Integrable pairing models in mesoscopic physics

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Brief History

- Cooper pair and BCS Theory (1956-57)
- Richardson exact solution (1963).
- Gaudin magnet (1976).
- Proof of Integrability. CRS (1997).
- Recovery of the exact solution in applications to ultrasmall grains (2000).
- SU(2) Richardson-Gaudin models (2001). Rational and Hyperbolic families.
- Applications of rational RG model to superconducting grains, atomic nuclei, cold atoms, quantum dots, etc…
- Generalized RG Models for r>1 (2006-2009). SO(6) Color pairing . SO(5) T=1 and SO(8) T=0,1 p-n pairing model and spin 3/2 cold atoms.
- Realization of the hyperbolic family in terms of a p-wave integrable pairing Hamiltonian (2010). Applications to nuclear structure (2011).

The Cooper Problem

PHYSICAL REVIEW VOLUME 104, NUMBER 4 NOVEMBER 15, 1956

Letters to the Editor

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Bound Electron Pairs in a Degenerate Fermi Gas*

Leon N. Cooper
Physics Department, University of Illinois, Urbana, Illinois
(Received September 21, 1956)

IT has been proposed that a metal would display superconducting properties at low temperatures if the one-electron energy spectrum had a volume inde-

$=(1/V)\exp[i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)]$ which satisfy periodic boundary conditions in a box of volume V, and where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of electron one and electron two. (One can use antisymmetric functions and obtain essentially the same results, but alternatively we can choose the electrons of opposite spin.) Defining relative and center-of-mass coordinates, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, $\mathbf{r}=(\mathbf{r}_2-\mathbf{r}_1)$, $\mathbf{K}=(\mathbf{k}_1+\mathbf{k}_2)$ and $\mathbf{k}=\frac{1}{2}(\mathbf{k}_2-\mathbf{k}_1)$, and letting $\mathcal{E}_K+\epsilon_k=(\hbar^2/m)(\frac{1}{4}K^2+k^2)$, the Schrödinger equation can be written

$$(\mathcal{E}_K+\epsilon_k-E)a_{\mathbf{k}}+\sum_{\mathbf{k}'}a_{\mathbf{k}'}(\mathbf{k}|H_1|\mathbf{k}')\times\delta(\mathbf{K}-\mathbf{K}')/\delta(0)=0 \quad (1)$$

where

$$\Psi(\mathbf{R},\mathbf{r})=(1/\sqrt{V})e^{i\mathbf{K}\cdot\mathbf{R}}\chi(\mathbf{r},K),\quad \chi(\mathbf{r},K)=\sum_{\mathbf{k}}(a_{\mathbf{k}}/\sqrt{V})e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (2)$$

and

$$(\mathbf{k}|H_1|\mathbf{k}')=\left(\frac{1}{V}\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}H_1e^{i\mathbf{k}'\cdot\mathbf{r}}\right)_{0\text{ phonons}}.$$

<u>Problem</u> : A pair of electrons with an attractive interaction on top of an inert Fermi sea.

$$|\phi\rangle=\sum_{k>k_F}\frac{1}{2\varepsilon_k-E}c_{k\uparrow}^{+}c_{-k\downarrow}^{+}|FS\rangle,\quad \frac{1}{G}=\sum_{k>k_F}\frac{1}{2\varepsilon_k-E}$$

“Bound” pair for arbitrary small attractive interaction. The FS is unstable against the formation of these pairs

If the many-body system could be considered (at least to a lowest approximation) a collection of pairs of this kind above a Fermi sea, we would have (whether or not the pairs had significant Bose properties) a model similar to that proposed by Bardeen which would display many of the equilibrium properties of the superconducting state.

Bardeen-Cooper-Schrieffer

PHYSICAL REVIEW VOLUME 108, NUMBER 5 DECEMBER 1, 1957

Theory of Superconductivity*

J. BARDEEN, L. N. COOPER,† AND J. R. SCHRIEFFER‡
Department of Physics, University of Illinois, Urbana, Illinois

(Received July 8, 1957)

A theory of superconductivity is presented, based on the fact that the interaction between electrons resulting from virtual exchange of phonons is attractive when the energy difference between the electrons states involved is less than the phonon energy, $\hbar\omega$. It is favorable to form a superconducting phase when this attractive interaction dominates the repulsive screened Coulomb interaction. The normal phase is described by the Bloch individual-particle model. The ground state of a superconductor, formed from a linear combination of normal state configurations in which electrons are virtually excited in pairs of opposite spin and momentum, is lower in energy than the normal state by amount proportional to an average $(\hbar\omega)^2$, consistent with the isotope effect. A mutually orthogonal set of excited states in one-to-one correspondence with those of the normal phase is obtained by specifying occupation of certain Bloch states and by using the rest to form a linear combination of virtual pair configurations. The theory yields a second-order phase transition and a Meissner effect in the form suggested by Pippard. Calculated values of specific heats and penetration depths and their temperature variation are in good agreement with experiment. There is an energy gap for individual-particle excitations which decreases from about $3.5kT_c$ at $T=0°$K to zero at T_c. Tables of matrix elements of single-particle operators between the excited-state superconducting wave functions, useful for perturbation expansions and calculations of transition probabilities, are given.

$$|\Psi\rangle \equiv e^{\Gamma^+}|0\rangle, \quad \Gamma^+ = \sum_k \frac{v_k}{u_k} c^+_{k\uparrow} c^+_{-k\downarrow}$$

Richardson's Exact Solution

ıme 3, number 6 PHYSICS LETTERS 1 February 1963

A RESTRICTED CLASS OF EXACT EIGENSTATES OF THE PAIRING-FORCE HAMILTONIAN *

R. W. RICHARDSON
H. M. Randall Laboratory of Physics,
University of Michigan, Ann Arbor, Michigan

Received 23 November 1962

Exact Solution of the BCS Model

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

Eigenvalue equation:

$$H_P \left|\Psi\right\rangle = E \left|\Psi\right\rangle$$

Ansatz for the eigenstates (generalized Cooper ansatz)

$$\left|\Psi\right\rangle = \prod_{\alpha=1}^{M} \Gamma^{\dagger}_{\alpha} \left|0\right\rangle, \quad \Gamma^{\dagger}_{\alpha} = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c^{+}_{k\uparrow} c^{+}_{-k\downarrow}$$

Richardson equations

$$1+g\sum_{k=0}\frac{1}{2\varepsilon_k - E_\alpha}+2g\sum_{\beta(\neq\alpha)=1}^{M}\frac{1}{E_\alpha - E_\beta}=0, \qquad E=\sum_{\alpha=1}^{M}E_\alpha$$

