BAROCLINIC INSTABILITY IN DIFFERENTIALLY ROTATING STARS

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Origin of the baroclinic instability

Eigenvalue equations

Excitation of *r*- and *g* - modes

The effect of composition gradient

Helicity and possibility of dynamos

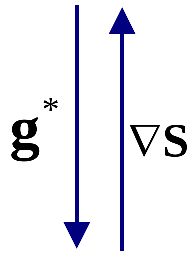
Is it observable?

Hydrodynamical equilibrium in rotating star

$\boldsymbol{V} = \boldsymbol{e}_\phi r \sin\theta\Omega$ Equilibrium condition: $\frac{1}{\rho}\boldsymbol{\nabla} P = \boldsymbol{g}^*\,, \quad \boldsymbol{g}^* = \boldsymbol{g} + r\sin\theta\Omega\,\boldsymbol{e}_\phi \times \boldsymbol{\Omega}$

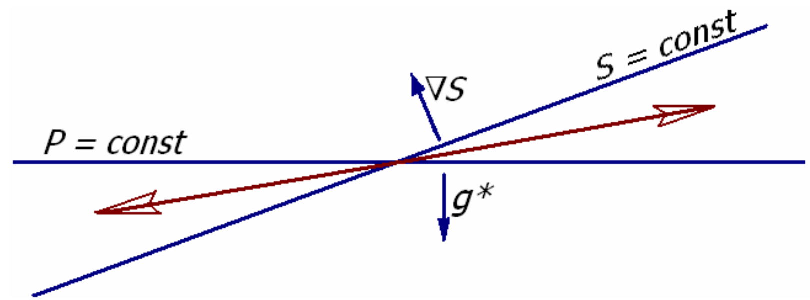
$$\sin\theta\frac{\partial\Omega^2}{\partial z} = -\frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}P)_\phi = \frac{1}{c_\mathrm{p}}(\boldsymbol{\nabla}s \times \boldsymbol{g}^*)_\phi$$

Barotropic stratification



Sufficiently large differential rotation, $\Delta\Omega \sim 0.1\Omega$, can be unstable (Watson 1981; Dziembowski & Kosovichev 1987; Charbonneau et al. 1999;....)

Baroclinic stratification



The “stable” stratification - if baroclinic - can provoke an instability (Shibahashi 1980; Tassoul & Tassoul 1983; ...)

Basic assumptions/approximations

- Shellular rotation

$$\Omega(r), \quad q = -\frac{r}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}r}$$

- Disturbances are global in horizontal dimentions but short-scaled in radius

$$\boldsymbol{u}, S \sim \exp\left(-\mathrm{i}\omega t + \mathrm{i}m\phi + \mathrm{i}kr\right)$$
$$\mu = \cos\theta$$

- Non-compressive disturbances but entropy perturbations due to radial displacements are allowed (Boussinesq approximation)

- Scalar potentials are used to specify toroidal (W) and poloidal (V) parts of the flow:

$$\boldsymbol{u} = \frac{\boldsymbol{e}_\phi}{r^2}\left(\hat{L}V\right) - \frac{\boldsymbol{e}_\theta}{r}\left(\frac{1}{\sin\theta}\frac{\partial W}{\partial\phi} + \frac{\partial^2 V}{\partial r\partial\phi}\right) + \frac{\boldsymbol{e}_\phi}{r}\left(\frac{\partial W}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial^2 V}{\partial r\partial\phi}\right)$$

$$\hat{L} = \frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\phi^2}$$

Equations of linear stability (eigenvalue) problem

- Toroidal flow

$$\hat{\omega}(\hat{L}W) = -\mathrm{i}\frac{\epsilon_\nu}{\hat{\lambda}^2}(\hat{L}W) + 2mW - 2\mu(\hat{L}V) - 2(1-\mu^2)\frac{\partial V}{\partial\mu}$$

- Poloidal flow

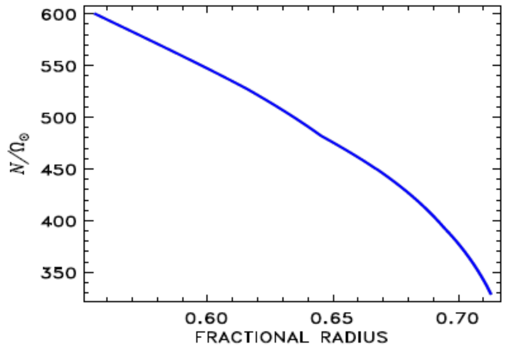
$$\hat{\omega}(\hat{L}V) = -\mathrm{i}\frac{\epsilon_\nu}{\hat{\lambda}^2}(\hat{L}V) - \hat{\lambda}^2(\hat{L}S) + 2mV - 2\mu(\hat{L}W) - 2(1-\mu^2)\frac{\partial W}{\partial\mu}$$

