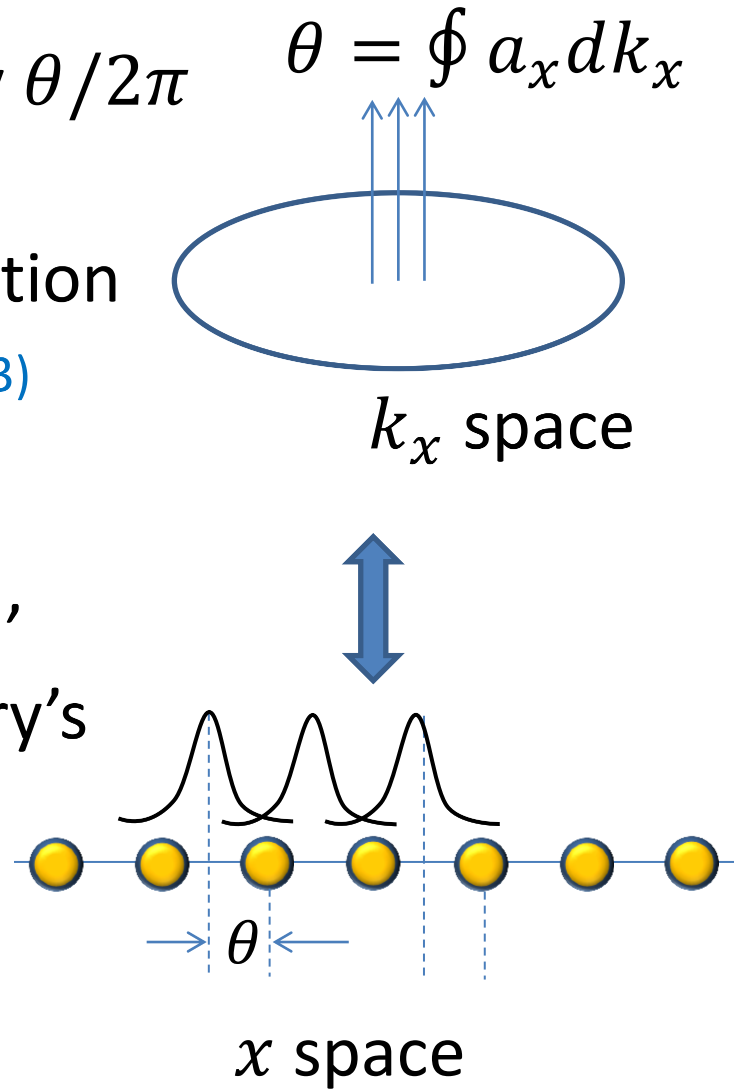
$$(\Delta x)^2 = \langle W_{nk_y}|x^2|W_{nk_y}\rangle - \langle W_{nk_y}|x|W_{nk_y}\rangle^2$$

- In 1D, the maximally localized Wannier function (MLWF) can be obtained by diagonalizing the projected x operator (Kivelson 1982):
- $\hat{x} = P\_xP\_$ with $P\_ = \sum_{\boldsymbol{k}} |\boldsymbol{k}\rangle\langle\boldsymbol{k}|$ projection to the occupied band.
- $\hat{x}|W_{nk_y}\rangle = x_{nk_y}|W_{nk_y}\rangle$
- $x_{nk_y} = n - \theta(k_y)/2\pi$



1D Wannier functions

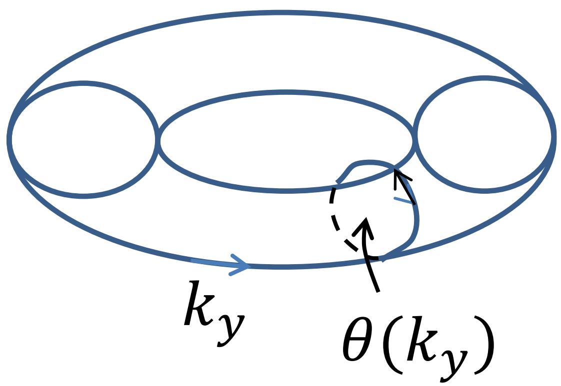
- Wannier functions are shifted by $\theta/2\pi$ with respect to the lattice sites
- Correspondingly, charge polarization $P = -\theta/2\pi$. (King-Smith&Vanderbilt '93)
- Since $x = i\partial/\partial k_x$, the projected position operator $\hat{x} = i\frac{\partial}{\partial k_x} - a_x$, $a_x(\boldsymbol{k}) = -i\langle \boldsymbol{k}|\partial_{k_x}|\boldsymbol{k}\rangle$ is the Berry's phase gauge field
- The shift of eigenvalues of $\hat{x}$ is determined by the flux of a_x



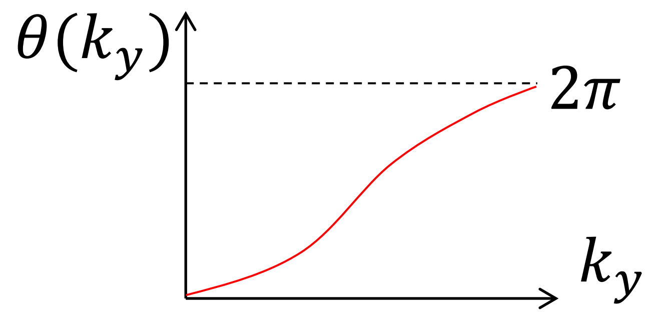
Coh & Vanderbilt, 2009 PRL

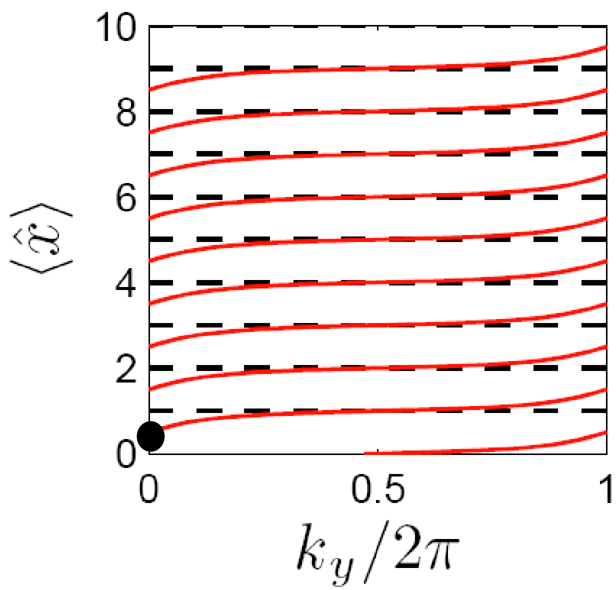
1D Wannier functions and the Chern number

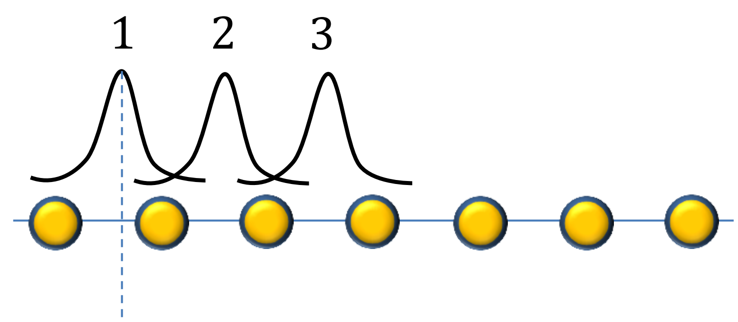
- Chern number on the Brillouin zone torus is the winding number of the flux $\theta(k_y)$



$$\frac{\sigma_H}{h/e^2} = C_1 = -\frac{1}{2\pi}\int_0^{2\pi} \partial_{k_y}\theta(k_y)\, dk_y$$