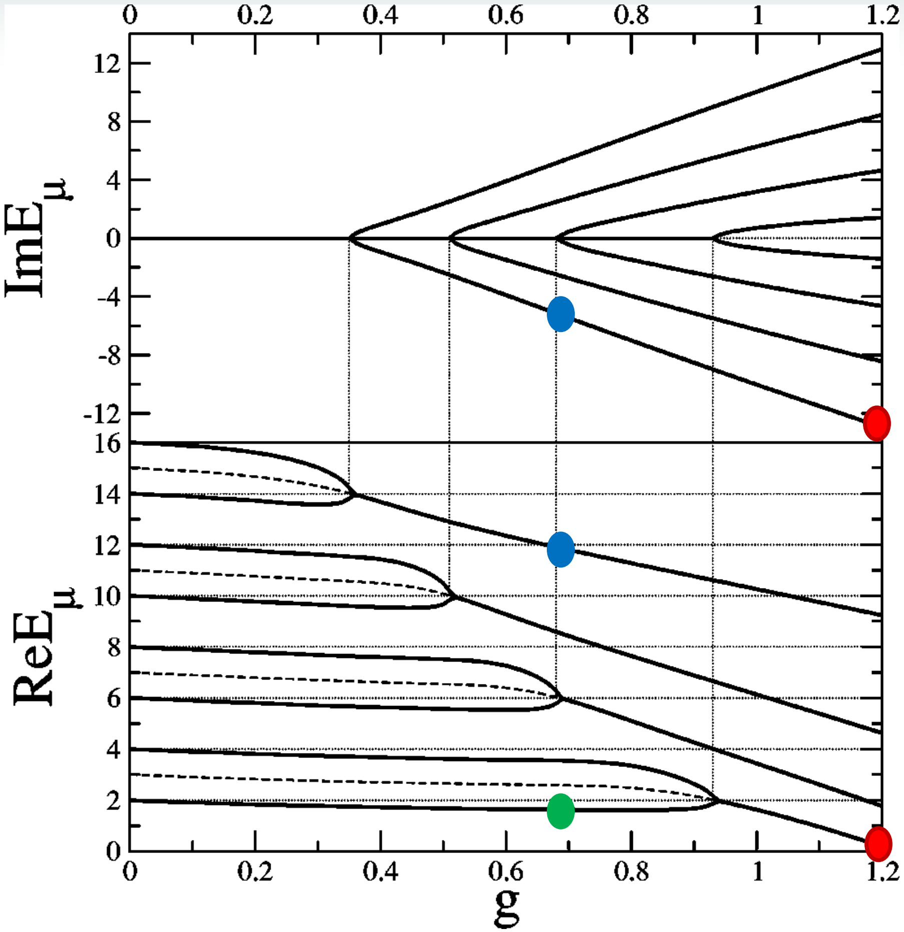
Properties:

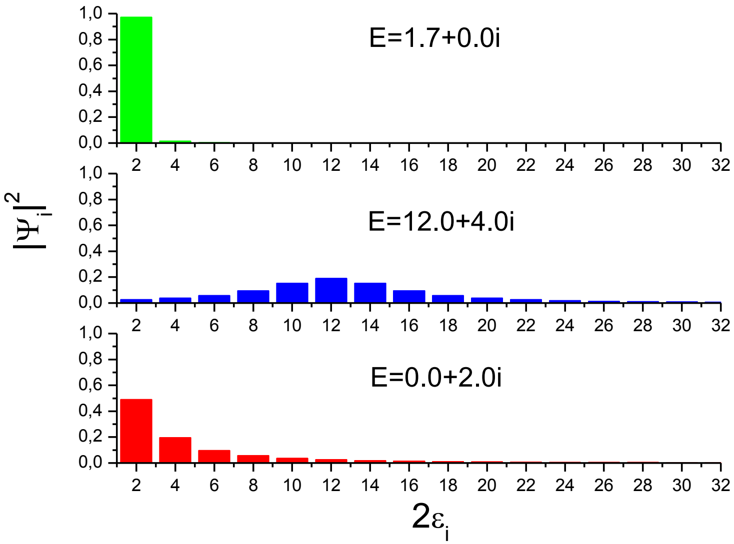
This is a set of *M* nonlinear coupled equations with *M* unknowns (E_α).

The pair energies are either real or complex conjugated pairs.

There are as many independent solutions as the dimension of the Hilbert space. The solutions can be classified in the weak coupling limit ($g\rightarrow 0$).

Exact solvability reduces an exponential complexity of the many-body problem to an algebraic problem.



