Cosmic magnetogenesis: from spontaneously emitted aperiodic turbulent to ordered equipartition fields

R. Schlickeiser
Institut für Theoretische Physik
Lehrstuhl IV: Weltraum- und Astrophysik
Ruhr-Universität Bochum, Germany

April 11, 2013

Topics:

1. Introduction
2. Plasma fluctuations
3. From disordered to ordered magnetic field structures
4. Summary and conclusions
5. Fluctuation theory in magnetized plasmas (Back-up material)

References:

Spontaneous electromagnetic fluctuations in unmagnetized plasmas I: General theory and nonrelativistic limit; RS, P. H. Yoon, 2012, Physics of Plasmas 19, 022105

Spontaneous electromagnetic fluctuations in unmagnetized plasmas II. Relativistic form factors of aperiodic thermal modes; T. Felten, RS, P. H. Yoon, M. Lazar, 2013, Physics of Plasmas, submitted

Cosmic magnetization: from spontaneously emitted aperiodic turbulent to ordered equipartition fields; RS, 2012, PRL 109, 261101

1. Introduction

Magnets have practically become everyday objects. Permanent ferromagnetism is a property of only a few densely packed materials, such as iron, in which the spin exchange interactions of individual atoms naturally line up in the same direction and create a residual persistent magnetic field. In the early universe, before iron and other magnetic materials had been created inside stars, such permanent magnetism did not exist.

Scientists have long wondered where the observed cosmic magnetization came from, given that the fully ionized gas of the early universe contained no ferromagnetic particles.

The detection of linearly polarized synchrotron emission at mm-UV frequencies from high redshift FERMI/LAT gamma-ray blazars and FSRQs (special classes of active galactic nuclei with powerful nonthermal relativistic jet emission) up to $z \leq 1.5$ (Böttcher and Reimer 2013) indicates the presence of partially ordered magnetic fields at $z = 1.5$. Beck and Wielebinski (2013) argue that magnetic fields existed already in quasars at epochs with $z \simeq 5$ and in starburst galaxies at $z \simeq 4$, based on Faraday rotation measurements and synchrotron emission. More systematic detections are expected from the future SKA-project and its pathfinders such as LOFAR with the eventual detection of Zeeman splitting of the redshifted 21-cm line emission.

Many astrophysicists believe that galactic magnetic fields are generated and maintained by dynamo action, whereby the energy associated with the differential rotation of spiral galaxies is converted into magnetic field energy.

However, the dynamo mechanism is only a means of amplification and dynamos require seed magnetic fields. These seed field are needed for possible instabilities from anisotropic plasma particle distribution functions, MHD instabilities (such as the Magneto-Rotational Instability) and/or the MHD dynamo process, which grow according to $d(\delta B)^2/dt = 2\gamma(\delta B)^2$, so that

$$(\delta B)^2(t) = (\delta B)^2(t=0)e^{2\gamma t} \tag{1}$$

Obviously, a nonvanishing initial $(\delta B)^2(t=0) \neq 0$ seed magnetic field energy density is required. Neither the dynamo process nor plasma instabilities generate magnetic fields without such seed fields.

Before the formation of the first stars, the luminous proto-interstellar matter consisted only of a fully ionized gas of protons, electrons, helium nuclei and lithium nuclei which were produced during the Big Bang.

The physical parameters that describe the state of this gas are, however, not constant. Density and pressure fluctuate around certain mean values, and consequently electric and magnetic fields fluctuate around vanishing mean values, due to the thermal motion of the plasma particles.

This small but finite dispersion in the form of random magnetic fields has now been calculated, specifically for the proto-interstellar gas densities and temperatures that occurred in the plasmas of the early universe at redshifts $z = 7 - 20$ of the reionization epoch, when something, probably the light from the first stars, provided the energy needed to break up the previously neutral gas that existed in the universe during the recombination era. The protons and electrons inside the plasma would have moved around continuously, simply by virtue of existing at a finite temperature. It is the finite variance of the resulting magnetic fluctuations, that subsequently leads to the creation of a stronger magnetism across the universe.

There have been alternative proposals for cosmic seed magnetic fields. Biermann (1950) proposed that the centrifugal force generated in a rotating plasma cloud will separate out heavier protons from lighter electrons, thereby creating a separation of charge that leads to tiny electric and magnetic fields. However, this scheme suffers from a lack of suitable rotating objects everywhere, meaning that it could only ever generate the magnetic fields in a small portion of the medium.

2. Plasma fluctuations

All plasmas, including unmagnetized and those in thermal equilibrium, have fluctuations. Because of the large sizes of astrophysical systems compared to the plasma Debye length, the fluctuations are descibed by real wave vectors ($\vec{k}$) and complex frequencies $\omega(\vec{k}) = \omega_R(\vec{k}) + \imath\gamma(\vec{k})$, implying for the space- and time-dependence of e.g. magnetic fluctuations the superposition of

$$\delta\vec{B}(\vec{x},t) \propto \exp[\imath(\vec{k}\cdot\vec{x} - \omega_R t) + \gamma t] \tag{2}$$

One distinguishes between

- **collective** modes with a fixed dispersion relation $\omega = \omega(\vec{k})$, e.g. electromagnetic waves in vacuum $\omega_R^2 = c^2k^2$ and $\gamma = 0$,
- **non-collective** modes with no dispersion relation $\omega = \omega(\vec{k})$,

and, regarding the real (ω_R) and imaginary (γ) part of the frequency,

- **weakly damped/amplified** wave-like modes with $|\gamma| \ll |\omega_R|$, e.g. collective Alfven and magnetosonic waves,
- **weakly propagating** modes with $|\omega_R| \ll |\gamma|$, e.g. collective mirror und firehose fluctuations,
- **aperiodic** modes with $\omega_R = 0$ fluctuate only in space, do not propagate as $\omega_R = 0$, but permanently grow or decrease in time depending on the sign of γ, e.g. collective Weibel fluctuations. And $|\delta B|^2 \gg |\delta E|^2$!

Because of their comparably low gas densities, all cosmic fully and partially ionized non-stellar plasmas are collision-poor, as indicated by the very small values of the plasma parameter

$$g = \nu_{ee}/\omega_{p,e} = 7.3 \cdot 10^{-4} \left(n_e/\mathrm{cm}^{-3}\right)^{1/2} (T_e/\mathrm{K})^{-3/2} \le \mathcal{O}\left(10^{-10}\right), \qquad (3)$$

given by the ratio of the electron-electron Coulomb collision frequency ν_{ee} to the electron plasma frequency $\omega_{p,e}$, characterizing interactions with electromagnetic fields, so that fully kinetic plasma descriptions are necessary.

