Extreme Value Statistics, Integer Partitions and Bose Gas

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- A brief review on Extreme Value Statistics of i.i.d random variables

$\Longrightarrow$ three limiting distributions: GUMBEL, FRÉCHET & WEIBULL

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- Summary and Conclusions

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Prob. density: $G'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \Longrightarrow$ GAUSSIAN

$\Longrightarrow$ LAW OF AVERAGES

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  a_N, b_N $\rightarrow$ Scale factors dependent on $p(x)$

$F(z) \rightarrow$ Scaling function: only of 3 possible varieties (depending on the tails of $p(x)$) $\Longrightarrow$ LAW OF EXTREMES

[Fréchet (1927), Fisher and Tippet (1928), Gnedenko (1943), Gumbel (1958)...] $\rightarrow$ Several applications (Climate, Finance, Oceanography.....)

Three types of Scaling Functions:

Type I (GUMBEL): If $p(x)$ is unbounded with faster than power law tail (e.g., exponential)

$$F_I(z) = \exp[-e^{-z}]$$

Type II (FRÉCHET): If $p(x)$ has power law tails: $p(x) \sim x^{-(\gamma+1)}$

$$\begin{aligned} F_{II}(z) &= 0 & z \leq 0 \\ &= \exp[-z^{-\gamma}] & z \geq 0 \end{aligned}$$

Type III (WEIBULL): If $p(x)$ is bounded: $p(x) \sim (1-x)^{(\gamma-1)}$

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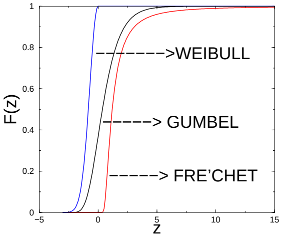

Integer Partition Problem

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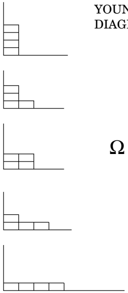

YOUNG DIAGRAMS

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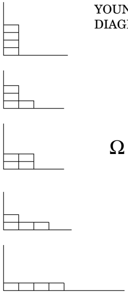

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Classical result of Hardy-Ramanujam (1918): for large E,

$$\Omega(E) \sim \exp\left[\pi\sqrt{\frac{2}{3}}E^{1/2}\right]$$

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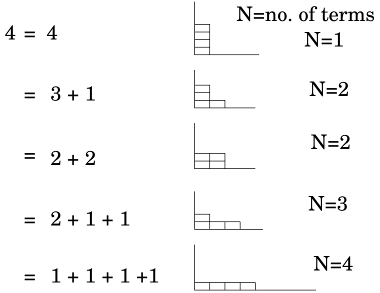

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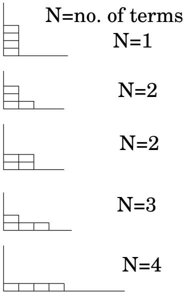
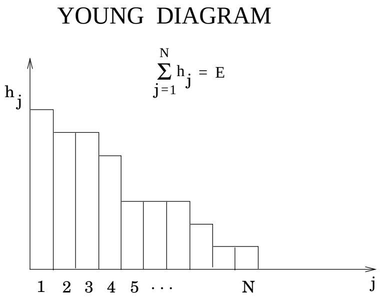


Question: $\Omega(N, E) =$ No. of partitions of E with N terms $=$?
Example: $\Omega(1, 4) = 1$, $\Omega(2, 4) = 2$, $\Omega(3, 4) = 1$, $\Omega(4, 4) = 1$

Young Diagram: A Combinatorial Problem

Arrange N columns of non-increasing heights: $h_1 \geq h_2 \geq \ldots \geq h_N > 0$



• $\Omega(N, E) =$ No. of ways of arranging N non-increasing columns such that

$$\sum_{j=1}^{N} h_j = E$$

• $\Omega(E) = \sum_{N} \Omega(N, E)$

All Partitions Equally Likely: Distribution of N

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$$Q(N|E) = \sum_{N'=0}^{N} P(N'|E) \to F_I\left[\frac{\pi}{\sqrt{6E}}(N - N^*(E))\right]$$

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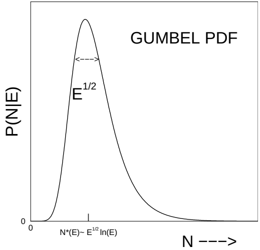

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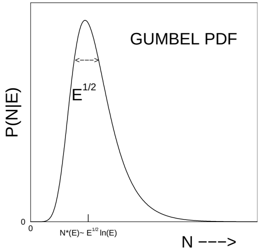

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If Yes: What is the connection between EVS and the partition problem, if there is any!