- Entropy

$$\hat{\omega}S = -\mathrm{i}\frac{\epsilon_\chi}{\hat{\lambda}^2}S + \hat{L}V + \mathrm{i}\frac{Q}{\hat{\lambda}}\mu\left(mW - (1-\mu^2)\frac{\partial V}{\partial\mu}\right)$$

Two governing parameters:

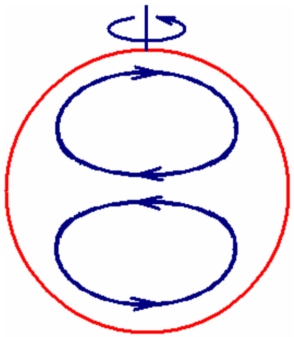
$$\hat{\lambda} = \frac{N}{\Omega k r}, \quad Q = 2q\frac{\Omega}{N}$$



Normalized diffusivities: $\epsilon_\chi = 10^{-4}, \quad \epsilon_\nu = 2\times 10^{-10}$

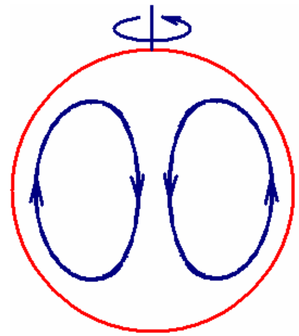
Symmetry properties

Equator-symmetric Sm-modes



$$V(\mu) = V(-\mu)$$
$$W(\mu) = -W(-\mu)$$
$$S(\mu) = S(-\mu)$$

Equator-antisymmetric Am-modes



$$V(\mu) = -V(-\mu)$$
$$W(\mu) = W(-\mu)$$
$$S(\mu) = -S(-\mu)$$

Eigenvalue equations are symmetric under the transformation:

$$(q, m, \omega, W, V, S) \rightarrow (-q, -m, -\omega^*, -W^*, V^*, -S^*)$$

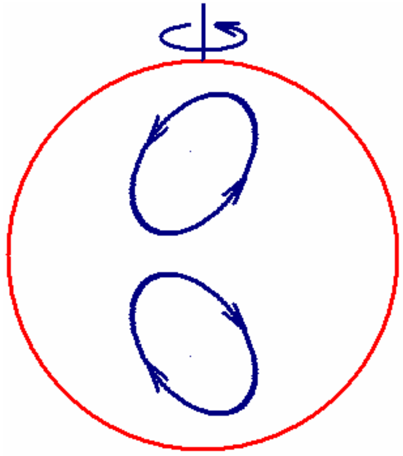
Unstable modes are expected to possess kinetic helicity

$$H_{\rm rel} = \langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle / (k\overline{u^2})$$

$$\langle X \rangle = \frac{1}{2\pi} \int_0^{2\pi} X \mathrm{d}\phi \ , \qquad \overline{u^2} = \frac{1}{2} \int_{-1}^{1} \langle u^2 \rangle \mathrm{d}\mu$$

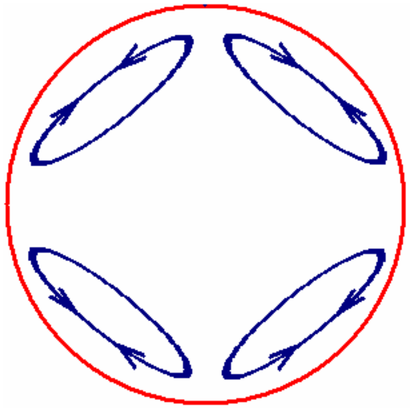
Two modes of stable oscillations
(uniform rotation, zero diffusion, N >> Ω)