Unlike for weakly amplified/damped modes (see Salpeter 1960, Sitenko 1967, Ichimaru 1973, Kegel 1998), however, for aperiodic fluctuations the expected fluctuation level has never been calculated quantitatively.

Only recently general expressions for the electromagnetic fluctuation spectra (electric and magnetic field, charge and current densities) from uncorrelated plasma particles in unmagnetized plasmas for arbitrary frequencies have been derived (Schlickeiser and Yoon 2012) using the system of the Klimontovich and Maxwell equations, which are appropriate for fluctuations wavelengths longer than the mean distance between plasma particles, i.e. $k \le k_{\max} = 2\pi n_e^{1/3}$. These general expressions are covariantly correct within the theory of special relativity, and hold for arbitrary momentum dependences of the plasma particle distribution functions and for collective and non-collective fluctuations.

2.1. Thermal aperiodic noise in the intergalactic medium

The most important unmagnetized cosmic plasma is the early intergalactic medium, photoionized from the earliest generation of stars at redshift $z \simeq 20$, which transformed the Universe from darkness after recombination to light. Modeling the photoionization by the first forming stars (Hui and Gnedin 1997, Hui and Haiman 2003) indicates IGM temperatures of about $T_e = T_p = T = 10^4 T_4$K and ionized gas densities of $n_e = 10^{-7} n_{-7}$ cm^{-3} at redshift $z = 4$.

The aperiodic transverse magnetic field fluctuation spectrum over the whole complex frequency plane for nonrelativistic plasma temperatures $u_a \ll c$, is shown in Fig. 1 and given by (Felten et al. 2013)

$$< \delta B_\perp^2 >_{k,\gamma} = \sum_a \frac{\omega_{p,a}^2 m_a u_a}{4\pi^3 k^3 c^2}$$

$$\times \frac{|\pi^{1/2}[(1+\frac{\gamma^2}{c^2k^2})D(|\frac{\gamma}{ku_a}|) - \sigma(1+\frac{\gamma^2}{c^2u_{a,\parallel}^2})]|}{|1+\frac{\gamma^2}{c^2k^2} + \sum_a \frac{\omega_{p,a}^2}{k^2c^2}|\frac{\gamma}{ku_a}|\pi^{1/2}[(1+\frac{\gamma^2}{c^2k^2})D(|\frac{\gamma}{ku_a}|) - \sigma(1+\frac{\gamma^2}{c^2u_{a,\parallel}^2})]|^2}, \quad (4)$$

where $u_a = (2k_BT_a/m_a)^{1/2}$ denotes the thermal velocity, $\sigma = 0, 1, 2$ for $\gamma > 0$, $\gamma = 0$, $\gamma < 0$, respectively (correct analytical continuation), and

$$D(x) = e^{x^2} \text{erfc}\,(x) = e^{x^2}\left[1 - \frac{2}{\pi^{1/2}}\int_0^x dt\, e^{-t^2}\right] \quad (5)$$

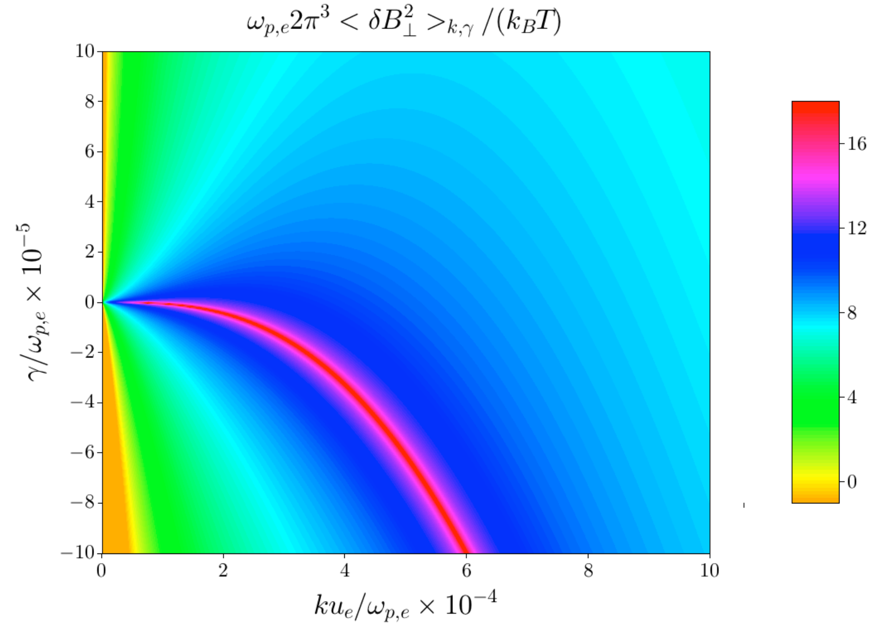

Figure 1: Contour plot of the spontaneously emitted aperiodic magnetic field fluctuation spectrum in a thermal nonrelativistic electron-proton distribution in units of $k_B T_e/(2\pi^3 \omega_{p,e})$. Equal electron and proton temperatures ($T_i = T_e$) and the value $\beta_e = 10^{-3}$ are adopted. The colour scale is logarithmic in powers of e. From Felten et al. (2013).

Integrating over all values of γ and k provides the energy density of spontaneously emitted fully random magnetic fluctuations

$$(\delta B)^2 = 4\pi \int_0^{k_{\max}} dk\, k^2 \; < \delta B^2 >_k \tag{6}$$

with

$$< \delta B_\perp^2 >_k = \int_{-\infty}^{\infty} d\gamma \;\; < \delta B_\perp^2 >_{k,\gamma} \tag{7}$$

RS (2012) calculated a lower limit to this integral by setting $\sigma = 0$ in Eq. (4), omitting the collective damped mode at $\gamma < 0$, yielding

$$\begin{aligned} < \delta B^2 >_k &= 2\int_0^\infty d\gamma\, U(k,\gamma) \\ &= \frac{\omega_{p,e}^2 m_e \beta_e^2}{2\pi^{5/2} k^2} \int_0^\infty dx\, \frac{F(x,\mu)}{[1+\beta_e^2 x^2]\left[1+\frac{\pi^{1/2}\omega_{p,e}^2}{k^2 c^2}\mathtt{x} F(x,\mu)\right]^2}, \end{aligned} \tag{8}$$

where $F(x,\mu) = D(x) + \mu^{-1} D(x\mu)$, the mass ratio $\mu^2 = m_p/m_e = 1836$ and $\beta_e = u_e/c = 1.84 \cdot 10^{-3} T_4^{1/2}$.