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Example: Partitions of $E = 4$

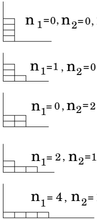
$4 = 4$ — $n_1=0, n_2=0, n_3=0, n_4=1$

$= 3 + 1$ — $n_1=1, n_2=0, n_3=1, n_4=0$

$= 2 + 2$ — $n_1=0, n_2=2, n_3=0, n_4=0$

$= 2 + 1 + 1$ — $n_1=2, n_2=1, n_3=n_4=0$

$= 1 + 1 + 1 + 1$ — $n_1=4, n_2=n_3=n_4=0$



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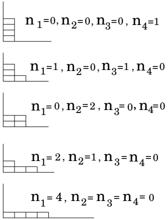
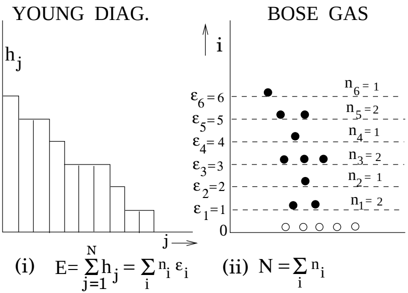


$E =$ Total excitation energy of the Bose gas $= \sum_i n_i \epsilon_i$ where $\epsilon_i = i$.

- $\epsilon_i = i \rightarrow$ single particle energy levels (equidistant)
- $n_i \rightarrow$ no. of bosons at level ϵ_i where $n_i = 0, 1, 2, \ldots$.

Integer Partition and Ideal Bose Gas

Integer Partition Problem $\longrightarrow$ Excitation Spectrum of Ideal Bose Gas



- $E = \sum_{i=1}^{\infty} n_i \epsilon_i =$ Total excitation energy
- $N = \sum_{i=1}^{\infty} n_i =$ Total no. of excited particles.

Hardy-Ramanujam via Bose Gas

$\Omega(E)$= No. of partitions of $E = \sum_{\{n_i\}} \delta\left(E - \sum_i n_i \epsilon_i\right)$

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Saddle point (for large E): $\frac{\partial S}{\partial \beta} = 0 \Rightarrow E = \sum_i \frac{\epsilon_i}{e^{\beta \epsilon_i} - 1} \to \int_0^\infty \frac{\rho(\epsilon) \epsilon\, d\epsilon}{e^{\beta \epsilon} - 1}$

Density of states: $\rho(\epsilon) = \nu \epsilon^{\nu - 1}$
Integer partition (special case): $\epsilon_i = i \Rightarrow \rho(\epsilon) = 1 \Rightarrow \nu = 1$

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Saddle point solution β_*: $\quad E = \frac{\nu \Gamma(\nu+1)\zeta(\nu+1)}{\beta_*^{\nu+1}}$.

$$\Omega(E) \sim e^{S(\beta_*, E)} \sim \exp\left[a_\nu E^{\nu/(1+\nu)}\right]$$

where $a_\nu = (\nu + 1)\nu^{-\nu/(\nu+1)}[\Gamma(\nu + 1)\zeta(\nu + 1)]^{1/(1+\nu)}$

For $\nu = 1$ (Hardy-Ramanujam): $\Omega(E) \sim \exp\left[\pi\sqrt{\frac{2}{3}}E^{1/2}\right]$

Single particle density of states:

Consider the case where for large ϵ:

$$\rho(\epsilon) \sim \nu \epsilon^{\nu-1} \text{ with } \nu > 0$$

.

Examples:

- Partition into sums of powers: $E = \sum n_i i^s$; Integer partition: $s = 1$

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- Particle in a one dimensional potential: $V(x) \sim |x|^\alpha$

  WKB approximation: $\rho(\epsilon) \sim \epsilon^{(2-\alpha)/2\alpha} \implies \nu = (\alpha + 2)/2\alpha$

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We will consider a general Bose gas with a density of states: $\rho(\epsilon) = \nu \epsilon^{\nu-1}$

Generating function–Grand canonical partition function

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Evidently, as $N \to \infty$, $C(\infty, E) = \Omega(E) =$ no. of config. with energy E.

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Generating function:

$$\sum_{N,E} C(N,E)\, e^{-\beta E}\, z^N = \prod_{i=0}^{\infty} \frac{1}{1 - z e^{-\beta \epsilon_i}}$$

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Saddle point solution (large E and N): $\frac{\partial S}{\partial \beta} = 0$ and $\frac{\partial S}{\partial z} = 0$ give

$$\begin{aligned} E &= \sum_{i=0}^{\infty} \frac{\epsilon_i}{\frac{1}{z} e^{\beta\epsilon_i} - 1} \to \int_0^{\infty} \frac{\epsilon \rho(\epsilon) d\epsilon}{\frac{1}{z} e^{\beta\epsilon} - 1} \\ N &= \sum_{i=0}^{\infty} \frac{1}{\frac{1}{z} e^{\beta\epsilon_i} - 1} \to \int_0^{\infty} \frac{\rho(\epsilon) d\epsilon}{\frac{1}{z} e^{\beta\epsilon} - 1} \end{aligned}$$

Asymptotic behaviour for large E and N

For large E and N, $C(N,E) \sim e^{S(\beta_*, z_*, E, N)}$ where β_* andz_* are solutions of the saddle points eqs.

