Large Deviations of the Extreme Eigenvalues of Random Matrices

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Plan

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- The problem & its origin

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- Top eigenvalue $\lambda_{\max}$ of a Gaussian random matrix and its large deviations

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- Extension to Wishart matrices (Laguerre ensemble)

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- Prob. distr. of $\lambda_{\max}$ $\Longleftrightarrow$ Coulomb gas with a wall
- Extension to Wishart matrices (Laguerre ensemble)
- Summary and Conclusions

A Trivial Problem

DIAGONAL MATRIX

$$\begin{bmatrix} x_{11} & & & & \\ & x_{22} & & 0 & \\ & & x_{33} & & \\ & 0 & & \ddots & \\ & & & & x_{NN} \end{bmatrix}$$

$$\Pr[x_{ii} = x] = (2\pi)^{-1/2} \exp[-x^2/2]$$

$\downarrow$

GAUSSIAN

N Eigenvalues: $\lambda_i = x_{ii} \longrightarrow$ Independent

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- $P_N = \mathrm{Prob}[\lambda_1 \leq 0,\ \lambda_2 \leq 0,\ \ldots,\ \lambda_N \leq 0] = 2^{-N} = \exp[-(\ln 2)\, N]$

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[R.M. May, Nature, 238, 413 (1972)——Ecosystems]

[Cavagna et. al. 2000, Fyodorov 2004, —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006— String theory]......

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More generally, for $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

$$\boxed{P_N \sim \exp[-\beta\theta N^2]} \text{ for large } N$$

Complex Landscapes

A particle moving in a N-dimensional landscape: $V(y_1, y_2, \ldots, y_N)$

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Spin glasses, Structural glasses, Supercooled liquids, String landscapes

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- Near a stationary point: $V(\{y_i\}) \approx \sum_{i,j} H_{i,j}(y_i - y_i^*)(y_j - y_j^*)$

Hessian matrix: $H_{i,j} \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

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- Eigenvalues of the Hessian (stability) matrix determines the nature of the stationary point:

$\Longrightarrow$ Local Minimum, Local Maximum & Saddles

Eigenvalues of the Hessian Matrix

Examples:

- $N = 1$-dimensional surface: Hessian matrix $H = \frac{\partial^2 V}{\partial^2 y}$

  If $\frac{\partial^2 V}{\partial^2 y} < 0 \rightarrow$ Local Maximum; if $\frac{\partial^2 V}{\partial^2 y} > 0 \rightarrow$ Local Minimum

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  Two real eigenvalues: (λ_1, λ_2)

  If $\lambda_1 < 0$ and $\lambda_2 < 0 \rightarrow$ Local Maximum

  If $\lambda_1 > 0$ and $\lambda_2 > 0 \rightarrow$ Local Minimum

  $$\left.\begin{array}{ll} \lambda_1 < 0, & \lambda_2 > 0 \\ \lambda_1 > 0, & \lambda_2 < 0 \end{array}\right\} \quad \rightarrow \text{Saddle}$$

Fraction of Local Maximum/Minimum in Random Hessian Model

- Random $(N \times N)$ Hessian Model: $H \equiv \left[\frac{\partial^2 V}{\partial y_i \partial y_j}\right]$

  $H \to (N \times N)$ real symmetric random matrix

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- In the random Hessian model

  $P_N =$ Fraction of Local Maxima/minima

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  $= \text{Prob}[\lambda_{\max} < 0]$

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- $\longrightarrow$ Most of the stationary points $\rightarrow$ Saddles

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- Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

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- where Z_N = Partition Function

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

Coulomb Gas interpretation

- $Z_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$

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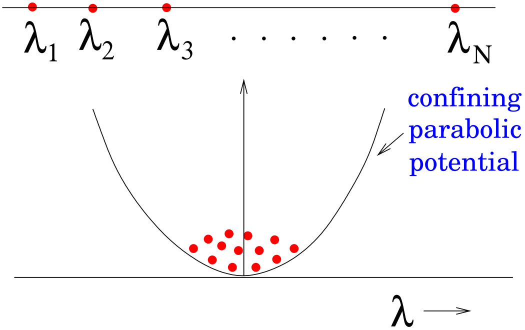
- 2-d Coulomb gas confined to a line (Dyson) with $\beta \to$ inverse temp.