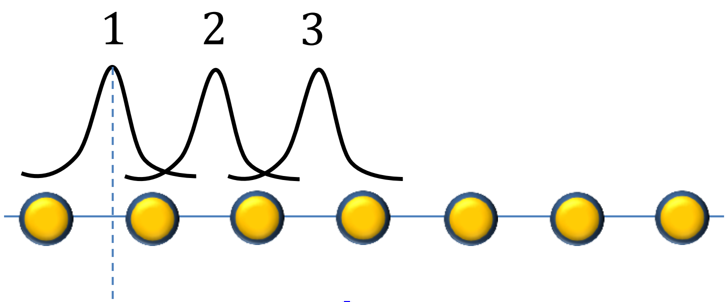


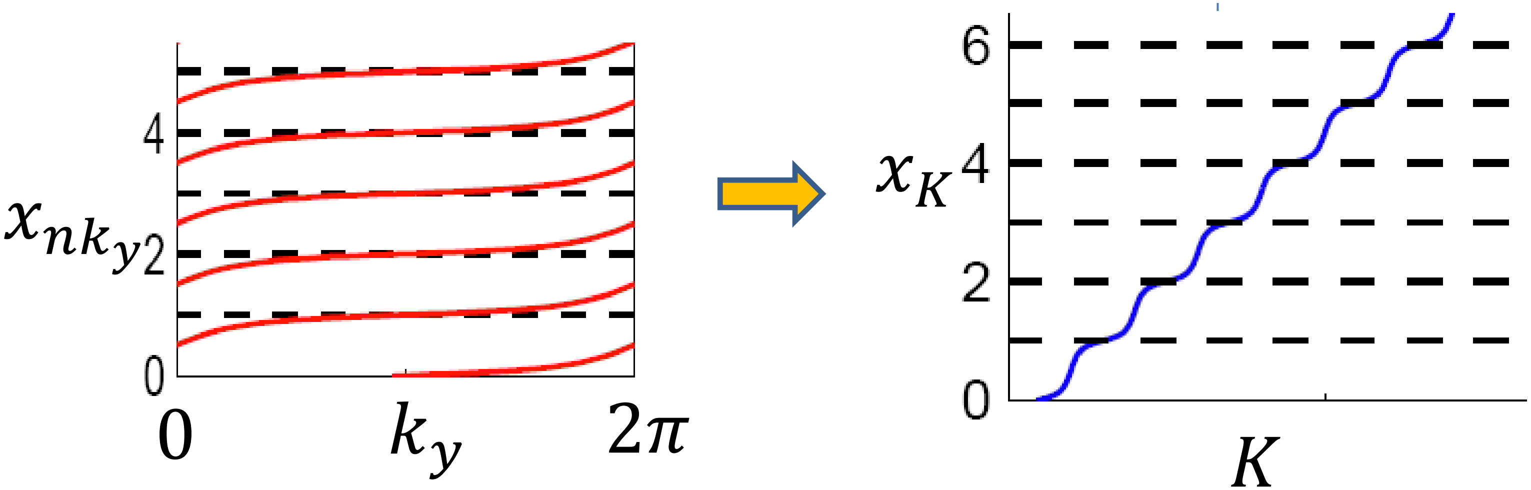




1D Wannier functions in QAH states

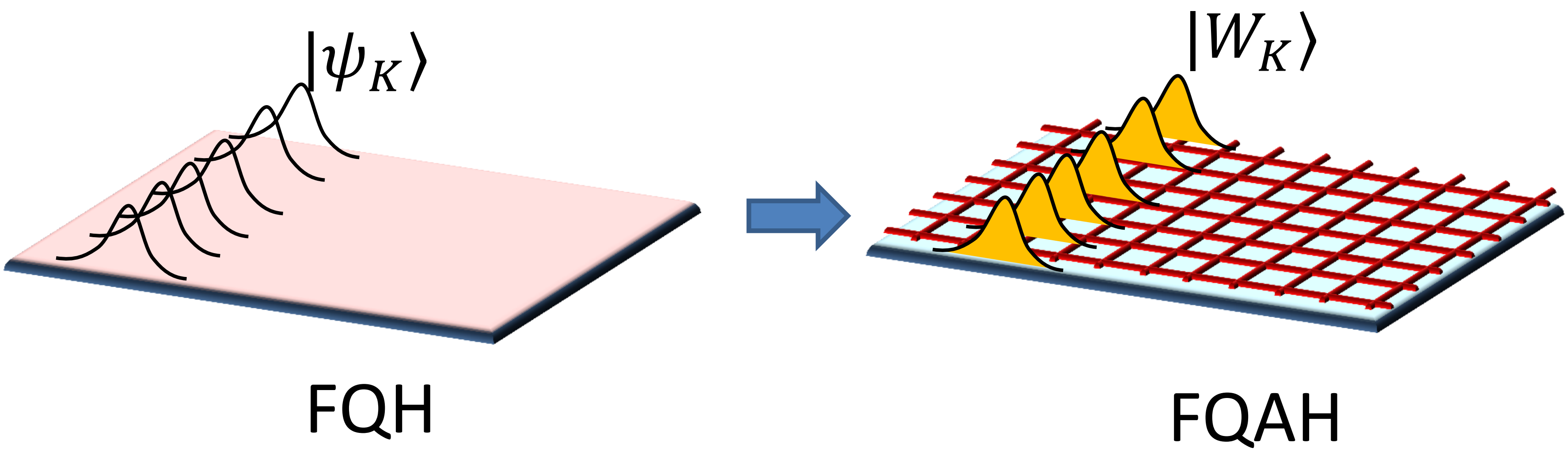
- "Twisted" boundary condition for Wannier functions
- $k_y \to k_y + 2\pi, |W_{nk_y}\rangle \to |W_{n+1,k_y}\rangle$
- A extended momentum K can be defined, only if the Chern number is nontrivial
- $|W_{nk_y}\rangle = |W_K\rangle$





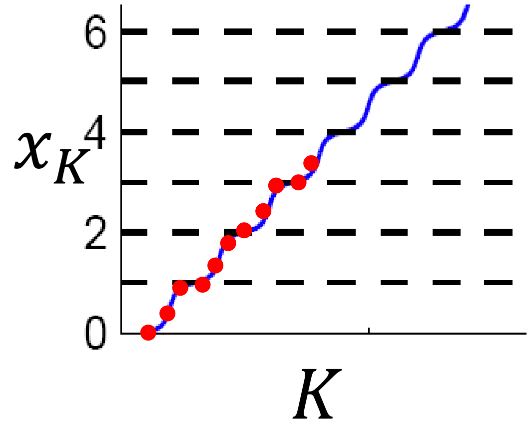
Using 1D Wannier functions to describe FQAH states

- After the redefinition, Wannier functions $|W_K\rangle$ are analog of Landau level wavefunctions
- $\psi_K(x,y) = e^{iky} e^{-\left(x - K l_B^2\right)^2 / 2 l_B^2}$



Using 1D Wannier functions to describe FQAH states

- Using this mapping of basis, every FQH wavefunction is mapped to the lattice FQAH states



$$n_K = 0,1$$

- FQH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1} |\psi_K\rangle$$

- ➔FQAH:

$$|\Psi\rangle = \sum_{\{n_K\}} \Phi(\{n_K\}) \prod_{n_K=1} |W_K\rangle$$

Using 1D Wannier functions to describe FQAH states

- The occupation number wavefunction $\Phi(\{n_K\})$ is known for many FQH states
- For the 1/3 Laughlin state (Rezayi&Haldane 1994 PRB)