For the case, $\rho(\epsilon) = \nu \epsilon^{\nu-1}$:

$$E = \frac{\nu \Gamma(\nu+1)}{\beta_*^{\nu+1}} \mathrm{Li}_{\nu+1}(z_*); \qquad N = \frac{\Gamma(\nu+1)}{\beta_*^{\nu}} \mathrm{Li}_{\nu}(z_*)$$

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large E and $N \Rightarrow \beta_* \rightarrow 0$ and $z_* \rightarrow 1^-$

As $z \rightarrow 1^-$:

$$\begin{aligned} \mathrm{Li}_\nu(z) &\approx \Gamma(1-\nu)\,(1-z)^{-(1-\nu)} && 0<\nu<1 \\ &= -\log(1-z) && \nu = 1 \\ &\approx \zeta(\nu) + \Gamma(1-\nu)\,(1-z)^{\nu-1} + \ldots && \nu > 1 \end{aligned}$$

Thus $\nu = 1$ plays a *critical* or *borderline* role

Three limiting distributions for $\nu = 1$, $0 < \nu < 1$ and $\nu > 1$: $\nu = 1 - - - - >$ Critical Case

Three limiting behaviours of the cumulative distribution of N given large E and single particle d.o.s $\rho(\epsilon) \sim \nu \epsilon^{\nu-1}$:

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The prob. density of N gets cut-off for $N > N_c = B_\nu E^{\nu/(1+\nu)}$ so that the cumulative distribution:

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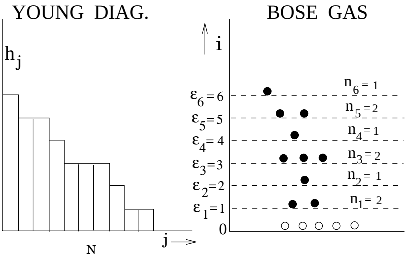
WEIBULL $\Longleftrightarrow$ Bose-Einstein condensation

$N_{ex} =$ no. of excited bosons can not exceed N_c. If the total no. of particles N exceeds N_c, all extra particles $N - N_c$ condense to the ground state $\epsilon_0 = 0$

Weibull $\longrightarrow$ Bose-Einstein Condensation



(i) $E = \sum_{j=1}^{N} h_j = \sum_i n_i \varepsilon_i$ (ii) $N = \sum_i n_i$

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Summary:

$Q(N|E)$= Prob[$N_{ex} \leq N$, given energy E]

The distribution of the no. of excited particles N_{ex} of an ideal gas of bosons with fixed total energy E and with single particle density of states $\rho(\epsilon) = \nu \epsilon^{\nu-1}$ has three possible limiting behaviours:

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Question: What is the connection to Extreme Value Statistics of i.i.d random variables, If any ?

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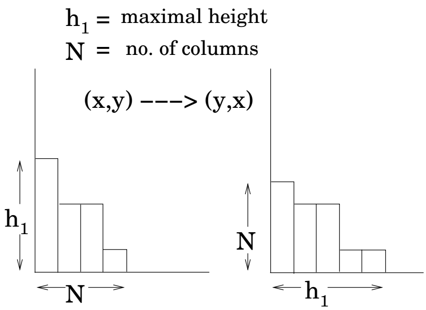

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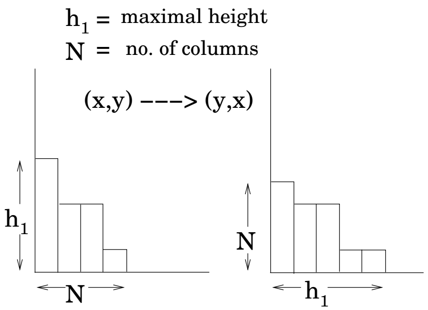

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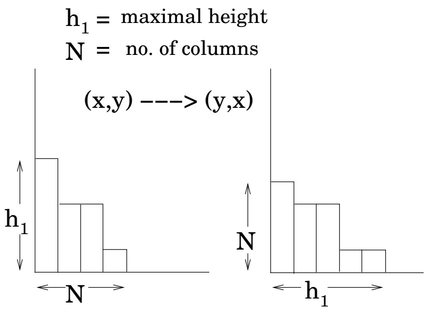

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- For $\nu \neq 1$: maximal height $h_1 \neq N$. While h_1 is still Gumbel distributed for all ν, N has Fréchet $(0 < \nu < 1)$ and Weibull $(\nu > 1)$ distributions respectively.

Origin of the three limiting distributions

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• Another example: sums of non-identical and correlated random variables can also have the 3 limiting distributions above (Bertin, 2005, Bertin and Clusel, 2006).

Summary and Conclusions:

- For Ideal Bose Gas:

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