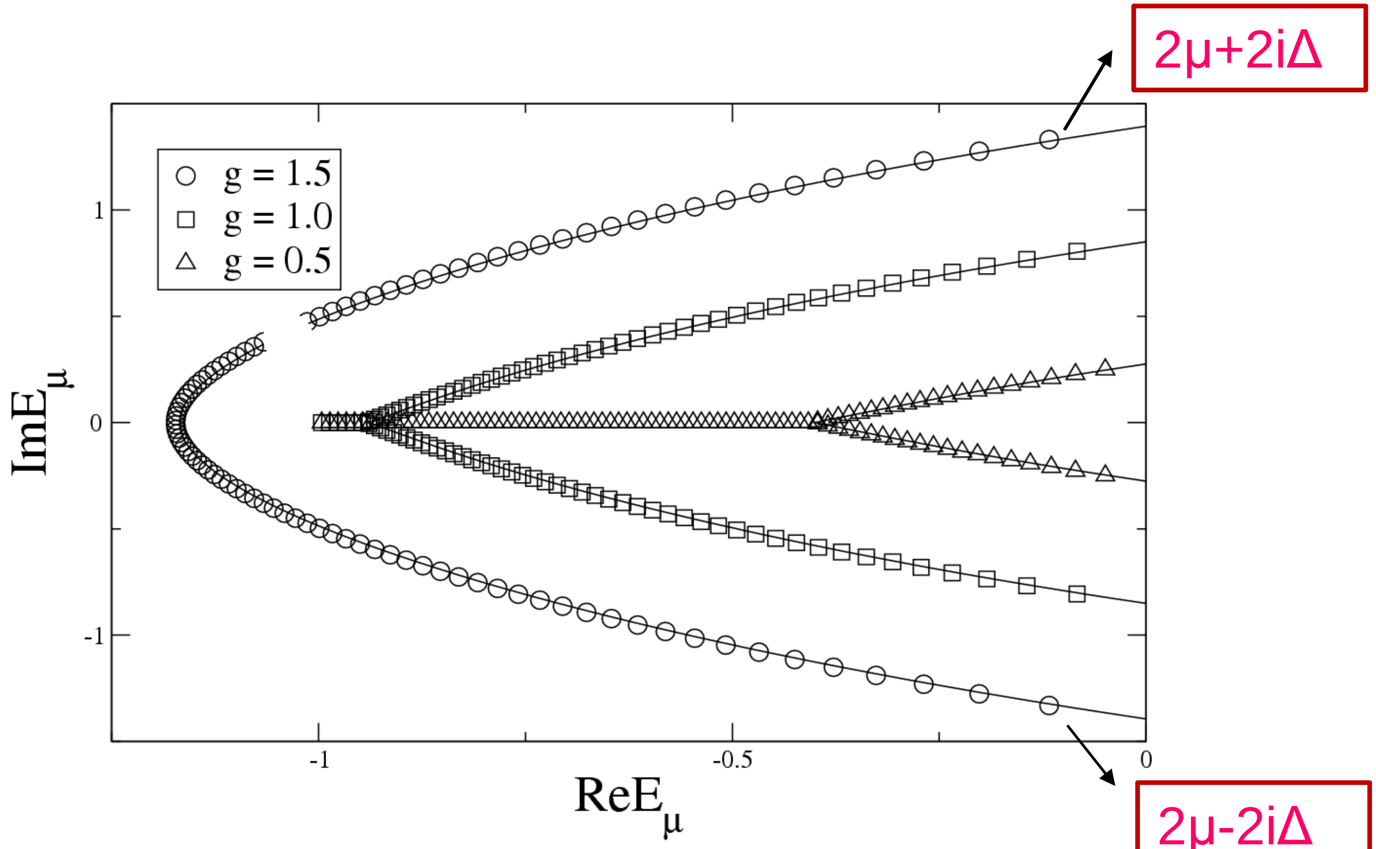


Evolution of the real and imaginary part of the pair energies with *g.*
L=16, M=8

Pair energies E for a system of 200 equidistant levels at half filling.

Hilbert space dimension: 9.06×10^{58} . Exact solution 100 non-linear coupled equation.

Solid lines represent the continuous limit of the Richardson equations.



Integrals of motion of the Richardson-Gaudin Models

L. Amico, A. Di Lorenzo, and A. Osterloh , Phys. Rev. Lett. 86, 5759 (2001).
J. D., C. Esebbag and P. Schuck, Phys. Rev. Lett. 87, 066403 (2001).

- Pair realization of the SU(2) algebra

$$S_j^z = \frac{1}{2}\sum_m a_{jm}^+ a_{jm} - \frac{\Omega_j}{4}\ ,\quad S_j^+ = \frac{1}{2}\sum_m a_{jm}^+ a^+_{j\bar{m}}$$

- The most general combination of linear and quadratic generators, with the restriction of being hermitian and number conserving, is

$$R_i = S_i^z + 2g\sum_{j(\neq i)}\left[\frac{X_{ij}}{2}\left(S_i^+S_j^- + S_i^-S_j^+\right) + Z_{ij}S_i^zS_j^z\right]$$

- The integrability condition $\left[R_i, R_j\right] = 0$ leads to

$$Z_{ij}X_{jk} + X_{jk}Z_{ki} + X_{ki}X_{ij} = 0$$

•These are the same conditions encountered by Gaudin (J. de Phys. 37 (1976) 1087) in a spin model known as the Gaudin magnet.

Gaudin (1976) found three solutions

XXX (Rational)

$$X_{ij} = Z_{ij} = \frac{1}{\eta_i - \eta_j}$$

XXZ (Hyperbolic)

$$X_{ij} = \frac{1}{Sinh\left(x_i - x_j\right)} = 2\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Z_{ij} = Coth\left(x_i - x_j\right) = \frac{\eta_i + \eta_j}{\eta_i - \eta_j}$$

Exact solution

$$R_i \left|\Psi\right\rangle = r_i \left|\Psi\right\rangle$$

Eigenstates of the Rational Model : Richardson Ansatz

$$\left|\Psi_{XXX}\right\rangle = \prod_\alpha \left(\sum_i \frac{1}{\eta_i - E_\alpha} S_i^+ \right) \left|0\right\rangle, \quad \left|\Psi_{XXZ}\right\rangle = \prod_\alpha \left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+ \right) \left|0\right\rangle$$

- Any function of the *R* operators defines an exactly solvable Hamiltonian..

- Within the pair representation two body Hamiltonians can be obtained by a linear combination of *R* operators:

$$H = \sum_{l=1}^{L} \varepsilon_l R_l \left(\eta, g\right) + C$$

- The parameters *g*, η´*s* and ε´*s* are arbitrary. There are 2 *L*+1 free parameters to define an integrable Hamiltonian in each of the families. (*L* number of single particle levels)

- The BCS Hamiltonian solved by Richardson can be obtained from the *XXX* family by choosing $\eta = \varepsilon$.

$$H_{BCS} = \sum_{i} 2\varepsilon_{\ddot{i}} S_i^z + g \sum_{ij} S_i^+ S_j^-$$

- An important difference between RG models and any other ES model is the large number of free parameters. They can be used to define physical interactions. They can even be chosen randomly.

Some models derived from **rational (XXX)** RG

- BCS Hamiltonian (Fermion and Boson).
- Generalized Pairing Hamiltonians (Fermion and Bosons).
- The Universal Hamiltonian of quantum dots.
- Central Spin Model.
- Generalized Gaudin magnets.
- Lipkin Model.
- Two-level boson models (IBM, molecular, etc..)
- Atom-molecule Hamiltonians (Feshbach resonances), or Generalized Jaynes-Cummings models,
- Breached superconductivity (Sarma state).
- Pairs with finite center of mass momentum, FFLO superconductivity.

Review: J.Dukelsky, S. Pittel and G. Sierra, Rev. Mod. Phys. 76, 643 (2004).

What is a Cooper pair in the superfluid is medium?