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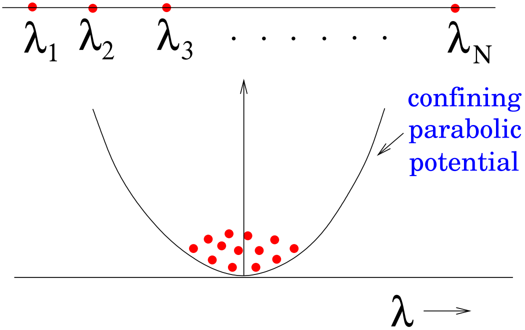

- Balance of energy $\longrightarrow N\,\lambda^2 \sim N^2$
- Typical eigenvalue: $\lambda_{\rm typ} \sim \sqrt{N}$ for large N

Spectral Density: Wigner's Semicircle Law

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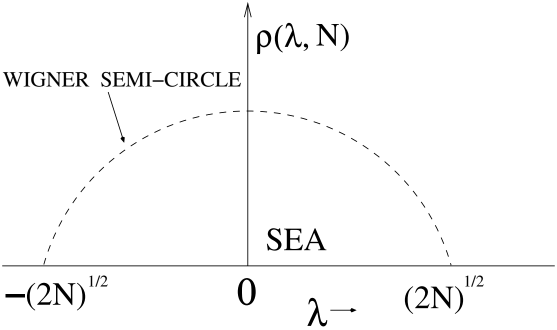

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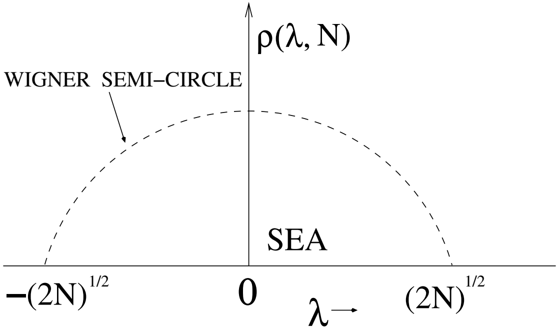
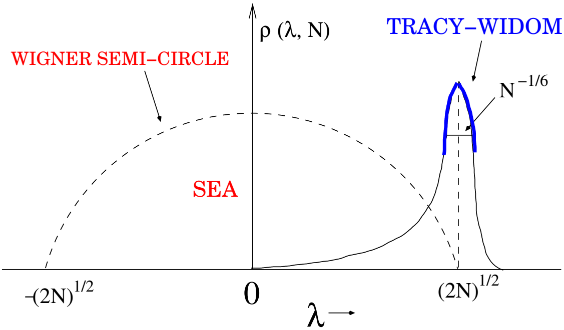


- $\langle \lambda_{\max} \rangle = \sqrt{2N}$ for large N.
- $\lambda_{\max}$ fluctuates from one sample to another. $\text{Prob}[\lambda_{\max}, N] = ?$

Tracy-Widom distribution for λ_{max}



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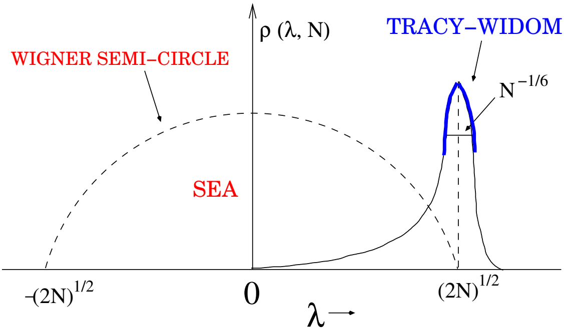

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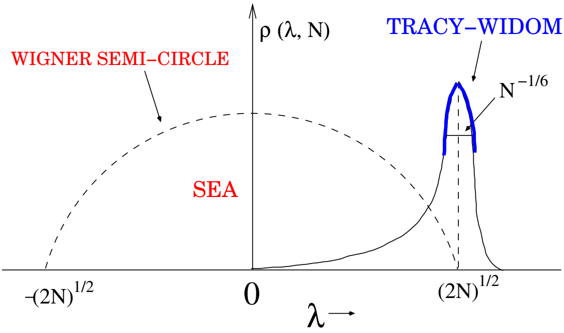

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- typical fluctuations are distributed via Tracy-Widom (1994):
- cumulative distribution:

$$\mathrm{Prob}[\lambda_{\max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$$

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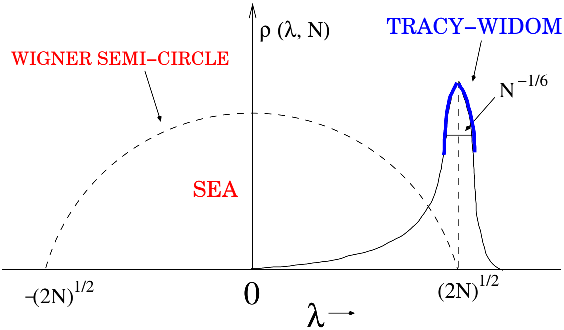

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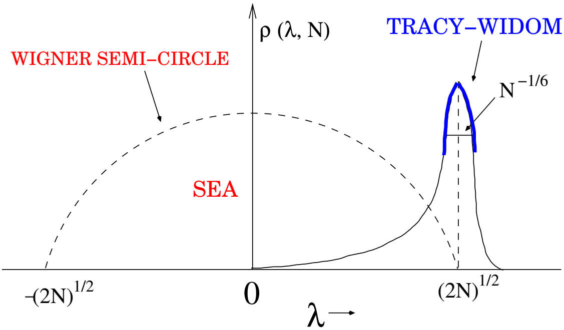