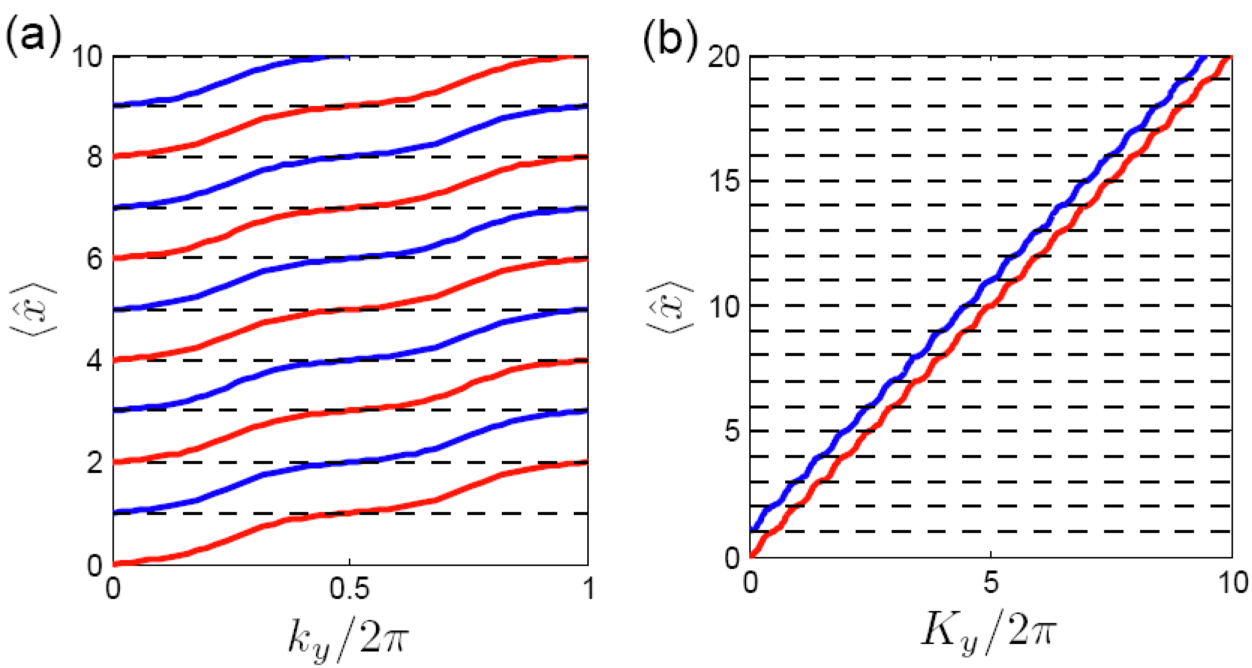
$$|\Psi_{1/3}\rangle = \bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ$$

$$+\ \bullet\circ\bullet\circ\circ\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\quad + \ldots\ldots$$

- A generic construction by Jack polynomials (Bernevig&Haldane 2008 PRL)
- All FQH wavefunctions can be mapped to FQAH states
- Numerical works confirmed the validity of Wannier representation in 1/2 and 1/3 (Wu et al 1207.5767, Scaffidi&Moller 1207.3539)
- Exact lattice Hamiltonians can be constructed by mapping the pseudopotential Hamiltonians (Haldane 1983) to the lattice system. (Lee, Thomale, XLQ, 1207.5587)

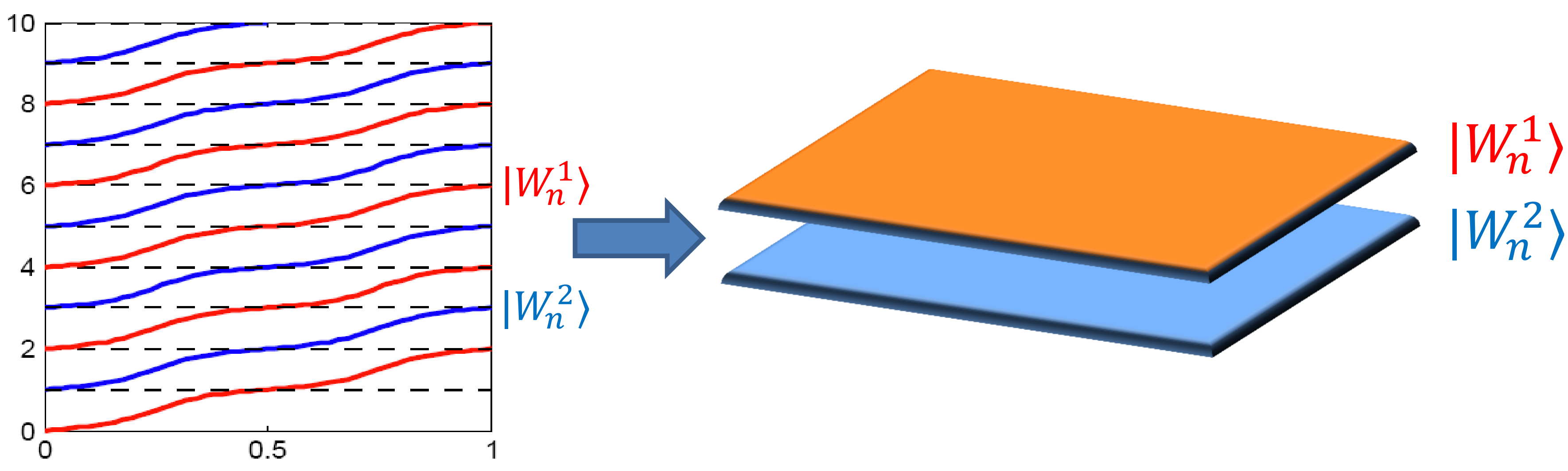
FQAH state with higher Chern number

- Are there new topological states in the FQAH system that are absent in the ordinary FQH?
- The Wannier approach can be generalized to bands with Chern number >1. (For an example of Chern number 2 model, see Wang&Ran PRB'11)
- Higher winding number of the Wannier state position



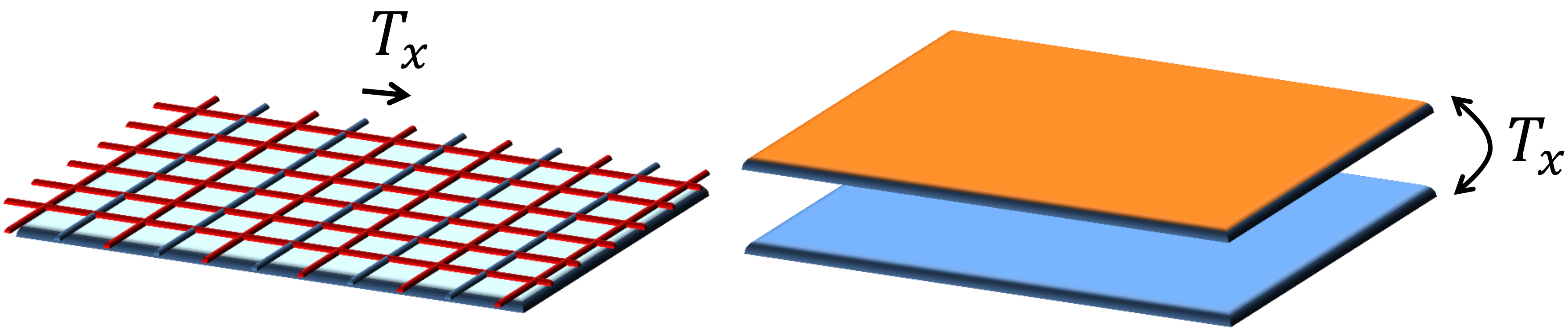
Realizing multi-layer FQH states in one band

- For Chern number $C_1 = 2$, the Wannier states form two groups $|W_n^1\rangle, |W_n^2\rangle$, with each group equivalent to a Landau level
- ➔Double-layer FQH states can be realized in a single band



Nontrivial representation of lattice translation symmetry

- Lattice translations T_x, T_y acts differently on this basis
- $T_x|W_n^1\rangle = |W_n^2\rangle,\ T_x|W_n^1\rangle = |W_n^1\rangle$
- $T_y\left|W_n^{1,2}\right\rangle = e^{in2\pi/L_y}|W_n^{1,2}\rangle$