G. Ortiz and J. Dukelsky, Phys. Rev. A 72, 043611 (2005)

$$\Psi = A\left[\varphi_1(r_1)\varphi_2(r_2)\cdots\varphi_{N/2}(r_{N/2})\right]$$

"Cooper" pair wavefunction $\varphi(r) = \frac{1}{V}\sum_k \varphi_k \, e^{ik\cdot r}$

- From MF BCS: $\varphi_k^{BCS} = C_{BCS}\frac{v_k}{u_k}$
- From pair correlations: $\varphi_k^{P} = \langle BCS | c_{-k\downarrow} c_{k\uparrow} | BCS \rangle = C_P u_k v_k$
- From Exact wavefuction: $\varphi_k^{E}(E) = \frac{C_E}{2\varepsilon_k - E}$

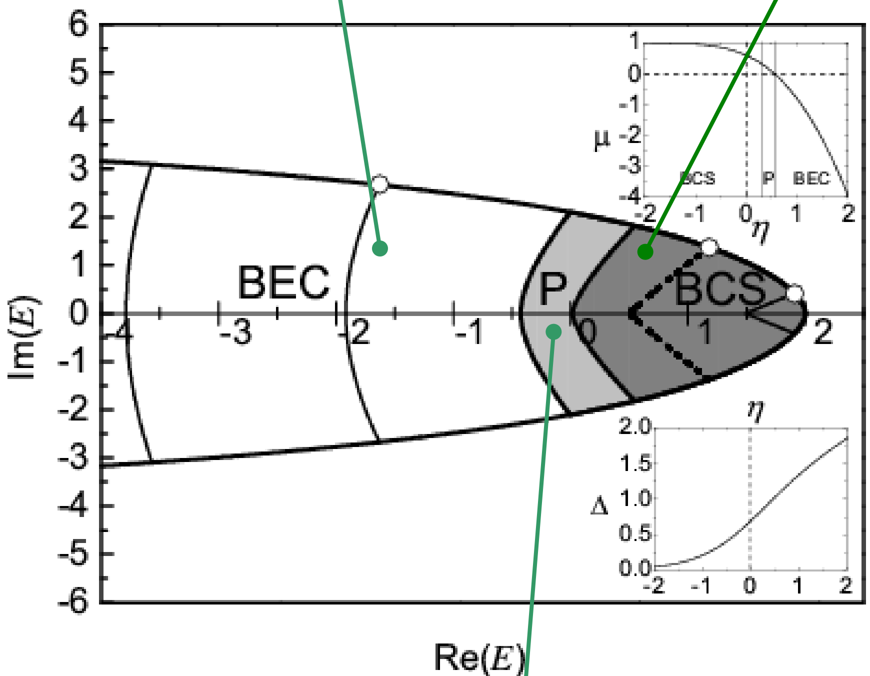
$$\varphi_E(r) = C_E \frac{e^{-r\sqrt{-E/2}}}{r}$$

- E real and <0, bound eigenstate of a zero range interaction parametrized by a.
- E complex and R (E) < 0, tightly bound molecule.
- E complex and R (E) > 0, weakly bound Cooper pair.
- E Real and >0 free two particle state.

BCS-BEC Crossover diagram

f=1 Re(E)<0

f pairs with Re(E) >0
1-f unpaired, E real >0



η= -1,   f = 0.35 (BCS)
η = 0,   f = 0.87 (BCS)
η = 0.37, f = 1 (BCS-P)
η = 0.55, f = 1 (P-BEC)
η = 1,2,   f=1 (BEC)