Because of the large value of $\mu = 43$ we can neglect the proton contribution, so that $F(x,\mu) \simeq D(x)$, implying in terms of the normalized wave vector $\kappa = kc/\omega_{p,e}$ that

$$< \delta B^2 >_k = \frac{m_e c^2 \beta_e^2}{2\pi^{5/2}\kappa^2} J_0(\beta_e, \kappa) \tag{9}$$

where we define the integrals

$$J_n(\beta, \kappa) = \int_0^\infty \frac{dx}{1+\beta^2 x^2} \frac{x^n D(x)}{\left[1 + \frac{\pi^{1/2}}{\kappa^2} x D(x)\right]^2} \tag{10}$$

for $n = 0, 1$. The approximative analytical evaluation of the integral (10) makes use of the rational approximation better than $2.5 \cdot 10^{-5}$ given by $D(x) \simeq a_1 t - a_2 t^2 + a_3 t^3$ with $t = 1/(1+px)$, $p = 0.47047$, $a_1 = 0.3480242$, $a_2 = 0.0958798$ and $a_3 = 0.7478556$. Given the smallness of a_2 we use as lower and upper limits $D_L(x) < D(x) < D_U(x)$ with

$$D_L(x) \simeq (a_0 + a_3 t^2) t, \quad D_U(x) \simeq (a_1 + a_4 t^2) t \tag{11}$$

where $a_0 = a_1 - a_2 = 0.2521444$ and $a_4 = a_3 - a_2 = 0.6519758$. The integral (10) then is well approximated by $J_n^L(\beta, \kappa) < J_n(\beta, \kappa) < J_n^U(\beta, \kappa)$ with

$$J_n^{U,L}(\beta, \kappa) = \int_0^\infty dx \frac{x^n D_{U,L}(x)}{[1 + \beta^2 x^2] \left[1 + \frac{\pi^{1/2}}{\kappa^2} x D_{L,U}(x)\right]^2} \tag{12}$$

After straigtforward but tedious algebra we derive

$$J_0^{L,U} \simeq \frac{a_{0,1}}{p} \frac{Y_{L,U} + \ln(1+Y_{L,U}) + \ln \frac{pe^{1/2}}{\beta}}{(1+Y_{L,U})^2}$$
$$\simeq \frac{a_{0,1}}{p} \begin{cases} \ln(pe^{1/2}/\beta) & \text{for } Y_{L,U} \ll 1 \\ \frac{Y_{L,U}+\ln(pY_{L,U}/\beta)}{Y_{L,U}^2} & \text{for } Y_{L,U} \gg 1 \end{cases} \quad (13)$$

and

$$J_1^{L,U} \simeq \frac{a_{0,1}}{p^2(1+Y_{L,U})^2} \left[\frac{2p}{\beta} + \frac{Y_{L,U}-1}{Y_{L,U}+1} \ln \frac{p(1+Y_{L,U})}{\beta}\right]$$
$$\simeq \frac{a_{0,1}}{p^2} \begin{cases} \frac{2p}{\beta} & \text{for } Y_{L,U} \ll 1 \\ \frac{(2p/\beta)+\ln(pY_{L,U}/\beta)}{Y_{L,U}^2} & \text{for } Y_{L,U} \gg 1 \end{cases} \quad (14)$$

with $Y_{L,U}(\kappa) = \pi^{1/2} a_{1,0}/[p\kappa^2]$. The asymptotic expansions for small and large values of $Y_{L,U}$ correspond to large and small values of the normalized wavenumber, respectively.
According to Eq. (9) the wavenumber power spectra $k^2 < \delta B^2 >_k^{L,U} \propto J_0^{U,L}(\beta,\kappa)$ to leading order increase $\propto \kappa^2$ at small normalized wavelength $\kappa \leq (a_{1,0}\pi^{1/2}/p)^{1/2}$ and approach constants at large κ. The constants at large values of κ provide the dominating contribution to the remaining κ-integral in Eq. (6). We find in this case

$$|\delta B|_L = \sqrt{\delta B_L^2} = 3.5\beta_e g^{1/3} W_e^{1/2} = 1.8 \cdot 10^{-16} T_4^{1/2} n_{-7}^{2/3}\ \mathsf{G}, \qquad (15)$$

providing $|\delta B| \simeq 2 \cdot 10^{-16} T_4^{1/2}$ G in cosmic voids and $|\delta B| \simeq 2 \cdot 10^{-10} T_4^{1/2}$ G in protogalaxies.

These guaranteed magnetic fields in the form of randomly distributed fluctuations, produced by the spontaneous emission of the isotropic thermal IGM plasma, may serve as seed fields for possible amplification by later possible plasma instabilities from anisotropic plasma particle distribution functions, MHD instabilities and/or the MHD dynamo process.

We note that the strength of the guaranteed spontaneously emitted magnetic seed fields (15) is significantly larger than the seed fields from the Biermann (1950) battery process (10^{-18} G) and cosmological phase transitions (Sigl et al. 1997) (10^{-20} G). They also have a nearly 100 percent volume filling factor.

These spontaneously emitted fluctuations have typical plasma scale lengths $\leq 10^{10} n_{-7}^{-1/2}$ cm, but as argued below, the first hydrodynamical compression generates considerably longer correlation lengths of the compressed magnetic fields determined by the spatial scale of the compressor.

2.2. Influence of viscous damping

While generated continuously by spontaneous emission, the turbulent magnetic field also experiences strong dissipation at small scales by viscous damping from collisional processes with the damping rate $\gamma_v(k) = 0.18\beta_e^2 k^2 c^2/(\omega_{p,e} g \mu^2)$ Hz (Braginskii 1965).