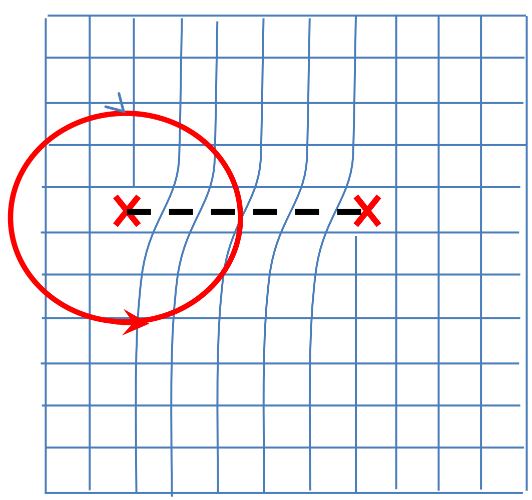


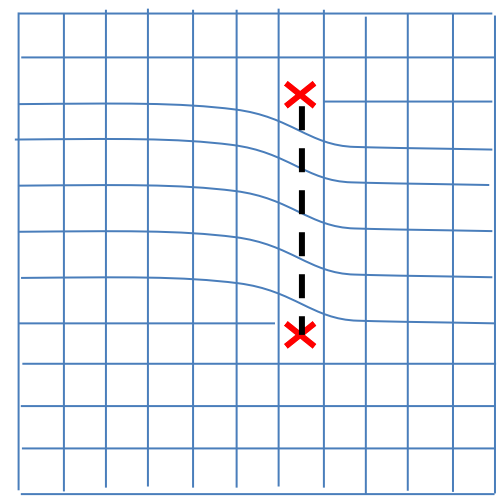
Topological nematic states

- Consider the Halperin (mnl) states (Halperin '83)
- $\Phi(z_i, w_j) = \prod_{i<j}(z_i - z_j)^m (w_i - w_j)^n \prod_{i,j}(z_i - w_j)^l$
- Lattice translation T_x exchanges the two "layers".
- For $m = n$ the state is translation invariant. However, the 4-fold lattice rotation symmetry (for a square lattice) is broken.
- We name such a state as a topological nematic state
- Lattice dislocations in a topological nematic state carry nontrivial topological degeneracy

Dislocations in topological nematic states

- Dislocations are described by the Burgers vector $\vec{b} = (b_x, b_y)$



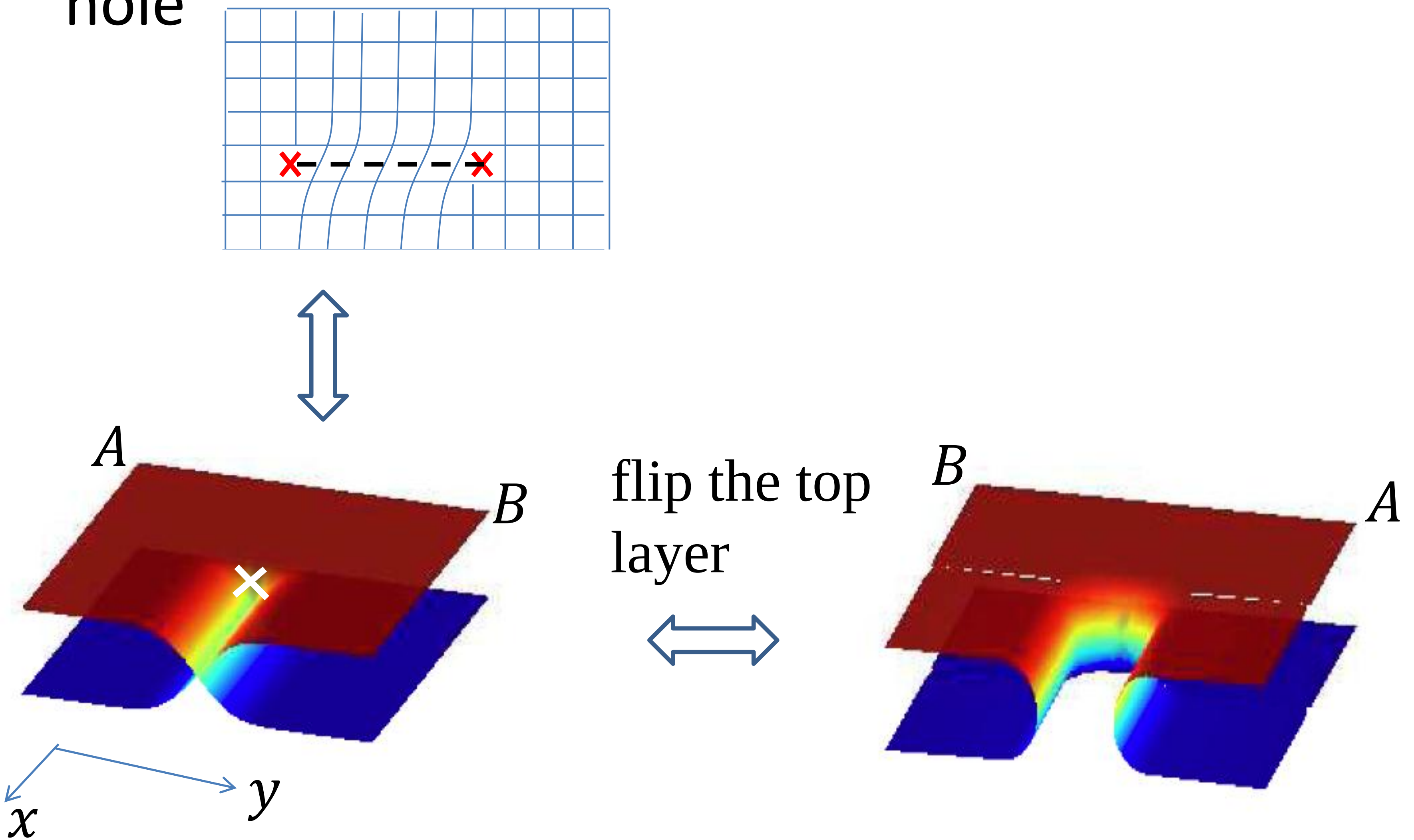


x-dislocation $\vec{b} = \hat{x}$ $\quad$ y-dislocation $\vec{b} = \hat{y}$

- Across the "branch-cut" of the x-dislocation, the two layers are exchanged!

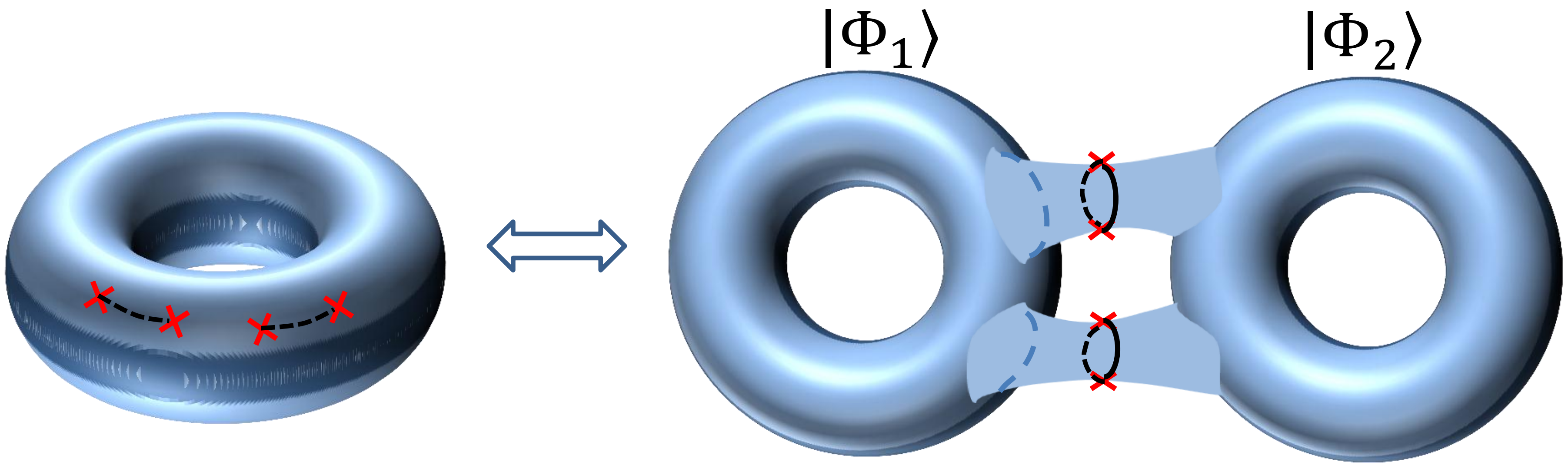
Dislocations in topological nematic states