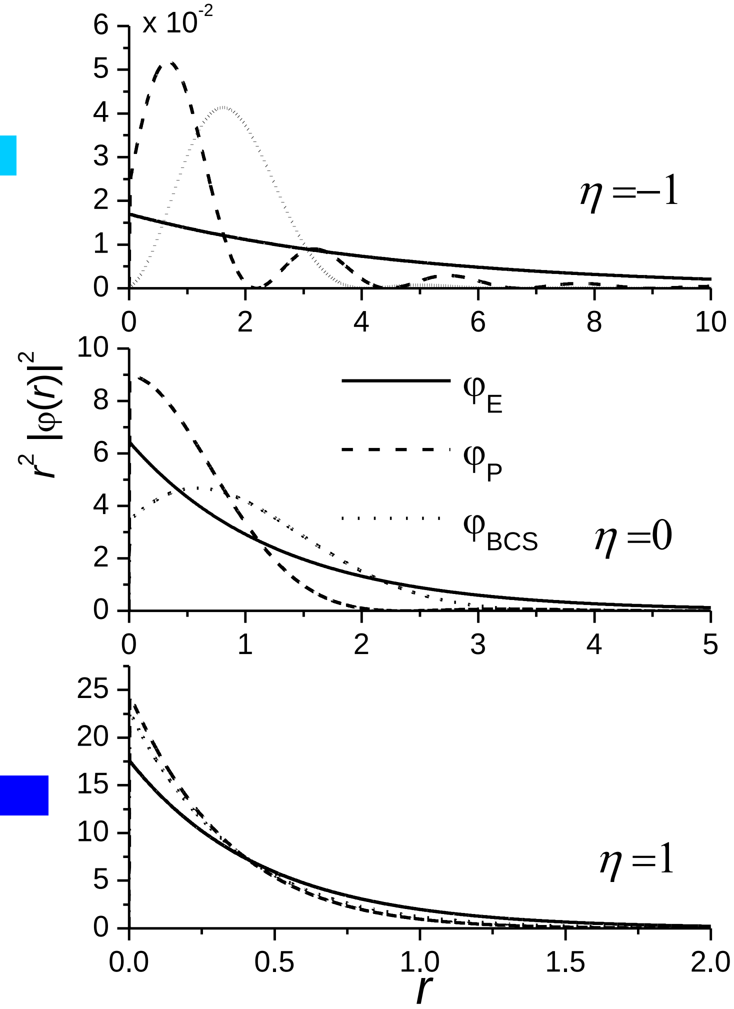
f=1 some Re(E)>0
others Re(E) <0

"Cooper" pair wave function

Weak coupling BCS

Strong coupling BCS

BEC



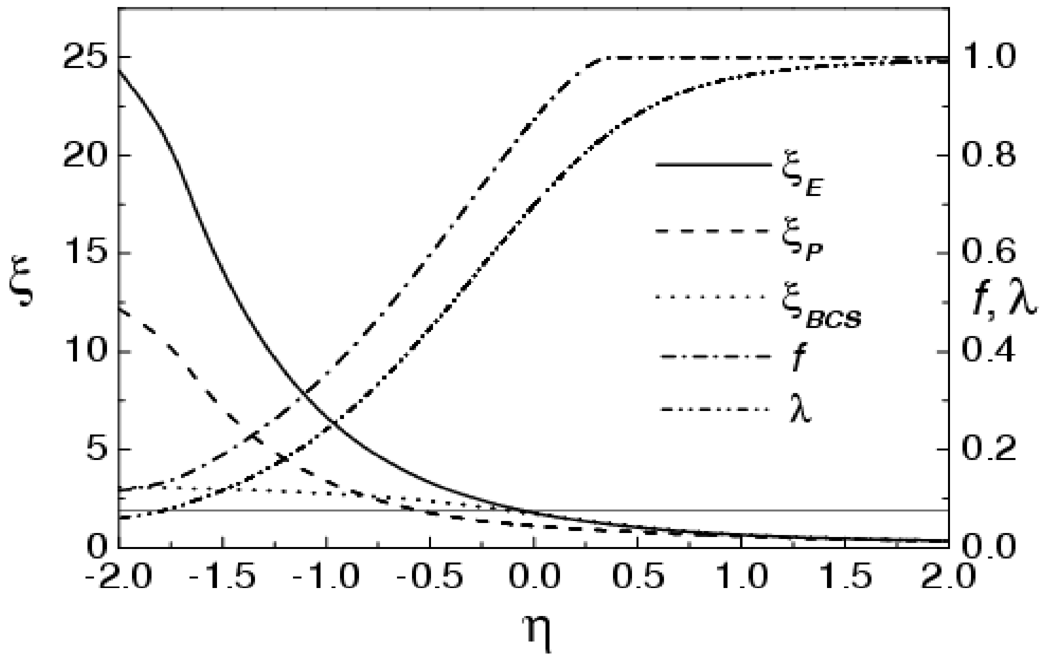
Sizes and fraction of the condensate

$$\xi = \sqrt{\langle \varphi | r^2 | \varphi \rangle}$$

$$\xi_E = 1/\operatorname{Im}\left(\sqrt{E}\right)$$

$$\lambda = \frac{2}{N}\int dr_1 dr_2 \left|\varphi_P(r_1, r_2)\right|^2$$

$$= \frac{3\pi}{16}\frac{\Delta^2}{\operatorname{Im}\left(\sqrt{\mu + i\Delta}\right)}$$



The Hyperbolic Richardson-Gaudin Model

A particular RG realization of the hyperbolic family is the separable pairing Hamiltonian:

$$H = \sum_i \eta_i R_i = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-$$

With eigenstates:

$$|\Phi_M\rangle = \prod_{\alpha=1}^{M} \left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+ \right) |0\rangle, \qquad E(\Phi_M) = \sum_{\alpha=1}^{M} E_\alpha$$

Richardson equations:

$$0 = \sum_i \frac{s_i}{\eta_i - E_\alpha} - \frac{Q}{E_\alpha} + \sum_{\alpha'(\neq\alpha)} \frac{1}{E_\alpha - E_{\alpha'}}, \quad 2Q = \frac{1}{G} - L + M - 2$$

The physics of the model is encoded in the exact solution. It does not depend on any particular representation of the Lie algebra

(p_x+ip_y) exactly solvable model

In 2D one can find a representation of the SU(2) algebra in terms of spinless fermions.

$$S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right), \qquad S_k^+ = \frac{k_x + ik_y}{|k|} c_k^\dagger c_{-k}^\dagger = \left(S_k^-\right)^\dagger$$

Choosing $\boldsymbol{\eta_k = k^2}$ we arrive to the p_x+ip_y Hamiltonian

$$H = \sum_{k(k_x>0)} \frac{k^2}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - G \sum_{\substack{k,k' \\ (k_x,k_x'>0)}} \left(k_x + ik_y\right)\left(k_x - ik_y\right) c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}$$

M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao, Phys. Rev. B 79, 180501 (2009).
C. Dunning, M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao,, J. Stat. Mech. P080025 (2010).
S. Rombouts, J. Dukelsky and G. Ortiz, Phys. Rev. B. 82, 224510 (2010).

Why *p*-wave pairing?

- p_x+ip_y paired phase has been proposed to describe the A1 superfluid phase of ^{3}He.

- N. Read and D. Green (Phys. Rev. B 61, 10267 (2000)), studied the p_x+ip_y model. They showed that *p*-wave pairing has a QPT (2º order?) separating two gapped phases: a) a non-trivial topological phase. **Weak pairing**; b) a phase characterized by tightly bound pairs. **Strong pairing**.

- Moreover, there is a particular state in the phase diagram (the Moore-Read Pfaffian) isomorphic to the $\nu=5/2$ fractional quantum Hall state.

- In polarized (single hyperfine state) cold atoms *p*-wave pairing is the most important scattering channel (*s*-wave is suppressed by Pauli). *p*-wave Feshbach resonances have been identified and studied. However, a *p*-wave atomic superfluid is unstable due to atom-molecule and molecule-molecule relaxation processes.

- Current efforts to overcome these difficulties. The great advantage is that the complete BCS-BEC transition could be explored.

From the exact solution

1) The Cooper pair wavefunction

$$\Gamma_\alpha^+ = \sum_k \frac{k_x + i k_y}{k^2 - E_\alpha} c_k^+ c_{-k}^+$$

- E_α real positive $\rightarrow$ uncorrelated pair
- E_α complex $\rightarrow$ Correlated Cooper pair
- E_α real negative $\rightarrow$ Bound state

2) All pair energies converge to zero (Moore-Read line)

$$G = \frac{1}{L - M + 1}, \qquad g = \frac{1}{1-\rho} \Rightarrow E = 0\,;$$

$$\left|\Phi_M\right\rangle_{Exact} = \left[\sum_{k, k_x > 0} \frac{1}{k_x - i k_y} c_k^\dagger c_{-k}^\dagger\right]^M \left|0\right\rangle = \left|\Phi_M\right\rangle_{PBCS}$$