<u>Toroidal *r* -modes</u>



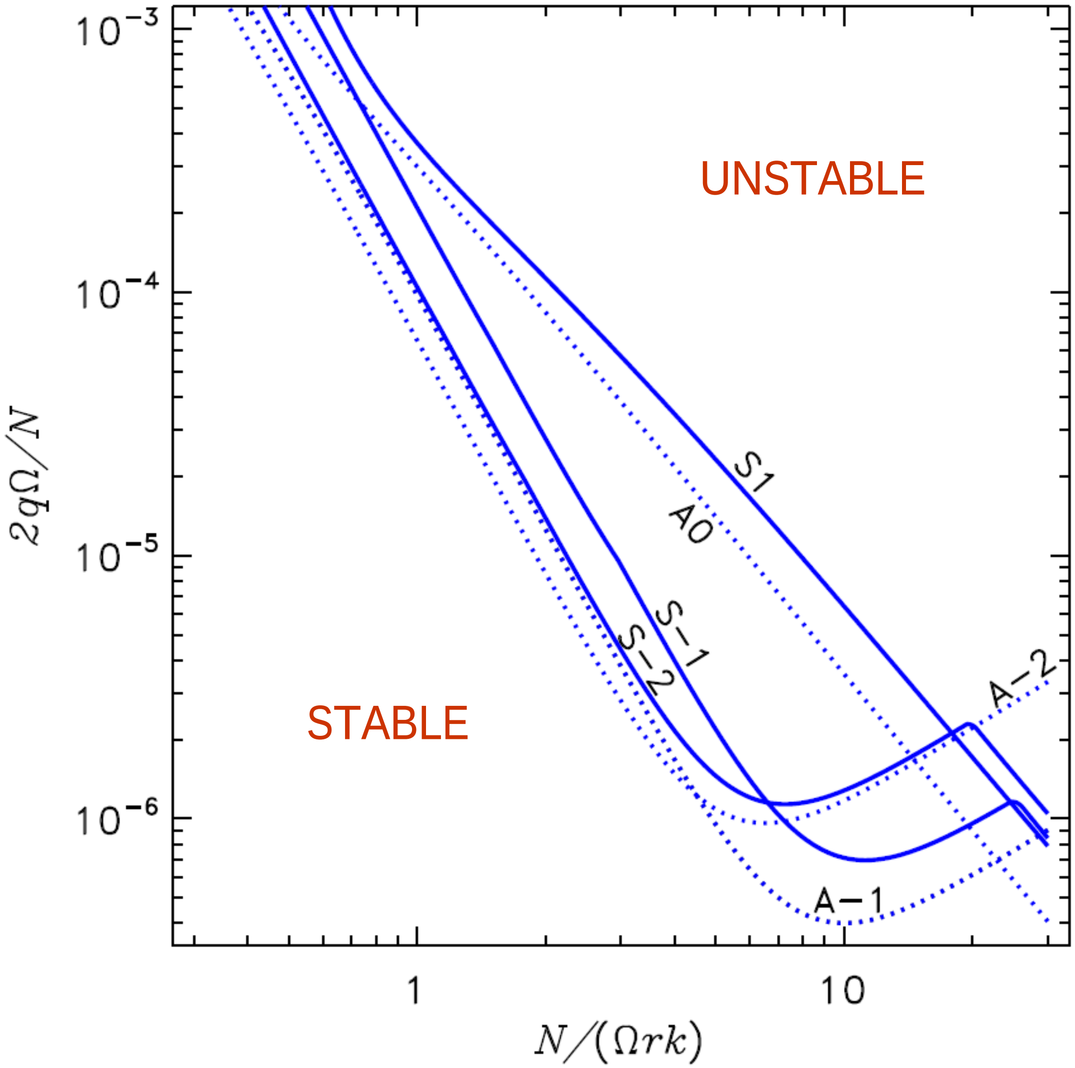
$$\omega_{lm}^{r} = -\frac{2m\Omega}{l(l+1)}$$

<u>Poloidal *g* -modes</u>



$$\omega_{lm}^{g} = \pm \frac{N}{kr}\sqrt{l(l+1)}$$

Stability map



$$\frac{\overline{S u_r}}{\sqrt{\overline{u_r^2}\ \overline{S^2}}} > 0$$ for all unstable modes

Baroclinic instability as a stability loss to excitation of *r*- and *g*-modes

Parameters of unstable disturbances for $\hat{\lambda} = 3$ and $Q = 10^{-3}$.