The equilibrium magnetic field fluctuation spectrum from continuous spontaneous emission and viscous damping is given by $< \delta B^2 >_{\rm eq} (k) = P(k)/\gamma_v(k)$, where

$$P(k) = 2\int_0^\infty d\gamma\, \gamma U(k,\gamma) = \frac{\omega_{p,e} m_e c^2 \beta_e^3}{2\pi^{5/2}\kappa} J_1(\beta_e,\kappa) \tag{16}$$

is the magnetic field power radiated per unit volume at k due to spontaneous emission, and J_1 the integral (10) for $n=1$. Consequently, the equilibrium magnetic field fluctuation power spectrum is given by

$$< \delta B^2 >_{\rm eq} (k) = \frac{\mu^2 m_e c^2 g \beta_e J_1(\beta_e,\kappa)}{0.36\pi^{5/2}\kappa^3}, \tag{17}$$

so that in this case

$$(\delta B)^2_{\rm eq} = 2 m_p c^2 \beta_e g \left(\frac{\omega_{p,e}}{c}\right)^3 \int_0^{\frac{2\pi c n_e^{1/3}}{\omega_{p,e}}} d\kappa\, \frac{J_1(\beta_e,\kappa)}{\kappa} \tag{18}$$

With the asymptotics (14) we find for the maximum and minimum equilibrium magnetic field strength $|\delta B_{\rm eq,U}| = 1.18 |\delta B_{\rm eq,L}|$ and

Introduction
Plasma fluctuations
From disordered to . . .
Summary and . . .
Fluctuation theory . . .

$$|\delta B_{\mathrm{eq,L}}| = 1.46 \cdot 10^{-10} g^{1/2} n_e^{3/4} \left[15.1 - \frac{\ln n_e}{6} \right]^{1/2} = 1.7 \cdot 10^{-21} n_{-7} T_4^{-3/4} \text{ G}, \tag{19}$$

providing $|\delta B| \simeq 2 \cdot 10^{-21} T_4^{-3/4}$ G in cosmic voids and $|\delta B| \simeq 2 \cdot 10^{-12} T_4^{-3/4}$ G in protogalaxies. Accounting for the additional viscous damping reduces the equilibrium magnetic field strength by about 5 and 2 orders of magnitude in cosmic voids and protogalaxies, respectively, as compared to the estimate (15).

In cosmic voids and protogalaxies the spontaneously emitted seed magnetic fields are too weak to affect the dynamics of the IGM plasma, as the small values of the associated turbulent plasma beta $\beta_t \geq 10^{13}$ in cosmic voids and $\beta_t \geq 10^{10}$ in protogalaxies indicate. Because of these ultrahigh turbulent plasma beta values, the seed fields are tied passively to the highly conducting IGM plasma as frozen-in magnetic fluxes.

3. From disordered to ordered magnetic field structures

As we demonstrated the unmagnetized, isotropic, thermal and steady IGM plasma by spontaneous emission generates steady tangled fields, isotropically distributed in direction, on small spatial scales $\leq 10^{10} n_7^{-1/2}$ cm (corresponding to $k \geq \omega_{p,e}/c$). Because of its ultrahigh turbulent plasma beta value, these seed fields are too weak to affect the dynamics of the IGM plasma, but are tied passively to the highly conducting IGM plasma as frozen-in magnetic fluxes.

Earlier analytical considerations and numerical simulations (Laing 1980, Hughes et al. 1985, Matthews and Scheuer 1990) showed that any shear and/or compression of the IGM medium enormously amplify these seed magnetic fields and make them anisotropic. Depending on the specific exerted compression and/or shear, even one-dimensional ordered magnetic field structures can be generated out of the original isotropically tangled field configuration.

Hydrodynamical compression or shearing of the IGM medium arises from the shock waves of the supernova explosions of the first stars at the end of their lifetime, or from supersonic stellar and galactic winds. Fig. 2 sketches the basic physical process.

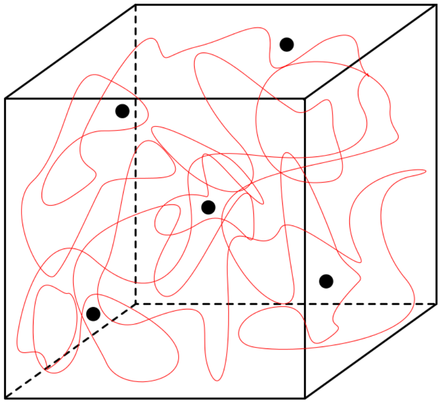
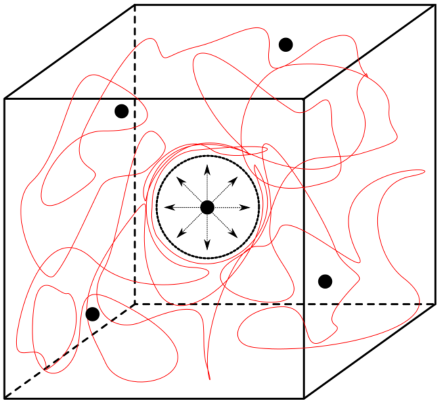

Figure 2: Illustration of the hydrodynamical stretching and ordering of cosmic magnetic fields. On the left figure a turbulent random magnetic field pervades the medium between five protostars. The right figure shows the ordering and stretching of the magnetic field as one of the stars explodes as a supernova. The outgoing shock wave compresses and orders the magnetic field in its vicinity.

The IGM seed magnetic field upstream of these shocks is random in direction, and by solving the hydrodynamical shock structure equations for oblique and conical shocks it has been demonstrated (Cawthorne and Comb 1990, Cawthorne 2006) that the shock compression enhances the downstream magnetic field component parallel to the shock, but leaving the magnetic field component normal to the shock unaltered. Consequently, a more ordered downstream magnetic field structure results from the randomly oriented upstream field.

Obviously, this magnetic field stretching and ordering occurs only in gas regions overrun frequently by shocks and winds. Each individual shock or wind (with speed V_s) compression orders the field on spatial scales R on time scales given by the short shock crossing time R/V_s, but signifant amplification requires multiple compressions. The ordered magnetic field filling factor is determined by the shock's and wind's filling factors which are large (80 percent) in the coronal phase of interstellar media (McKee and Ostriker 1977) and near shock waves in large-scale cosmic structures (Miniati et al. 2000). In cosmic regions with high shock/wind activity, this passive hydrodynamical amplification and stretching of magnetic fields continues until the magnetic restoring forces affect the gas dynamics, i.e. at ordered plasma betas near unity. As a consequence, magnetic fields with equipartition strength are not generated uniformly over the whole universe by this process, but only in localized cosmic regions with high shock/wind activity. It continues until the magnetic restoring forces affect the gas dynamics, i.e. at ordered plasma betas near unity.