Density $\rho = M / L$
Coupling $g = GL$

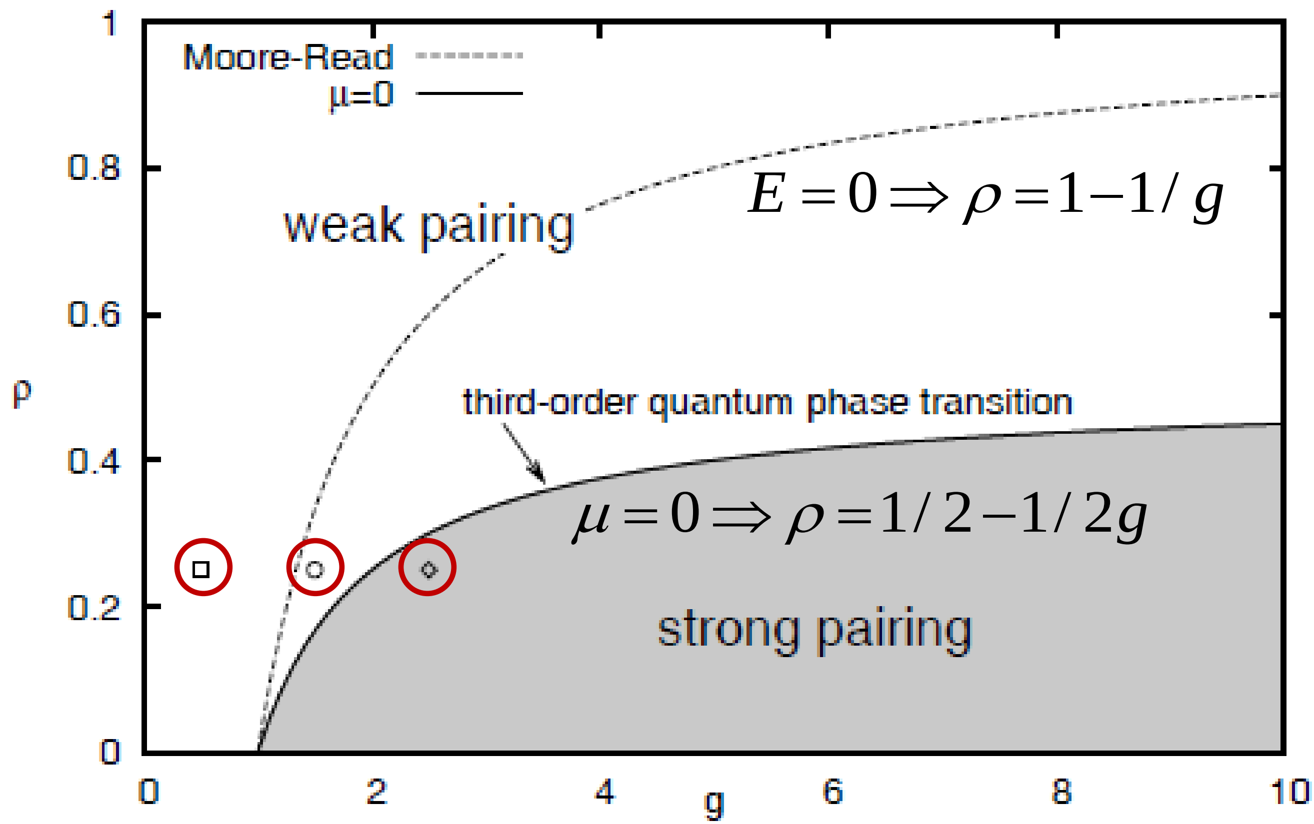
3) All pair energies real and negative (Phase transition)

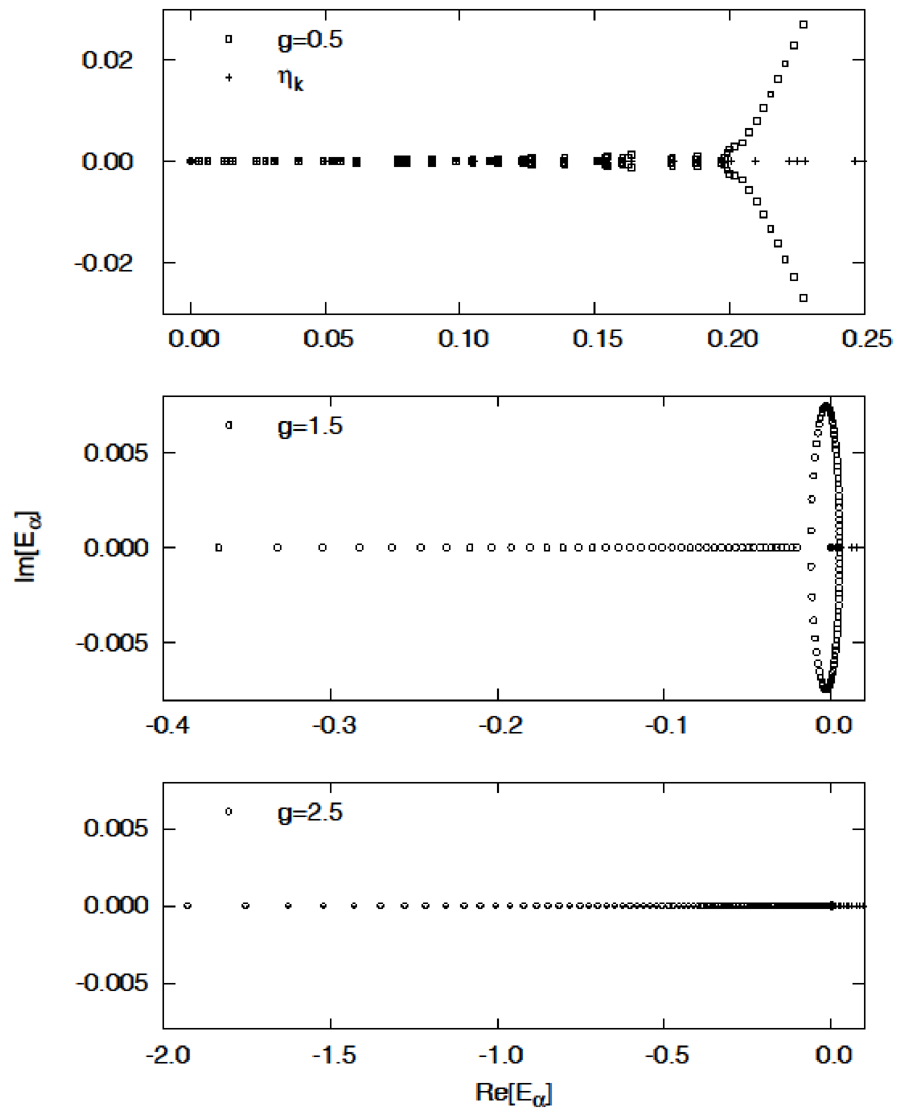
$$G \geq \frac{1}{L - 2M + 1}, \quad g \geq \frac{1}{1 - 2\rho}$$

$$for \quad g = 1/(1 - 2\rho) \Rightarrow E_1 = 0$$

Quantum phase diagram of the hyperbolic model

The phase diagram can be parametrized in terms of the density $\rho = M / L$
and the rescaled coupling $g = GL$
In the thermodynamic limit the Richardson equations $\longrightarrow$ BCS equations





Exact solution in a 2D lattice with disk geometry of R=18 with total number of levels L=504 and M=126. (quarter filling)