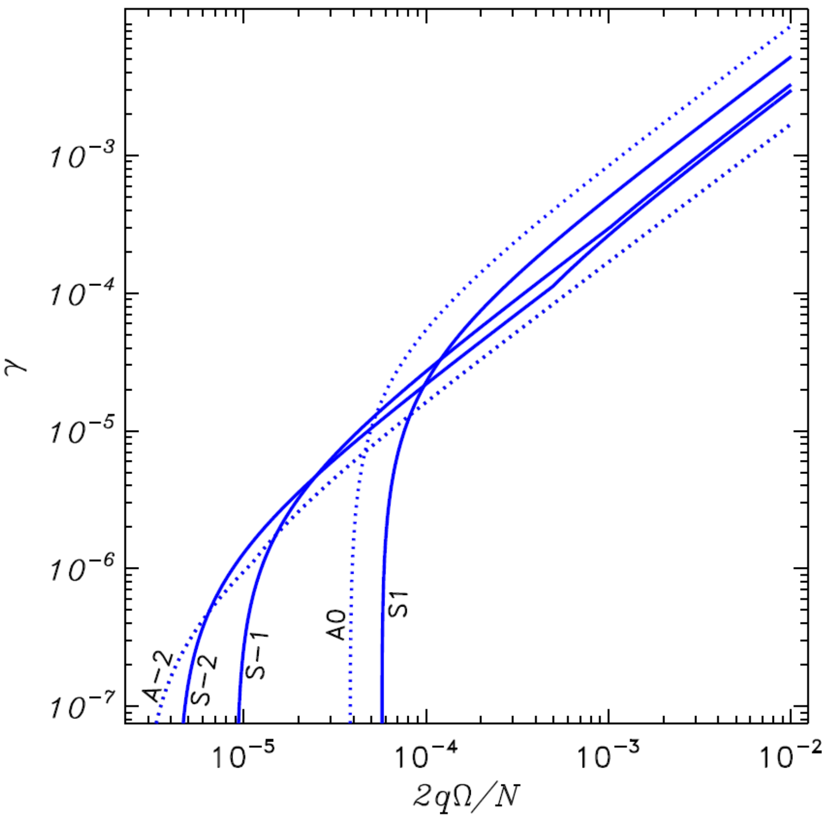
Mode	$\hat{\gamma}, 10^{-4}$	$\Re(\hat{\omega})$	$\hat{\omega}^r$	$\hat{\omega}^g$	$\overline{u_\mathrm{p}^2}/\overline{u_\mathrm{t}^2}$	$\overline{Su_r}/\sqrt{\overline{u_r^2}\ \overline{S^2}}$
A0	8.41	4.34		4.24	24.0	3.35×10^{-5}
A1	2.91	7.27		7.35	43.0	6.38×10^{-6}
A3	1.14	-13.6		-13.4	171	1.38×10^{-6}
A10	0.709	-34.6		-34.5	2487	3.32×10^{-7}
A-1	2.04	0.989	1		1.99×10^{-4}	2.99×10^{-3}
A-3	1.46	0.199	0.2*		9.62×10^{-7}	2.17×10^{-2}
A-10	0.507	0.0952	0.0952*		2.79×10^{-9}	0.155
S0	3.29	7.46		7.35	32.7	7.46×10^{-6}
S1	4.93	-4.85		-4.24	34.6	2.17×10^{-5}
S3	3.10	-10.7		-10.4	260	4.90×10^{-6}
S10	1.03	-31.5		-31.5	5727	5.24×10^{-7}
S-1	2.94	0.320	0.333		1.93×10^{-4}	3.67×10^{-3}
S-3	2.22	-10.2		-10.4	265	3.35×10^{-6}
S-10	0.874	-31.4		-31.5	5772	4.44×10^{-7}

Kinetic energy of disturbances is the sum of energies of their toroidal and poloidal parts:

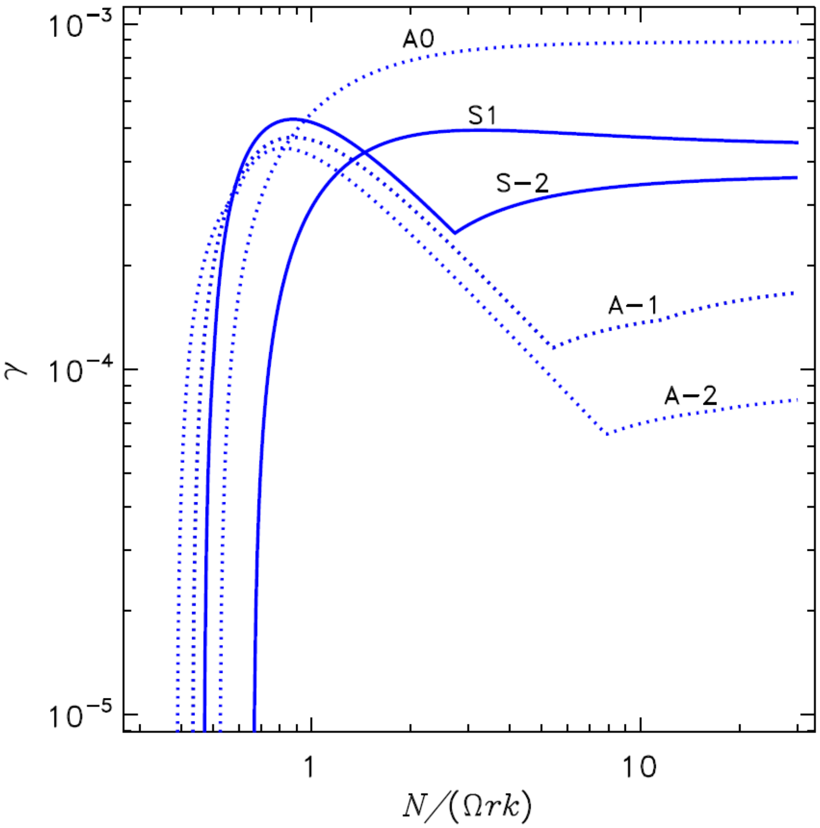
$$\overline{u^2} = \overline{u_\mathrm{p}^2} + \overline{u_\mathrm{t}^2} = \frac{1}{4}\sum_l l(l+1)\left(|\,V_l\,|^2 + |\,W_l\,|^2\right)$$

Growth rates

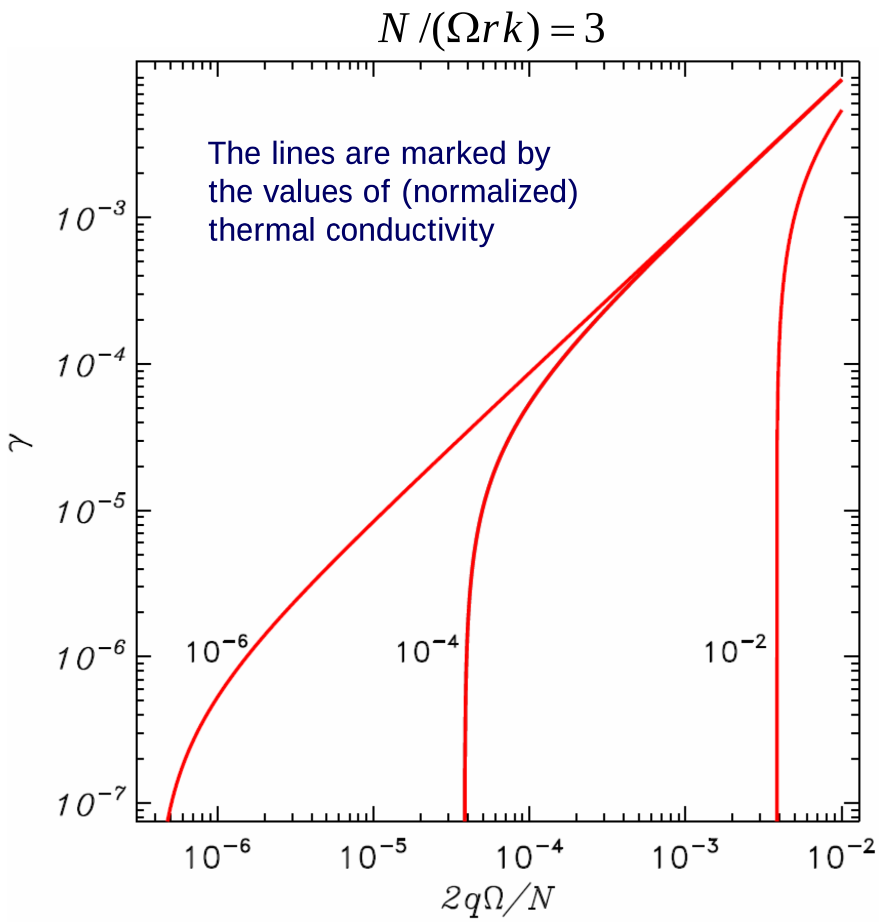
$N/(\Omega rk) = 3$



$2q\Omega / N = 0.001$



Dependence on the thermal diffusivity



Composition stratification changes the effective buoyancy frequency:

$$N^2 \longrightarrow N^2_{\rm eff} = N^2 + N^2_\mu ,$$

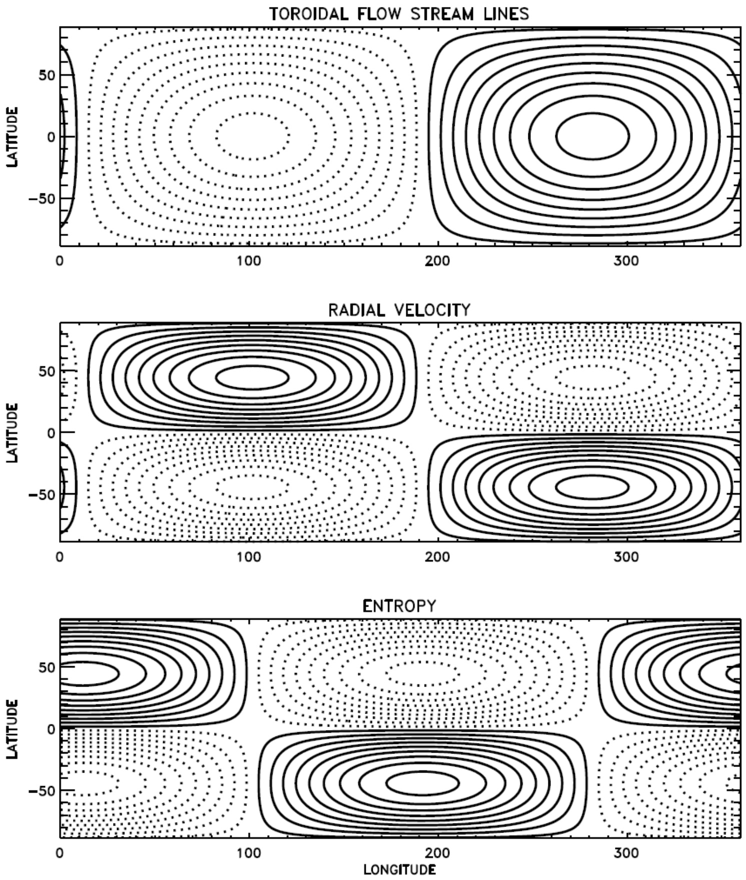
$$N^2_\mu = -\frac{g}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}r}$$