- A pairs of x-dislocations is equivalent to a “worm-hole”



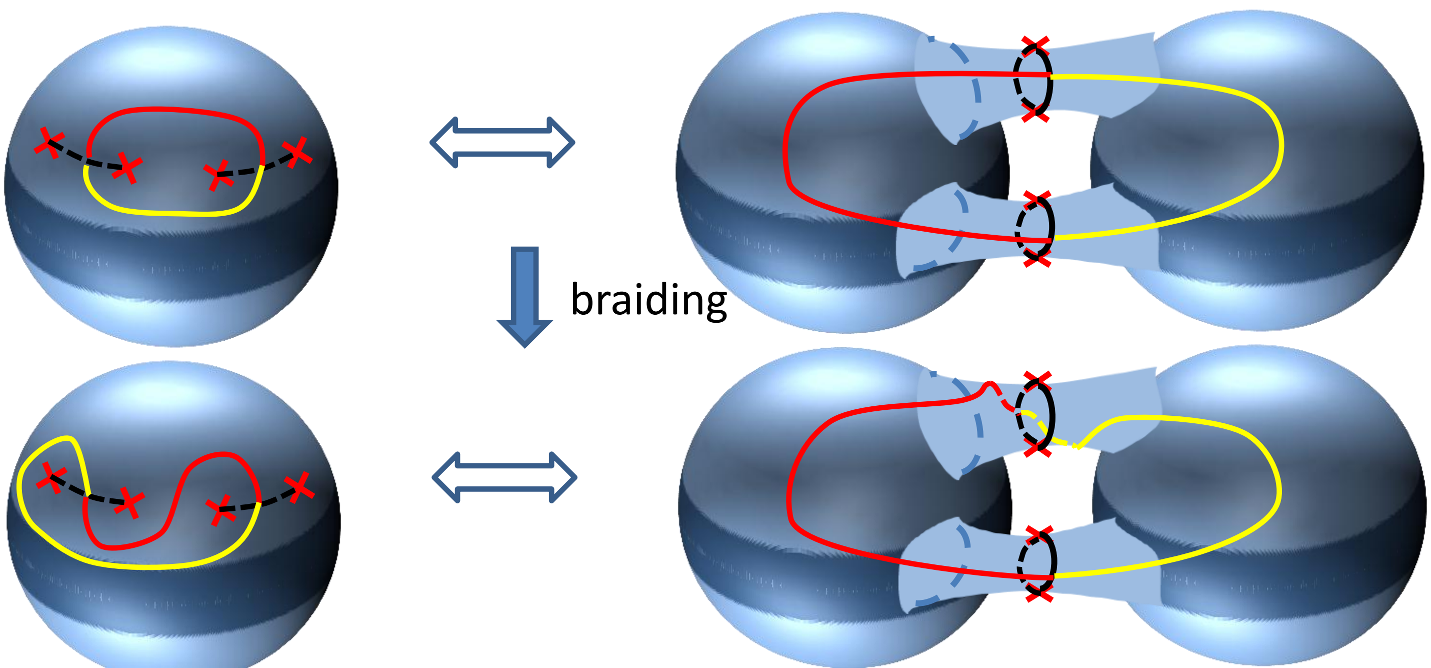
Dislocations in topological nematic states

- Consider a simple case of $(mm0)$ state, which is a direct product of two Laughlin states
- $|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$
- Consider two pairs of dislocations on the torus
- Ground state degeneracy $N = m^3$
- Degeneracy $d = \sqrt{m}$ per dislocation
- Dislocations become "genus generators"---genons



Properties of genons: braiding statistics

- Braiding of two genons corresponds to a “Dehn twist” on the high genus surface, which is a generator of the mapping class group of the surface
- ➔Braiding of the genons is determined by the modular transformation property of the underlying FQH state.
- <u>Braid group $B_{2n}(S^2)$ → Mapping class group MCG_{n-1} → Rep.</u>

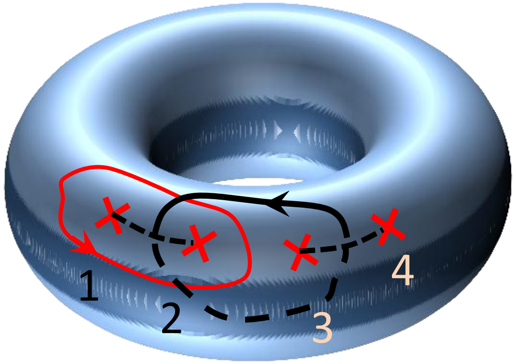


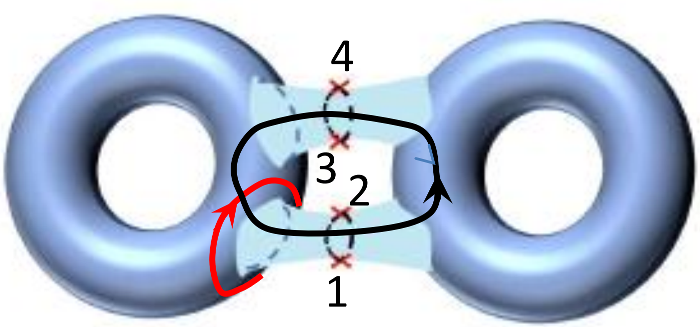
Properties of genons: braiding statistics

- Braiding of two defects corresponds to the Dehn twist along the loop surounding them
- Example: (220) state. 2 ground states for 4 defects. The braiding matrices are

$$U_a = e^{i\theta}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad U_b = e^{i\phi}\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

- Abelian phases are undetermined.
- The non-Abelian statistics is identical to Ising anyon!





Genons in more general nematic states

- For more general (mml) states, the topological degeneracy with n pairs of dislocation is
- $N = |m^2 - l^2| \cdot |m - l|^{n-1}$
- The degeneracy per dislocation, i.e., quantum dimension is $d = \sqrt{|m - l|}$
- This degeneracy can be obtained by studying the Chern-Simons theory with branch-cuts