$D \cong 10^{122}$

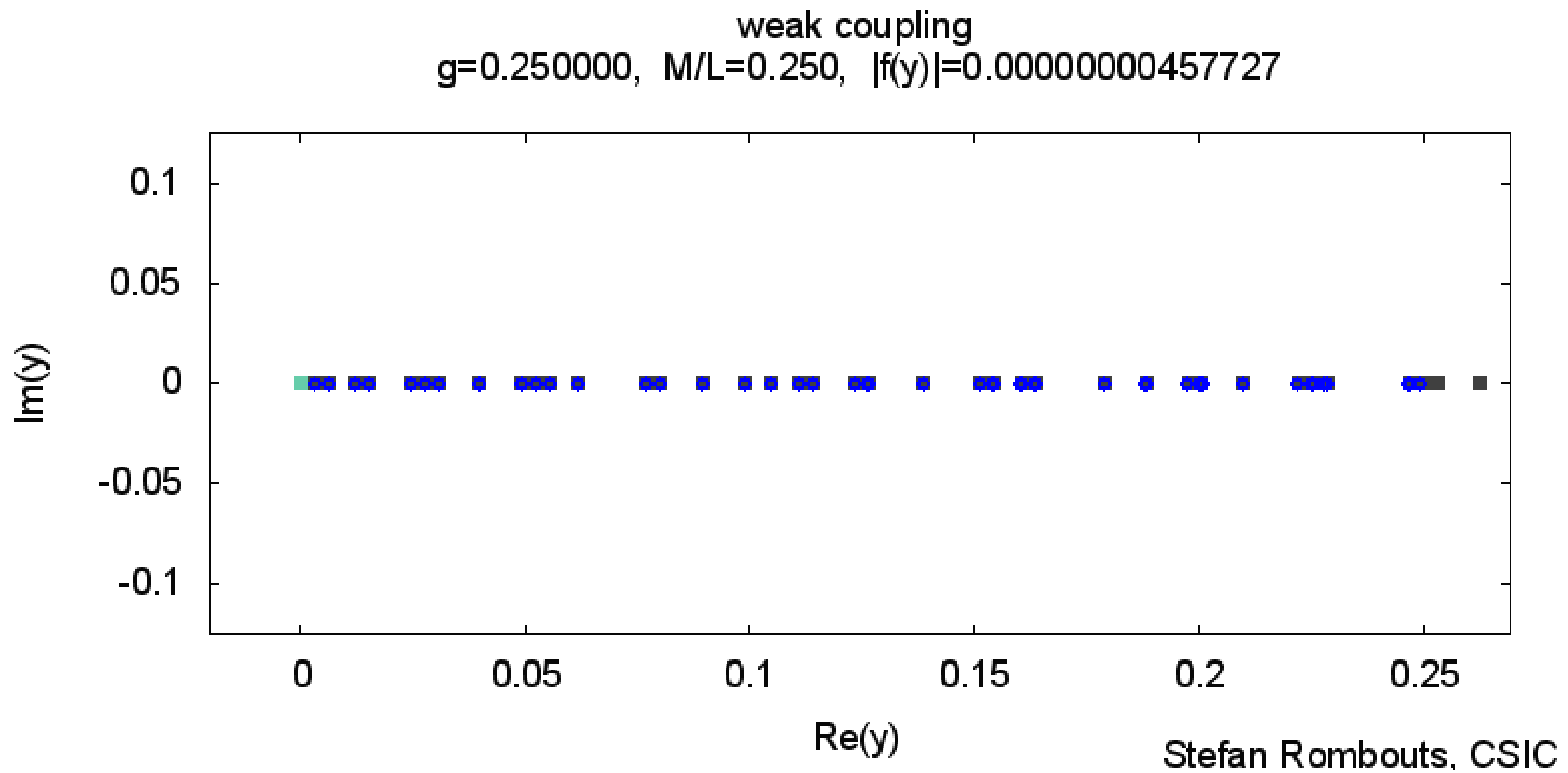
g=0.5 weak pairing

g=1.33 Moore-Read

g=1.5 weak pairing

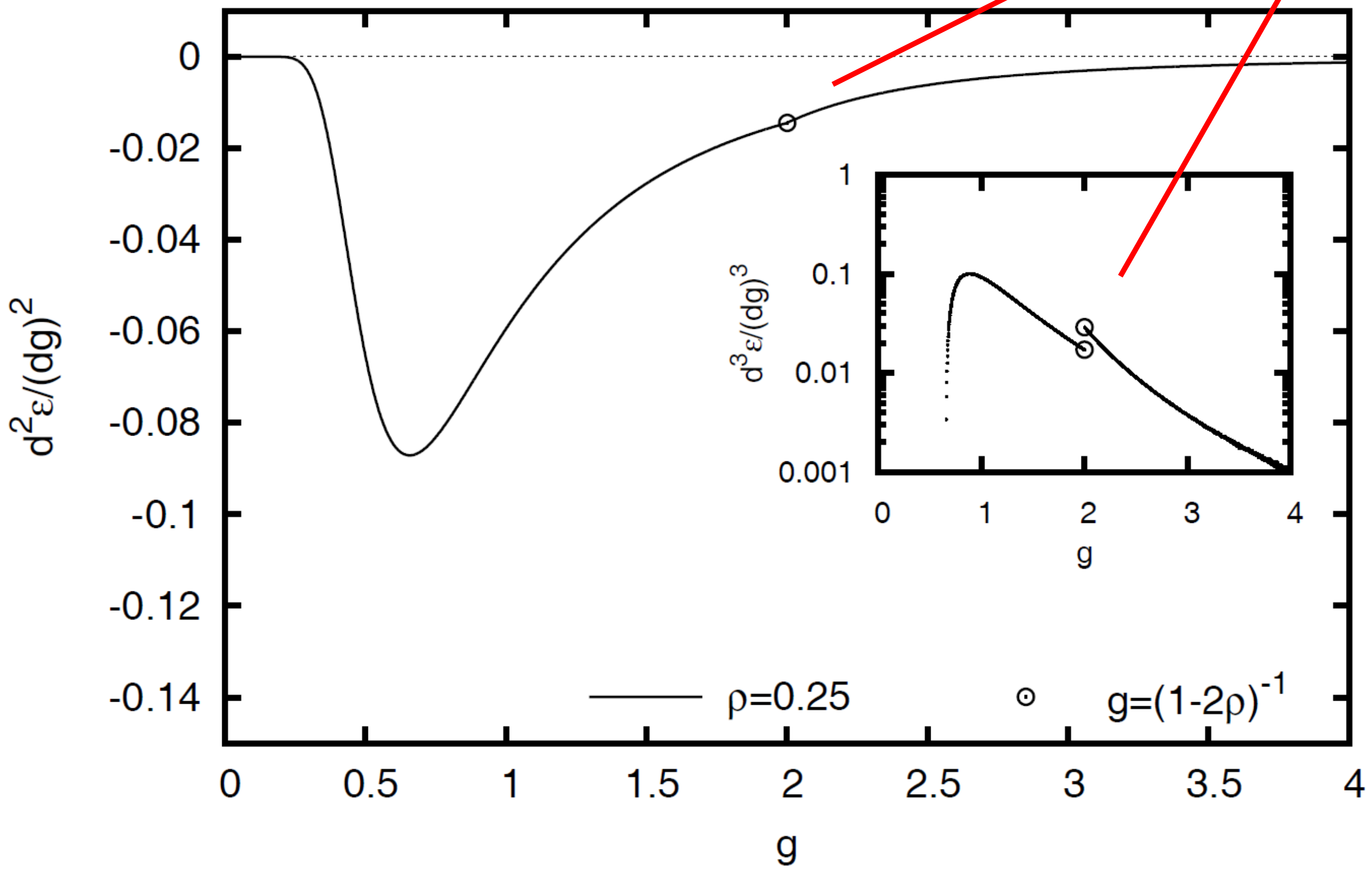
g=2.0 QPT

g=2.5 strong pairing


weak coupling
g=0.250000, M/L=0.250, |f(y)|=0.00000000457727
0.1
0.05
0
-0.05
-0.1
Im(y)
0
0.05
0.1
0.15
0.2
0.25
Re(y)
Stefan Rombouts, CSIC

Higher order derivatives of the GS energy in the thermodynamic limit

3º order QPT



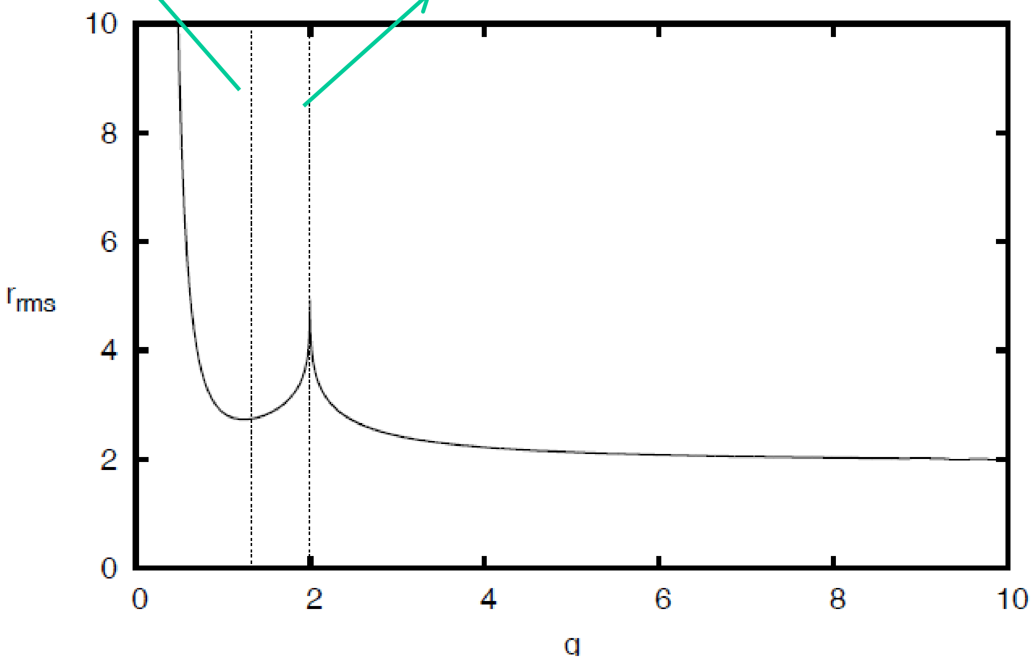
Characterization of the QPT

In the thermodynamic limit the pair wavefunction in k-space is:

$$\phi(k) = \langle \psi | c_k^\dagger c_{-k}^\dagger | \psi \rangle = u_k v_k$$

Moore-Read

QPT, $\mu = 0$



Size of the pair wavefunction

$$r_{rms}^2 = \frac{\int \left| \nabla \phi(k) \right|^2 dk}{\int \left| \phi(k) \right|^2 dk}$$

$$\lim_{\mu \to 0} \left(r_{rms}^2 \right) \approx Ln|\mu|$$

Accessible experimentally by quantum noise interferometry and time of flight analysis???

A similar analysis can be applied to the pairs in the exact solution

$$S^{+}\left(E_{\alpha}\right)=\sum_{k}\phi_{\alpha}\left(k\right)c_{k}^{\dagger}c_{-k}^{\dagger}, \quad \phi_{\alpha}\left(k\right)\propto\frac{k_{x}+ik_{y}}{k^{2}-E_{\alpha}}$$

The root mean square $r^{2}_{rms,exact}$ of the pair wavefunction is finite for *E* complex or real and negative.

However, $r^{2}_{rms,exact}\Rightarrow\infty$ for $E \text{ real and } \geq 0$

In strong pairing all pairs are bound and have finite radius.

At the QPT one pair energy becomes real an positive corresponding to a single deconfined Cooper pair on top of an ensemble of bound molecules.