In protogalaxies significant and rapid amplification of the spontaneously emitted aperiodic turbulent magnetic fields results from the small-scale kinetic dynamo process (Brandenburg and Subramanian 2005) generated by the gravitational infall motions during the formation of the first stars (Schleicher et al. 2010). Additional gaseous spiral motion may stretch and order the magnetic field on large protogalactic spatial scales.

Our suggested mechanism of spontaneously emitted aperiodic turbulent magnetic fields should also operate during earlier cosmological epochs before recombination, and in fully-ionized stellar interiors (Do you need seed fields for stellar dynamos, and if yes, where do they come from?).

Felten et al. (2013) also calculated the fluctuation spectra for ultrarelativistic plasma temperatures shown in Fig. 3. With $I=\gamma/kc$ and $\mu_a = m_a c^2/k_B T_a$

$$< \delta B_\perp^2 >_{k,I} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c}$$

$$\times \frac{|(1+I^2)[\text{arctan}\frac{\sqrt{1-\mu_a^2}}{I} - \frac{\pi\sigma}{2}] - I|}{|1+I^2+\sum_a \frac{\omega_{p,a}^2 \mu_a I}{2k^2c^2}((1+I^2)[\text{arctan}\frac{\sqrt{1-\mu_a^2}}{I} - \frac{\pi\sigma}{2}] - I)|^2}, \tag{20}$$

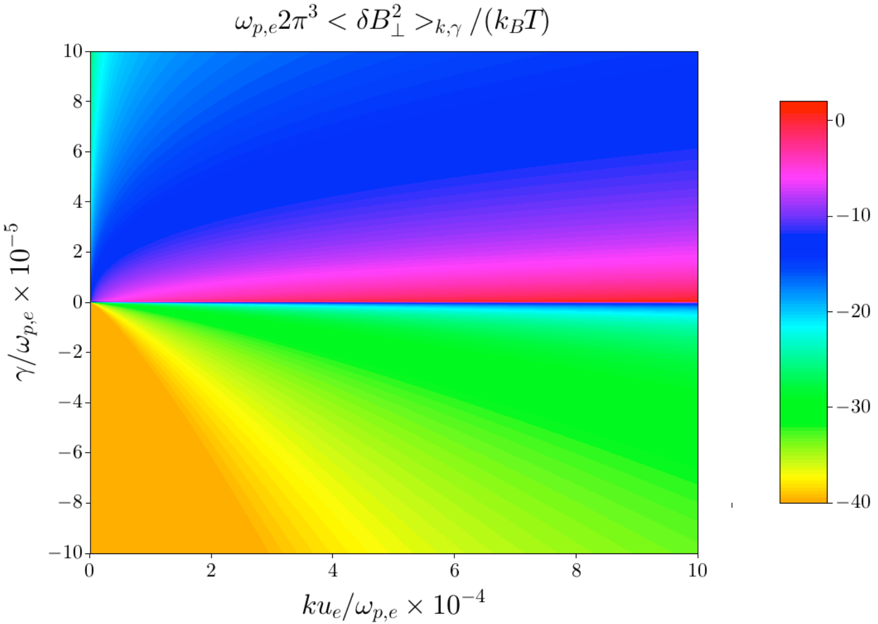

Figure 3: Colour plot of the spontaneously emitted aperiodic magnetic field fluctuation spectrum in a thermal ultrarelativistic electron-proton distribution in units of $k_B T_e/(2\pi^3 \omega_{p,e})$. Equal electron and proton temperatures ($T_i = T_e$) and $\beta_e = u_e/c = 10^5$ are adopted. The colour scale is logarithmic in powers of e.

4. Summary and conclusions

- An unmagnetized nonrelativistic thermal electron-proton plasma spontaneously emits aperiodic turbulent magnetic field fluctuations of strength $|\delta B| = 9\beta_e g^{1/3} W_e^{1/2}$ G, where β_e is the normalized thermal electron velocity in units of c, W_e the thermal plasma energy density and g the plasma parameter.

- For the unmagnetized intergalactic medium, immediately after the reionization onset, the field strength from this mechanism is about $4.7 \cdot 10^{-16}$ G, too weak to affect the dynamics of the plasma.

- The shear and/or compression of the intergalactic medium exerted by the first supernova explosions amplify these seed fields and make them anisotropic, until the magnetic restoring forces affect the gas dynamics at ordered plasma betas near unity.

- The suggested mechanism of spontaneously emitted aperiodic turbulent seed magnetic fields should also operate during earlier cosmological epochs before recombination and in fully-ionized stellar interiors.

- I thank my collaborators Tim Felten, Marian Lazar and Peter Yoon. Financial support by the Deutsche Forschungsgemeinschaft, Alexander-von-Humboldt-Stiftung, Projektförderung Mercator Research Center Ruhr (MERCUR) and Verbundforschung Astroteilchenphysik is gratefully acknowledged.

5. Fluctuation theory in magetized plasmas

5.1. Basic equations

The electromagnetic fields fulfill Maxwell equations

$$\nabla \times \vec{B}(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\,\vec{E}(\vec{x},t) = \frac{4\pi}{c}\sum_a q_a \int d^3p\; \vec{v}\, f_a(\vec{x},\vec{p},t), \tag{21}$$

$$\nabla \cdot \vec{B}(\vec{x},t) = 0, \quad \nabla \times \vec{E}(\vec{x},t) + \frac{1}{c}\frac{\partial}{\partial t}\,\vec{B}(\vec{x},t) = 0. \tag{22}$$

$$\nabla \cdot \vec{E}(\vec{x},t) = 4\pi \sum_a q_a \int d^3p\; f_a(\vec{x},\vec{p},t), \tag{23}$$

where the charged particle's phase space distributions $f_a(\vec{x},\vec{p},t)$ determine the current and charge densities on the right-hand side of Eqs. (21) and (23).