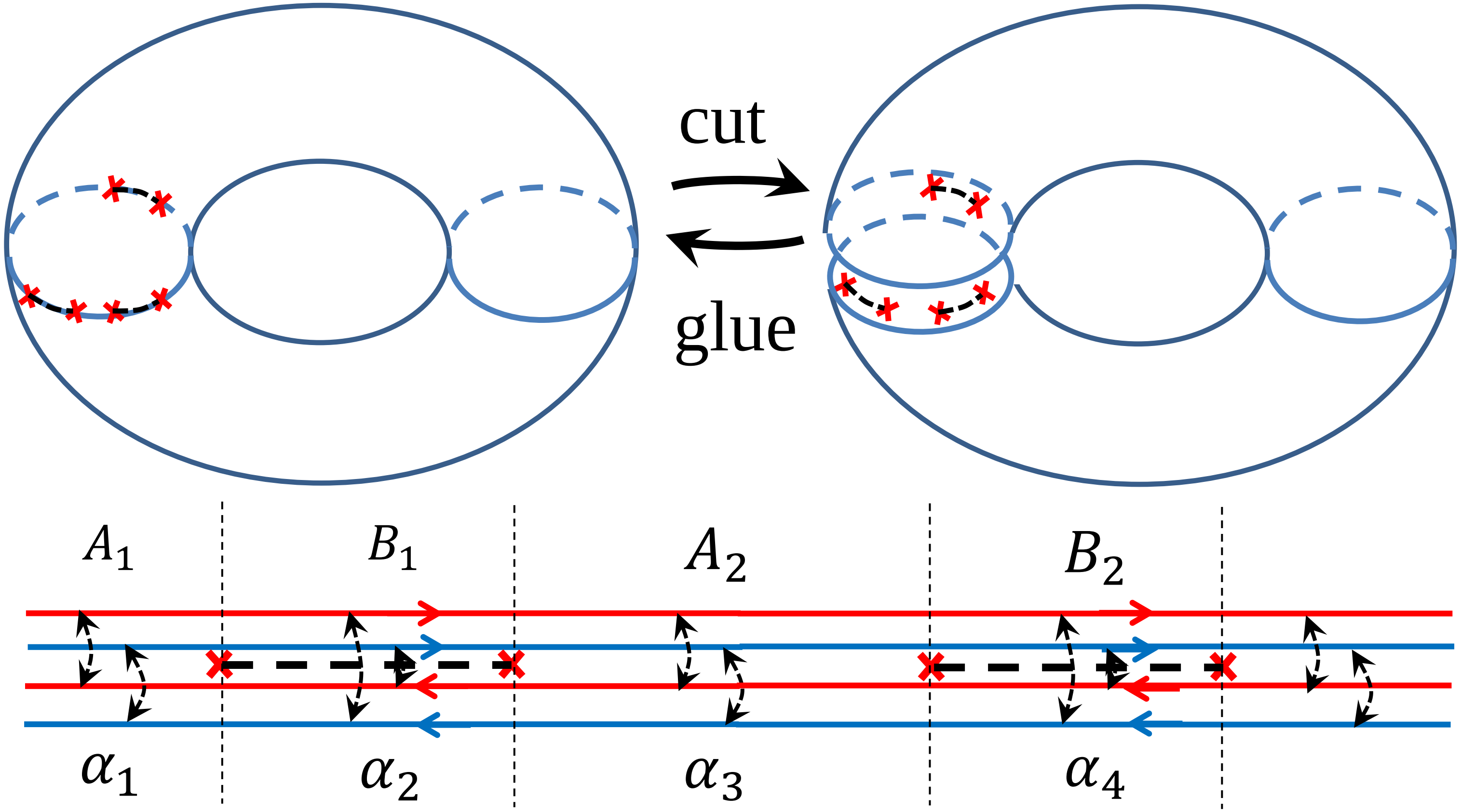
$$\mathcal{L} = \frac{1}{4\pi} a_\mu^I K_{IJ} \partial_\nu a_\tau^J$$

(Barkeshli&Wen 2010)

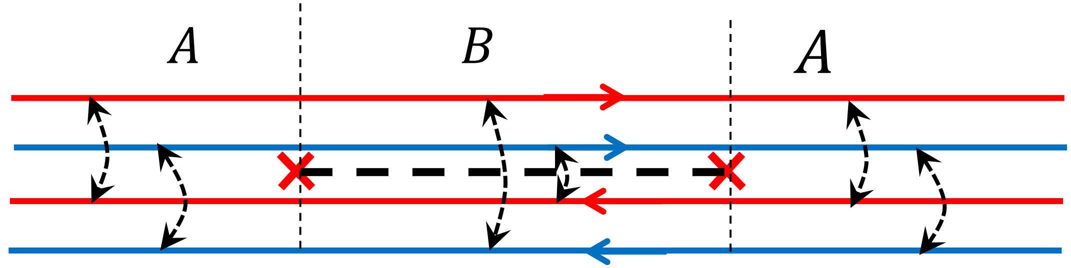
- Alternatively, it can be understood from an edge state picture

Edge state picture of dislocation-induced degeneracy

- Consider the torus as a cylinder glued along the edge
- Inter-edge tunneling exchanges the two layers across the branch-cut



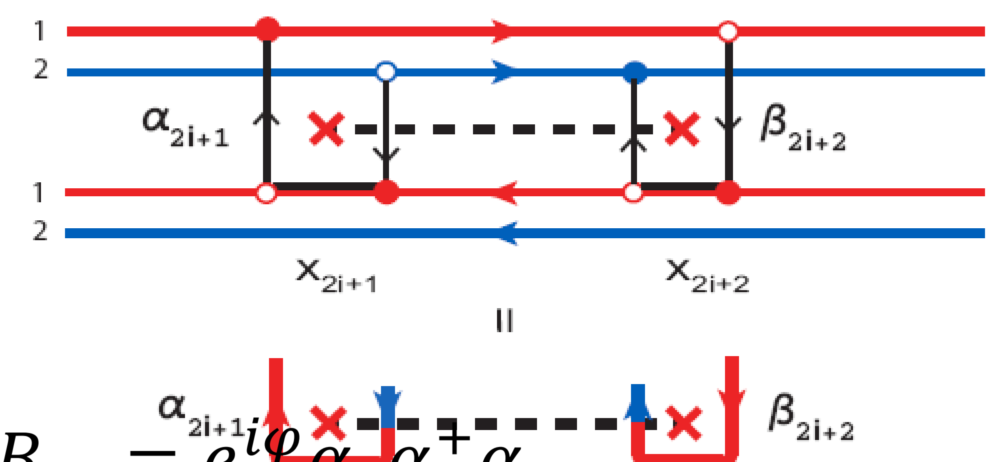
Edge state picture of dislocation-induced degeneracy



- The edge states are described by the chiral Luttinger liquid theory (wen 1990) with the boson fields $\phi_{1L}, \phi_{2L}, \phi_{1R}, \phi_{2R}$.
- Electron operator
  $c_{L,R}^{1} = e^{i(m\phi_{L,R}^{1} + l\phi_{L,R}^{2})}, c_{L,R}^{1} = e^{i(l\phi_{L,R}^{1} + m\phi_{L,R}^{2})},$
- Inter-edge electron tunneling determines the number of degenerate minima $(m^2 - l^2)^n (m + l)^n$
- There are $2n - 1$ independent charge on the $2n$ dislocations, each takes $m + l$ values.
- ➔Topological degeneracy
  $N = \frac{(m^2 - l^2)^n (m+l)^n}{(m+l)^{2n-1}} = (m^2 - l^2)(m - l)^{n-1}$

Edge state picture of genon braiding

- The braiding of genons can also be understood in the edge state picture (similar to Linder et al, Clarke et al '12, see next Pg)
- Zero mode operators can be defined as open path quasi-particle tunneling e.g. $\alpha_{2i-1} = e^{i\phi_1(x_{A_i})} e^{-i\tilde{\phi}_1(x_{B_i})}$,
- $[\alpha_i, H] = [\beta_i, H] = 0$
- Physical operators are quadratic combinations of α_i, β_i.
- The braiding matrix is determined by its action to the zero modes