Exactly Solvable Pairing Hamiltonians

1) SU(2), Rank 1 algebra

$$H_R = \sum_i \varepsilon_i S_i^z - g \sum_{ij} S_i^+ S_j^-$$

2) SO(5), Rank 2 algebra

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} P_{i\tau}^+ P_{j\tau}$$

J. Dukelsky, V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea y S. Lerma H. PRL 96 (2006) 072503.

3) SO(6), Rank 3 algebra

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\alpha} P_{i\alpha}^+ P_{j\alpha}$$

B. Errea, J. Dukelsky and G. Ortiz, PRA 79 05160 (2009)

4) SO(8), Rank 4 algebra

$$H_{ST} = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} P_{i\tau}^+ P_{j\tau} - g \sum_{ij\sigma} D_{i\sigma}^+ D_{j\sigma}$$

$$H_{3/2} = \sum_i \varepsilon_i n_i - g \sum_{ij} \left(P_{i00}^+ P_{j00} + \sum_{m=-2}^{2} P_{i2m}^+ P_{j2-m} \right)$$

S. Lerma H., B. Errea, J. Dukelsky and W. Satula. PRL 99, 032501 (2007).

Summary

- For finite systems, the exact solution incorporates mesoscopic fluctuations absent in BCS and PBCS.

- From the analysis of the exact Richardson wavefunction we proposed a new view to the nature of the Cooper pairs in the BCS-BEC transition for s-wave and p-wave pairing.

- The hyperbolic RG offers a unique tool to study a rare 3º order QPT in the p_x+ip_y paired superfluid.

- We found that the root mean square size of the pair wave function diverges at the critical point. It could be a clear experimental signature of the QPT.

- Extensions to larger rank algebras open new horizons for the RG models.