Each phase space distribution $f_a(\vec{x},\vec{p},t)$ of ionized particles fulfills

$$\frac{\partial f_a}{\partial t} + \;\vec{v}\cdot\frac{\partial f_a}{\partial \vec{x}} + q_a[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}]\cdot\frac{\partial f_a}{\partial \vec{p}} = Q_a(\vec{x},\vec{p},t), \tag{24}$$

where the source term $Q_a(\vec{x},\vec{p},t)$ accounts for sources and sinks of particles and other (than the Lorentz force) electromagnetic interactions. Via the Lorentz force the electromagnetic fields determine the behaviour of the particle distribution functions f_a.

5.2. Linear fluctuations

We linearize the system of Maxwell's equations and the collisionless Boltzmann equation by investigating small deviations

$$f_a(\vec{x},\vec{p},t) = f_{a0}(\vec{p}) + \delta f_a(\vec{x},\vec{p},t),\ \vec{B}(\vec{x},t) = \vec{B}_0(\vec{x},t) + \delta\vec{B}(\vec{x},t),$$
$$\vec{E}(\vec{x},t) = 0 + \delta\vec{E}(\vec{x},t) \tag{25}$$

around the adopted "equilibrium" state of a spatially uniform and stationary magnetized plasma $(f_{a0}(\vec{p}), \vec{B}_0(\vec{x},t))$ with vanishing ordered electric field $\vec{E}_0(\vec{x},t) = 0$. Small fluctuations mean that $|\delta f_a| \ll f_{a0}$ and $|\delta\vec{B}| \ll B_0$.

Using that the equilibrium state fulfils Eqs. (21) - (24), and neglecting higher order terms in fluctuating quantities in the collisionless Boltzmann equation, results in the linearized equations e.g.

$$\nabla\times\delta\vec{B}(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\delta\vec{E}(\vec{x},t) = \frac{4\pi}{c}\sum_a q_a\int d^3p\ \vec{v}\,\delta f_a(\vec{x},\vec{p},t), \tag{26}$$

$$\left[\frac{\partial}{\partial t} + \ \vec{v}\cdot\frac{\partial}{\partial\vec{x}} + q_a\frac{\vec{v}\times\vec{B}_0}{c}\cdot\frac{\partial}{\partial\vec{p}}\right]\left[\delta f_a(\vec{x},\vec{p},t) - \delta N_a^0(\vec{x},\vec{p},t)\right]$$
$$= -q_a[\delta\vec{E} + \frac{\vec{v}\times\delta\vec{B}}{c}]\cdot\frac{\partial f_{a0}(\vec{p})}{\partial\vec{p}}, \tag{27}$$

where the Klimontovich perturbation $\delta N_a^0(\vec{x},\vec{p},t)$ accounts for the near interactions of neighbouring uncorrelated charged particles in the plasma.

We introduce the Fourier-Laplace transforms of the fluctuations and its inverse

$$\begin{pmatrix} f_a(\vec{k},\omega,\vec{p}) \\ N_a(\vec{k},\omega,\vec{p}) \\ \vec{E}(\vec{k},\omega) \\ \delta\vec{B}(\vec{k},\omega) \end{pmatrix} = \int d^3x \int_0^\infty dt \begin{pmatrix} \delta f_a(\vec{x},\vec{p},t) \\ \delta N_a^0(\vec{x},\vec{p},t) \\ \delta\vec{E}(\vec{x},t) \\ \delta\vec{B}(\vec{x},t) \end{pmatrix} \exp\left[-\imath(\vec{k}\cdot\vec{x}-\omega t)\right], \quad (28)$$

$$\begin{pmatrix} \delta f_a(\vec{x},\vec{p},t) \\ \delta N_a^0(\vec{x},\vec{p},t) \\ \delta\vec{E}(\vec{x},t) \\ \delta\vec{B}(\vec{x},t) \end{pmatrix} = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \begin{pmatrix} f_a(\vec{k},\omega,\vec{p}) \\ N_a(\vec{k},\omega,\vec{p}) \\ \vec{E}(\vec{k},\omega) \\ \delta\vec{B}(\vec{k},\omega) \end{pmatrix} \exp\left[\imath(\vec{k}\cdot\vec{x}-\omega t)\right] \quad (29)$$

with the restriction $\Im\omega = \gamma > 0$ for convergence reasons.

Applying the inverse Fourier-Laplace transforms (29) the transformed linearized equations (26) - (27) provide the wave equation

$$\Lambda_{ij}(\vec{k},\omega)E_j(\vec{k},\omega) = E_i^0(\vec{k},\omega) = -\frac{4\pi\imath}{\omega}\sum_a q_a \int d^3p v_i N_a(\vec{k},\omega,\vec{p}) \qquad (30)$$

with the Maxwell operator in terms of the dielectric tensor $\psi_{ij}(\vec{k},\omega)$

$$\Lambda_{ij}(\vec{k},\omega) = \frac{k^2c^2}{\omega^2}\left[\frac{k_ik_j}{k^2} - \delta_{ij}\right] + \psi_{ij}(\vec{k},\omega) \qquad (31)$$

We adopt without loss of generality a coordinate system, where $\vec{B}_0 = B_0\vec{z}$ and

$$\vec{k} = (k_\perp, 0, k_\parallel) = k(\sin\Theta, 0, \cos\Theta) \qquad (32)$$

and introduce cylindrical momentum coordinates $p_x = p_\perp\cos\phi, p_y = p_\perp\sin\phi, p_z = p_\parallel$. The dielectric tensor in gyrotropic equilibrium plasma distributions $f_{a0}(\vec{p}) = n_a f_a^{(0)}(p_\parallel, p_\perp)$ then reads (RS 2010)

$$\psi_{ij}(\omega,\vec{k}) = \delta_{ij}+\delta_{i3}\delta_{j3}\frac{2\pi}{\omega^2}\sum_a \omega_{p,a}^2 m_a \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp v_\parallel \left(p_\perp \frac{\partial f_a^{(0)}}{\partial p_\parallel} - p_\parallel \frac{\partial f_a^{(0)}}{\partial p_\perp}\right)$$

$$+\frac{2\pi \sum_a \omega_{p,a}^2 m_a^2}{\omega}\int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \gamma \left[(1-\frac{k_\parallel v_\parallel}{\omega})\frac{\partial f_a^{(0)}}{\partial p_\perp} + \frac{k_\parallel v_\perp}{\omega}\frac{\partial f_a^{(0)}}{\partial p_\parallel}\right] T_{ij}(\omega,\vec{k}) \tag{33}$$

where

$$T_{ij}(\omega,\vec{k}) = \sum_{n=-\infty}^{\infty} \frac{1}{\omega - k_\parallel v_\parallel + n\Omega_a}$$
$$\times \begin{pmatrix} \frac{n^2 J_n^2(z)}{z^2} v_\perp^2 & -\frac{\imath n J_n(z) J_n'(z)}{z} v_\perp^2 & -\frac{n J_n^2(z)}{z} v_\parallel v_\perp \\ \frac{\imath n J_n(z) J_n'(z)}{z} v_\perp^2 & (J_n'(z))^2 v_\perp^2 & -\imath J_n(z) J_n'(z) v_\parallel v_\perp \\ -\frac{n J_n^2(z)}{z} v_\parallel v_\perp & \imath J_n(z) J_n'(z) v_\parallel v_\perp & J_n^2(z) v_\parallel^2 \end{pmatrix} \tag{34}$$