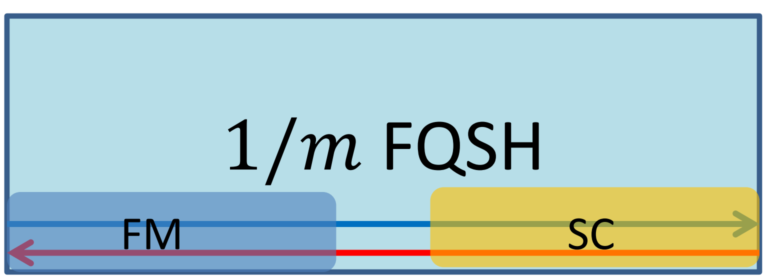


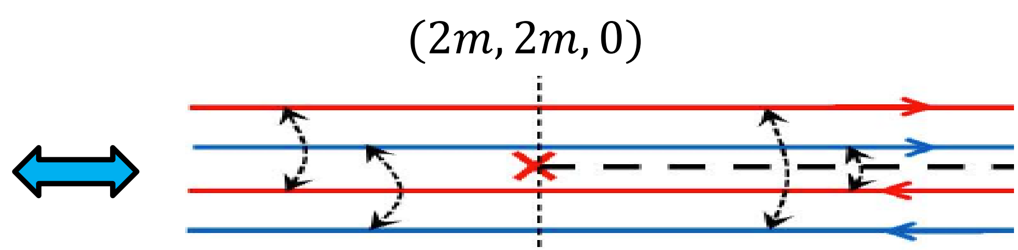
- $B_{12}^{-1}\alpha_1 B_{12} = e^{i\varphi}\alpha_2$

$$B_{12}^{-1}\alpha_2 B_{12} = e^{i\varphi}\alpha_2\alpha_1^{+}\alpha_2$$

Relation between genons and other twist defects

- Genons are related to another kind of extrinsic defect (Linder et al, Clarke et al, Cheng, Vaezi, 2012) at the domain wall between FM and SC regions of the FQSH edge states, or SC and ordinary inter-edge tunneling in FQH edge
- The edge theory for $\frac{1}{m}$ FQH is equivalent to the anti-symmetric sector $\phi_- = \phi_1 - \phi_2$ of the $(2m, 2m, 0)$ topological nematic states, if $\phi_+ = \phi_1 + \phi_2$ is gapped.





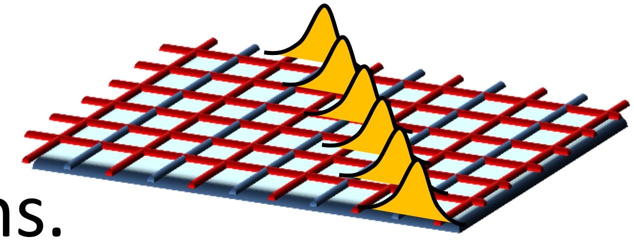
$\cos(m(\phi_L + \phi_R))$

$\cos(m(\phi_L - \phi_R))$

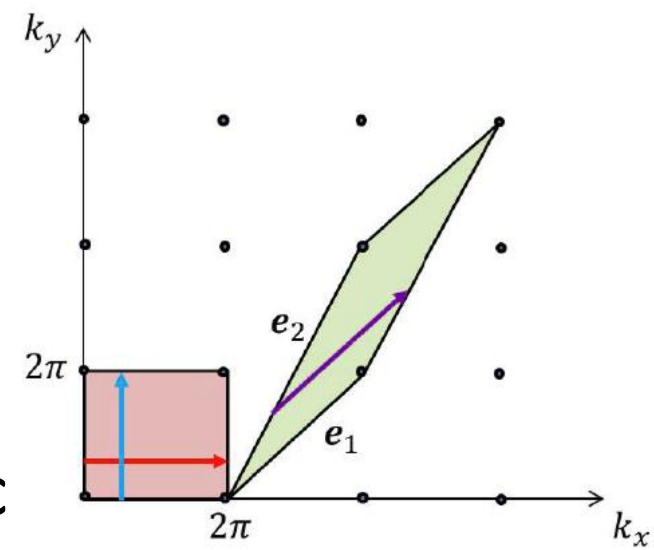
$\cos(m(\phi_{L-} + \phi_{R-})) \cdot \cos(m(\phi_{L+} + \phi_{R+}))$

$\cos(m(\phi_{L-} - \phi_{R-})) \cdot \cos(m(\phi_{L+} + \phi_{R+}))$

Classification of topological nematic states



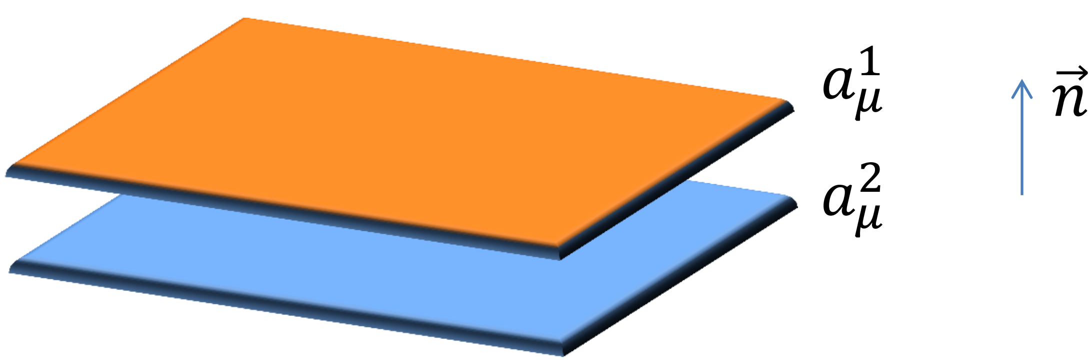
- The topological nematic states discussed above are sensitive to x-dislocations but not y-dislocations.
- Generically, 1D Wannier functions can be defined along any reciprocal lattice direction $\vec{K} = 2\pi(n_x, n_y)$



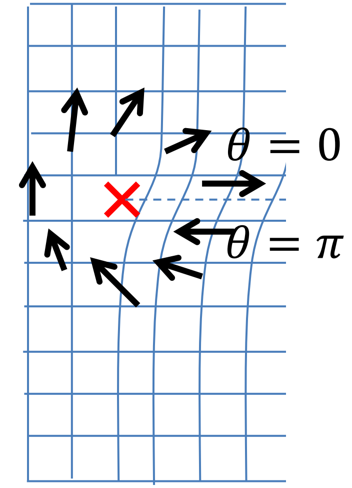
- The corresponding topological nematic states is sensitive to dislocations with burgers vector $\vec{b} \cdot \frac{\vec{K}}{2\pi}$ odd.
- ➔ 3 types of topological nematic states $(0,1), (1,0), (1,1)$
- The ordinary Halperin state can be viewed as a trivial class $(0,0)$

A topological field theory description

- Without dislocations, the effective theory is an Abelian $U(1) \times U(1)$ Chern-Simons theory (Wen&Zee'92) $\mathcal{L} = \frac{1}{4\pi} a_\mu^I K_{IJ} \partial_\nu a_\tau^J$
- Around a dislocation, a_μ^1 and a_μ^2 are exchanged
- To describe this effect we introduce a $U(2)$ gauge field A_μ and a Higgs field $H = \sigma \cdot \vec{n} e^{i\theta}$ which breaks $U(2) \to U(1) \times U(1)$. $\vec{n}$ should change sign when a reference point is moving around a dislocation.