Eigenmodes

$$\left(\hat{\lambda} = 3, \quad Q = 0.001\right)$$

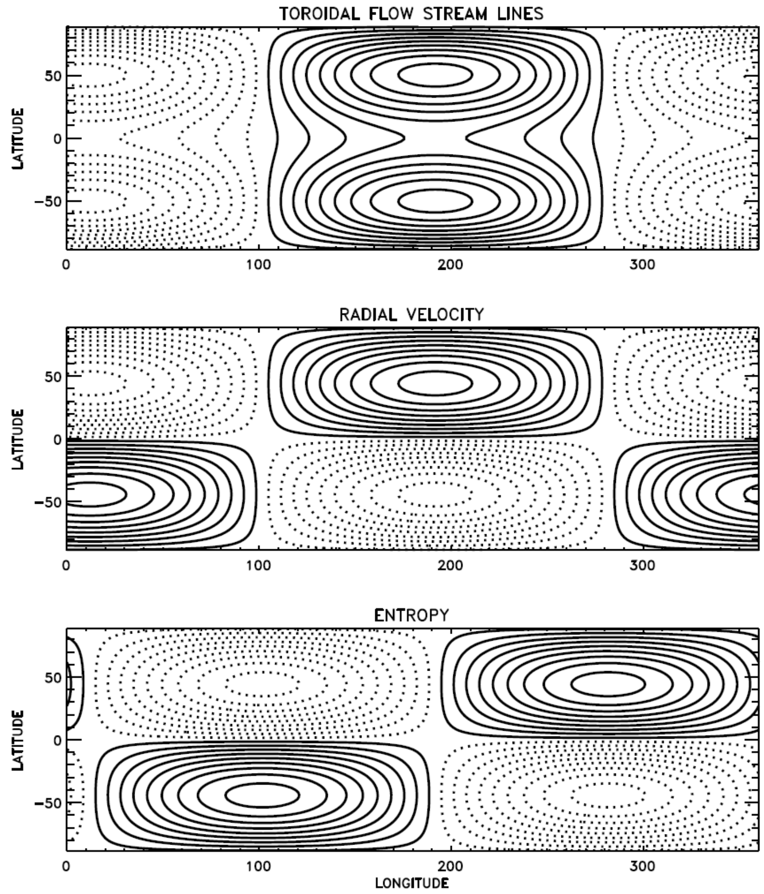
Toroidal r–mode A-1

$$\overline{u_{\rm p}^2}/\overline{u_{\rm t}^2} = 2\times10^{-4}, \quad \overline{Su_r}/\sqrt{\overline{u_r^2 S^2}} = 3\times10^{-3}$$



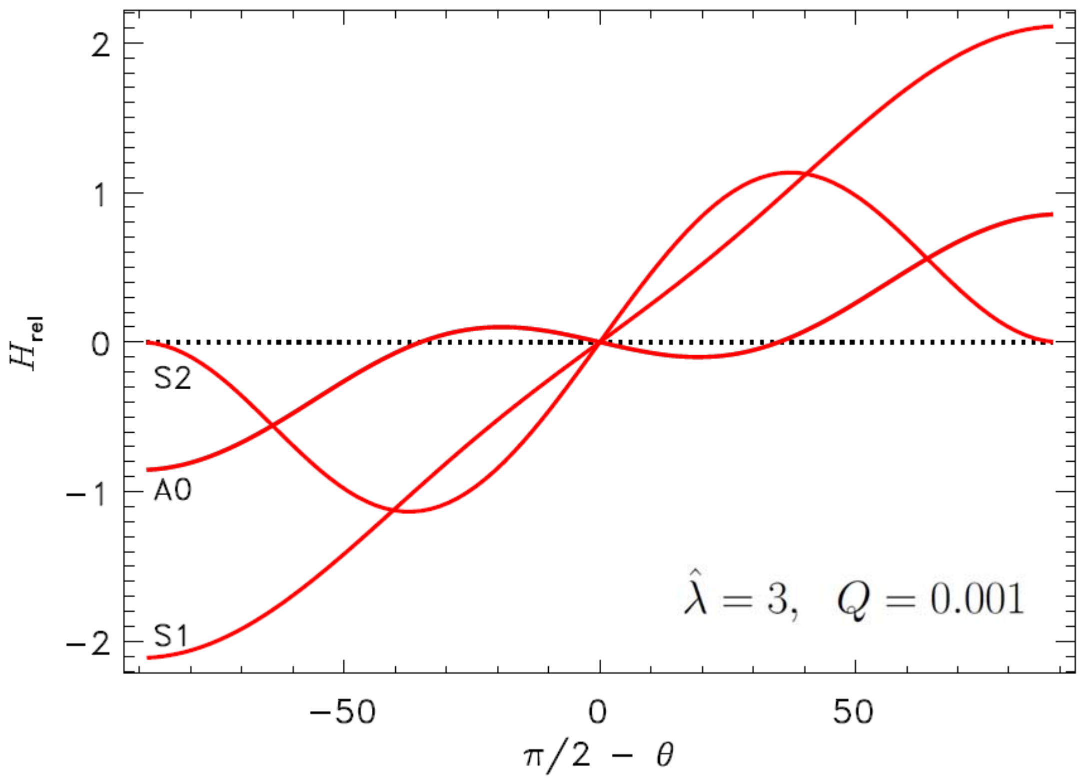
Poloidal g–mode A1

$$\overline{u_{\rm p}^2}/\overline{u_{\rm t}^2} = 43, \quad \overline{Su_r}/\sqrt{\overline{u_r^2 S^2}} = 6.4\times10^{-6}$$



Possibility of dynamos

$$H_{\rm rel} = \langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle / (k\overline{u^2})$$



Alecian et al. (2013) observed rapid (~10 yrs) changes in global magnetic field of Herbig star HD 190073

Conclusions

Even a very small radial differential rotation can provoke baroclinic instability in stellar radiation zones.

The instability can be understood as stability loss to excitation of *r* – and *g*-modes of global oscillations.

The unstable disturbances are helical. Turbulence resulting from the instability can be prone to dynamos.