Here $\omega_{p,a} = (4\pi n_a e_a^2/m_a)^{1/2}$ denotes the plasma frequency, $z = k_\perp v_\perp/\Omega_a$, where $\Omega_a = \epsilon_a|\Omega_a| = sign(q_a)\Omega_{0a}/\gamma$ is the relativistic gyrofrequency of particles of sort a, $\epsilon_a = e_a/|e_a|$ the charge sign, and $\Omega_{0a} = |e_a|B_0/m_a c$ is the absolute value of the nonrelativistic gyrofrequency of particles of sort a. $J_n(z)$ denotes the Bessel function of order n and $J_n'(z) = \frac{\partial J_n(z)}{\partial z}$ its first derivative.

We use the spatial two-time correlation function of uncorrelated particles in a magnetized plasma

$$(2\pi)^4 < N_a N_b^* > (\vec{k},\omega) =$$
$$\delta_{ab}\Re\left(\int_0^\infty d\tau e^{\imath\vec{k}\cdot(\vec{x}_a(-\tau)-\vec{x}_b)+\imath\omega\tau}\delta[\vec{p}_a(\tau)-\vec{p}_b]n_a f_a^{(0)}(p_\parallel,p_\perp)\right), \tag{35}$$

with the unperturbed gyromotion in the uniform magnetic field

$$\vec{p}_a(-\tau) = \begin{pmatrix} p_\perp \cos(\phi+\Omega_a\tau) \\ p_\perp \sin(\phi+\Omega_a\tau) \\ p_\parallel \end{pmatrix}, \tag{36}$$

implying

$$\vec{x}_a(-\tau)-\vec{x}_b = \begin{pmatrix} \frac{v_\perp}{\Omega_a}(\sin(\phi+\Omega_a\tau)-\sin\phi) \\ -\frac{v_\perp}{\Omega_a}(\cos(\phi+\Omega_a\tau)-\cos\phi) \\ -v_\parallel\tau \end{pmatrix} \tag{37}$$

to calculate according to Eq. (30) with the wavevector orientation (32)

$$< E_i^0 E_j^{0*} > (\vec{k},\omega) = \frac{16\pi^2}{|\omega|^2} \sum_a \sum_b q_a q_b \int d^3p_a \int d^3p_b < N_a N_b > (\vec{k},\omega) v_{ai} v_{jb}$$

$$= \frac{\sum_a \omega_{p,a}^2 m_a}{4\pi^3 |\omega|^2} |\Re\Big(\int d^3p f_a^{(0)}(p_\parallel, p_\perp) v_i v_j \int_0^\infty d\tau e^{\imath(\omega - k_\parallel v_\parallel)\tau + \imath \frac{k_\perp v_\perp}{\Omega_a}[\sin(\phi+\Omega_a\tau) - \sin\phi]}\Big)|$$

$$= \frac{\sum_a \omega_{p,a}^2 m_a}{4\pi^3 |\omega|^2} |\Re\Big(\int d^3p f_a^{(0)}(p_\parallel, p_\perp) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(z) J_m(z) e^{\imath(n-m)\phi}$$

$$\times \int_0^\infty d\tau v_i(\phi + \Omega_a\tau) v_j(\phi) e^{\imath(\omega - k_\parallel v_\parallel + n\Omega_a)\tau}\Big)|, \tag{38}$$

where we used the Bessel function identity

$$e^{\imath z \sin\eta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{\imath n\eta} \tag{39}$$

With the gyrotropic distribution function

$$f_a^{(0)}(p_\parallel, p_\perp) = \frac{1}{2\pi} F_a(p_\parallel, p_\perp) \tag{40}$$

we obtain

$$< E_i^0 E_j^{0*} > (\vec{k}, \omega) = \frac{\sum_a \omega_{p,a}^2 m_a}{4\pi^3 |\omega|^2} |K_{ij}(\vec{k}, \omega)| \tag{41}$$

with the form factors defined by

$$K_{ij}(\vec{k}, \omega) = \frac{1}{2\pi} \Re \Big(\int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp p_\perp F_a(p_\parallel, p_\perp) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(z) J_m(z)$$

$$\times \int_0^\infty d\tau e^{\imath(\omega - k_\parallel v_\parallel + n\Omega_a)\tau} \int_0^{2\pi} d\phi v_i(\phi + \Omega_a \tau) v_j(\phi) e^{\imath(n-m)\phi} \Big) \tag{42}$$

Performing the ϕ-integration in Eq. (42) provides

$$K_{ij}(\vec{k}, \omega) = \Re \left(\imath \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp p_\perp F_a(p_\parallel, p_\perp) T_{ij}(\vec{k}, \omega) \right) \tag{43}$$

with the same tensor (34), which reflects the general dissipation-fluctuation theorem.