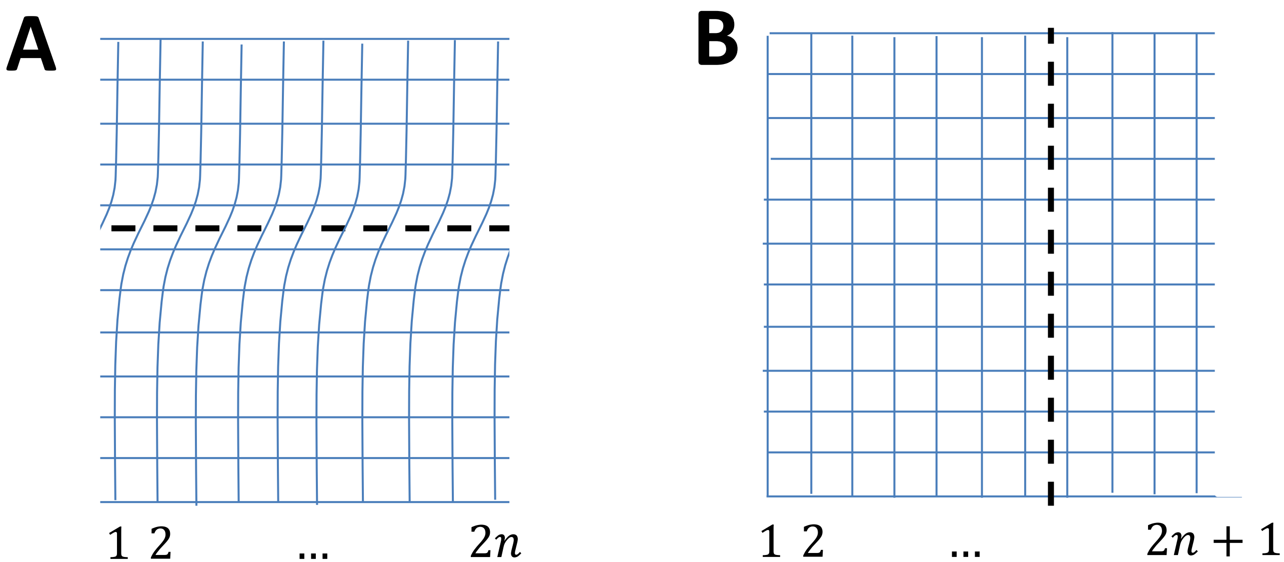
A topological field theory description



- We propose the Lagrangian
- $$\mathcal{L} = \frac{m-l}{4\pi}\epsilon^{\mu\nu\tau} tr\left[A_\mu \partial_\nu A_\tau + \frac{2}{3} A_\mu A_\nu\ A_\tau\right] + \frac{l}{4\pi}\epsilon^{\mu\nu\tau} tr\left[A_\mu\right]\partial_\nu tr[A_\tau] + J tr[D_\mu H^\dagger D_\mu H]$$
- $\theta = \vec{u} \cdot \vec{K}/2$ with $\vec{u}$ the lattice displacement field, and $\vec{K} = 2\pi\left(n_x, n_y\right)$ the type of the topological nematic state.
- Around a dislocation $\vec{u}$ changes by the Burgers vector $\vec{b}$. Thus if $\vec{b} \cdot \vec{K}/2\pi$ is odd, the dislocation corresponds to a half vortex of θ which must be associated with a half vortex of $\vec{n}$.
- ➔The two $U(1)$ Chern-Simons fields are exchanged around the dislocation

Numerical probe of topological nematic states

- For a (1,0) topological nematic state on a torus with hopping shifted like in A, or odd number of sites in x direction, the ground state degeneracy is reduced from $|m^2 - l^2|$ to $|m + l|$.
- This can be used as a numerical probe to topological nematic states in exact diagonalization.

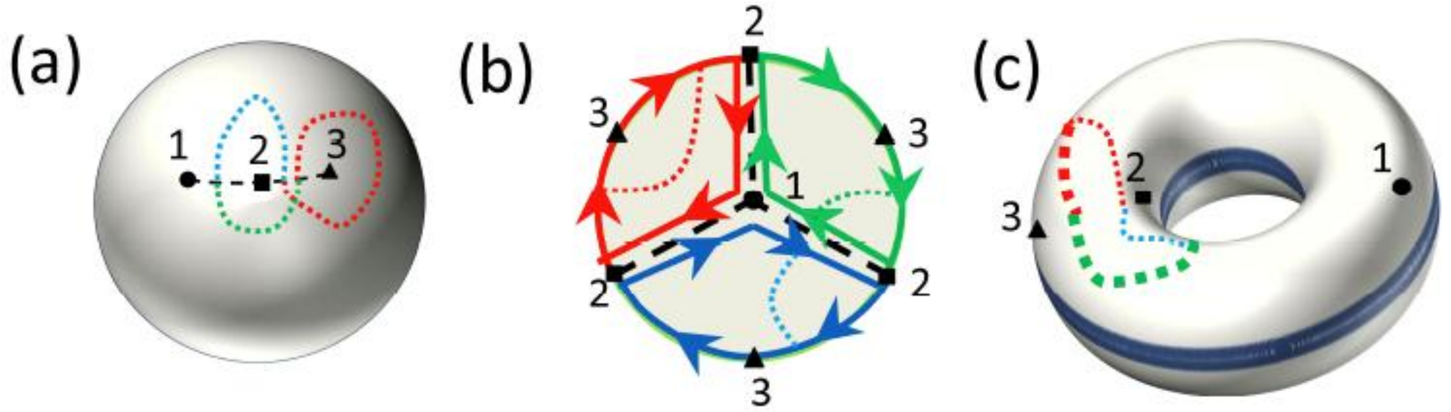


More general discussion on genons and twist defects

- Genons can be defined in any bilayer topological state $G \times G$ where G is an Abelian or non-Abelian state.
- In general the quantum dimension of genon is $d = D_G = \sqrt{\sum_i d_i^2}$ with D_G the total quantum dimension of the single-layer theory
- If we take G to be Ising TQFT, the genon braiding realize Dehn twists in Ising TQFT which leads to universal topological quantum computation (Bravyi&Kitaev '00, Freedman et al '06)

More general discussion on genons and twist defects

- In Chern number N topological nematic states, translation acts as a Z_N operation to the N effective layers. Dislocations become Z_N genons.
- For Z_3 case, in an Abelian FQH state with K matrix $\mathrm{K} = \begin{pmatrix} m & l & l \\ l & m & l \\ l & l & m \end{pmatrix}$, the quantum dimension of genon is $d = |m - l|$. The braiding is also equivalent to some Dehn twists on high genus surfaces.



More general discussion on genons and twist defects

- In general, twist defect can be defined whenever a topological state has a global symmetry
- Many different twist defects can be mapped to genons

Examples of Symmetries of Topological States

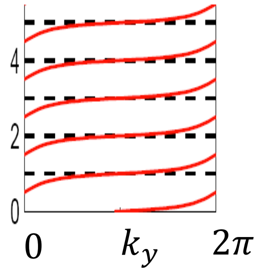
Topological states	Symmetries	Transformation of quasiparticles
Z_N states	layer permutation Z_2	$(a,b) \to (b,a)$
	particle-hole symmetry Z_2	$(a,b) \to (N-a, N-b)$
N-layer FQH states	layer permutation S_N	$(a_1, a_2, .., a_N) \to (a_{P_1}, a_{P_2}, ..., a_{P_N})$
$1/k$-Laughlin FQH state	particle-hole symmetry Z_2	$a \to (k-a)$.

Toric code and generalization (Bombin, You&Wen)

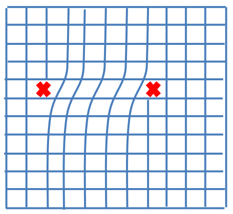
topological nematic states, or real multi-layer FQH (Barkeshli&Qi)

FM/SC domain wall (Linder et al, Clarke et al, Cheng, Vaezi, 2012)

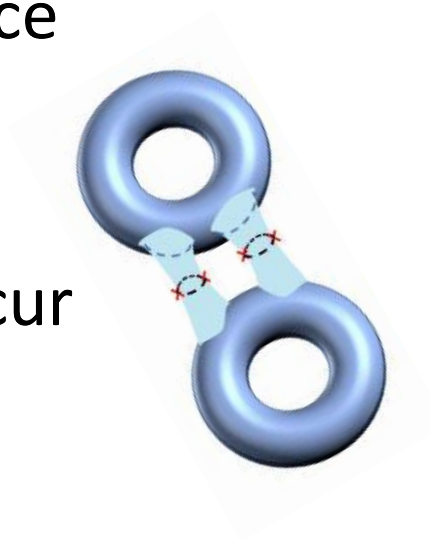
Summary and discussions



- 1D Wannier functions provide the proper basis for characterizing fractional quantum anomalous Hall states.



- Besides known FQH states, the lattice models realize new states such as topological nematic states, with nontrivial interplay between lattice translation symmetry and topology.



- Even in an Abelian topological state, non-Abelian genons or other twist defects can occur
- New possibility of topological quantum computation.

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Thanks!