5.3. Unmagnetized plasma

In a vanishing magnetic field, it is not necessary to adopt two non-zero components ($k_\perp$ and $k_\parallel$) of the wave vector. We rather set $k_\perp = 0$ and identify the z-direction with the wave vector direction. We therefore consider the limit $k_\perp \to 0$, corresponding to $z \to 0$ and $R \to 0$, together with $\Omega_a = 0$. We find for the form factors

$$K_{11} = K_{22} = \frac{1}{2}K_\perp = \Re\Big[\imath \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\perp v_\perp^2 F_a(p_\parallel, p_\perp)}{\omega - k_\parallel v_\parallel}\Big], \tag{44}$$

$$K_{33} = K_\parallel = \Re\Big[\imath \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\perp v_\parallel^2 F_a(p_\parallel, p_\perp)}{\omega - k_\parallel v_\parallel}\Big] \tag{45}$$

Likewise the only non-vanishing components of the dielectric tensor are

$$\psi_{ij}(\omega, \vec{k}) = \delta_{ij} + \chi_{ij} = \delta_{ij} + \delta_{i3}\delta_{j3}\frac{\sum_a \omega_{p,a}^2}{\omega^2}\int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\parallel}{\gamma}\left(p_\perp \frac{\partial F_a}{\partial p_\parallel} - p_\parallel \frac{\partial F_a}{\partial p_\perp}\right)$$

$$+ \frac{\sum_a \omega_{p,a}^2}{2\omega^2}\int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\perp^2}{\gamma}\left[\frac{\partial F_a}{\partial p_\perp} + \frac{k_\parallel v_\perp}{\omega - k_\parallel v_\parallel}\frac{\partial F_a}{\partial p_\parallel}\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2p_\parallel^2}{p_\perp^2} \end{pmatrix}, \tag{46}$$

providing for the Maxwell operator (31) with $k = k_\parallel$

$$\Lambda_{ij} = \begin{pmatrix} \Lambda_T & 0 & 0 \\ 0 & \Lambda_T & 0 \\ 0 & 0 & \Lambda_L \end{pmatrix} \tag{47}$$

with the transverse

$$\begin{aligned} \Lambda_T(k,\omega) &= 1 - \frac{c^2k^2}{\omega^2} \\ &+ \frac{\sum_a \omega_{p,a}^2}{2\omega^2} \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\perp^2}{\gamma} \left[\frac{\partial F_a}{\partial p_\perp} + \frac{k v_\perp}{\omega - k v_\parallel} \frac{\partial F_a}{\partial p_\parallel} \right] \end{aligned} \tag{48}$$

and longitudinal dispersion functions

$$\Lambda_L(k,\omega) = 1 + \frac{\sum_a \omega_{p,a}^2}{\omega} \int_{-\infty}^{\infty} dp_\parallel p_\parallel \int_0^\infty dp_\perp \frac{p_\perp}{\gamma(\omega - k v_\parallel)} \frac{\partial F_a}{\partial p_\parallel} \tag{49}$$

5.4. Spontaneously emitted fluctuations

The solution of the wave equation (30) together with Eq. (41) then provide for the energy in electric and magnetic field fluctuations in unmagnetized plasmas generated per unit volume at k and ω due to spontaneous emission

$$\begin{pmatrix} < \delta E_{\parallel}^2 >_{k,\omega} \\ < \delta E_{\perp}^2 >_{k,\omega} \\ < \delta B^2 >_{k,\omega} \end{pmatrix} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3} \begin{pmatrix} \frac{|K_{\parallel}(k,\omega)|}{|\omega\Lambda_L(k,\omega)|^2} \\ \frac{|K_{\perp}(k,\omega)|}{|\omega\Lambda_T(k,\omega)|^2} \\ \frac{c^2k^2|K_{\perp}(k,\omega)|}{|\omega^2\Lambda_T(k,\omega)|^2} \end{pmatrix}, \tag{50}$$

where we have introduced the total transverse electric field component

$$< \delta E_{\perp}^2 >_{k,\omega} = < \delta E_{11,k,\omega}^2 > + < \delta E_{22,k,\omega}^2 > \tag{51}$$

and used the induction law $\vec{B}(k,\omega) = (c/\omega)\vec{k} \times \vec{E}(k,\omega)$.

The form factors (44) - (45) are the generalizations of the standard expressions found in the literature (Salpeter 1960, Sitenko 1967, Ichimaru 1973, Kegel 1998) in which the weak amplification limit of $\Im(\omega) = \gamma \to 0^+$ is taken at the outset to approximate by the Dirac-formula

$$\lim_{\gamma \to 0^+} \frac{1}{\omega_R + \imath\gamma - \vec{k}\cdot\vec{v}} = \pi\, \delta(\omega_R - \vec{k}\cdot\vec{v}) \tag{52}$$

Coulomb's law then provides for the charge density fluctuations

$$< \delta\rho^2 >_{k,\omega} = \frac{k^2 < \delta E_\parallel^2 >_{k,\omega}}{(4\pi)^2} = \sum_a \frac{\omega_{p,a}^2 m_a k^2}{2(2\pi)^5} \frac{|K_\parallel(k,\omega)|}{|\omega\Lambda_L(\vec{k},\omega)|^2}, \tag{53}$$

and the continuity equation $\dot{\rho} + \text{div}\vec{j} = 0$ yields for the parallel current density fluctuations

$$< \delta J_\parallel^2 >_{k,\omega} = \frac{|\omega|^2 < \delta\rho^2 >_{k,\omega}}{k^2} = \sum_a \frac{\omega_{p,a}^2 m_a}{2(2\pi)^5} \frac{|K_\parallel(k,\omega)|}{|\Lambda_L(\vec{k},\omega)|^2} \tag{54}$$

Ampere's law gives the perpendicular current density fluctuations

$$\begin{aligned} < \delta J_\perp^2 >_{k,\omega} &= \frac{|\omega|^2}{(4\pi)^2} \left[1 + \frac{c^4 k^4}{|\omega|^4}\right] < \delta E_\perp^2 >_{k,\omega} \\ &= \sum_a \frac{\omega_{p,a}^2 m_a}{2(2\pi)^5} \left[1 + \frac{c^4 k^4}{|\omega|^4}\right] \frac{|K_\perp(k,\omega)|}{|\Lambda_T(\vec{k},\omega)|^2} \end{aligned} \tag{55}$$

The form factors and dispersion functions determine all electromagnetic fluctuation spectra. Most of the earlier work concentrated on weakly damped fluctuations.

Using the isotropic thermal Maxwell-Jüttner distribution functions

$$F_a(E) = e^{-\mu_a E}, \quad \mu_a = \frac{m_a c^2}{k_B T}, \quad E = \sqrt{1 + \frac{p_\parallel^2 + p_\perp^2}{m_a^2 c^2}}, \tag{56}$$

with the normalization factor

$$N_a = \frac{\mu_a}{2(m_a c)^3 K_2(\mu_a)}, \tag{57}$$

to calculate the form factors and the dispersion functions then provides in the limits $\mu_a \gg 1$ and $\mu \ll 1$ the respective fluctuation spectra for all cases of interest.