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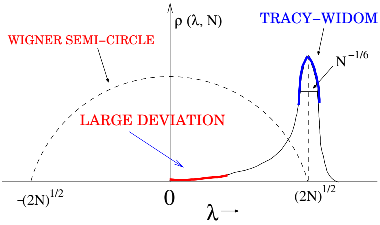
- $F_\beta(z) \to$ solution of Painlevé equation

Probability of Large Deviation of λ_{max}:



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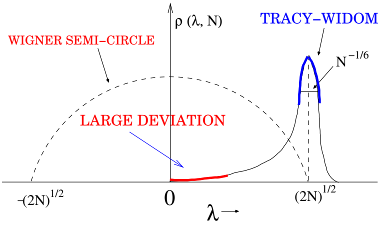

- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t-\sqrt{2N})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-1/6})$ around the mean $\sqrt{2N}$, i.e., when $|\lambda_{\max} - \sqrt{2N}| \sim N^{-1/6}$

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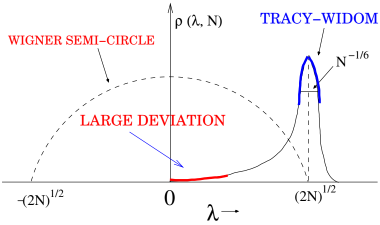

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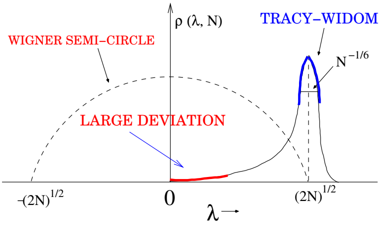

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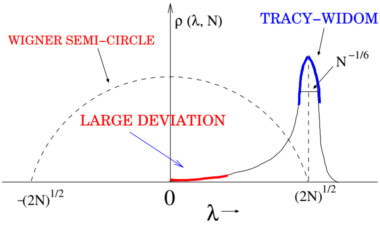

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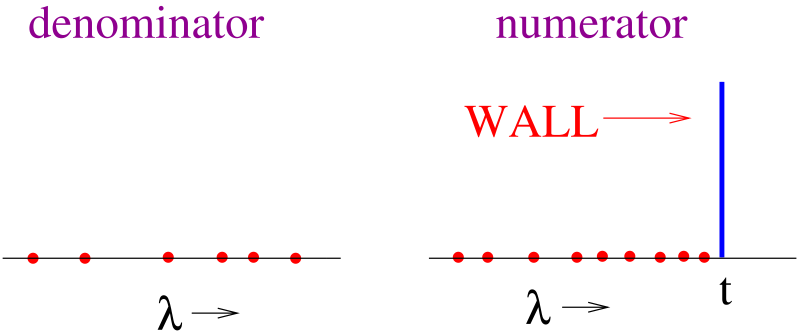

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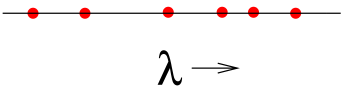

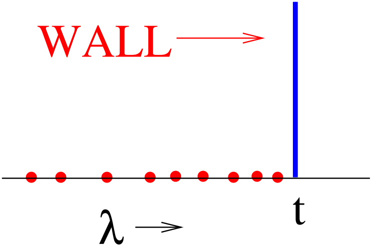

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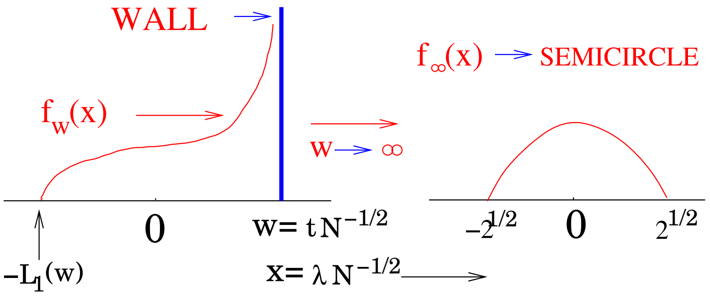

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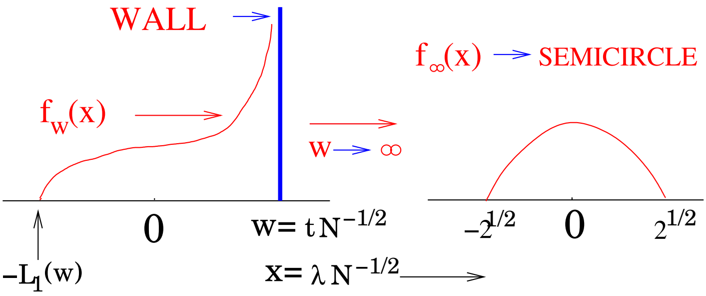

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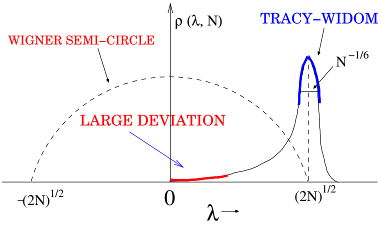

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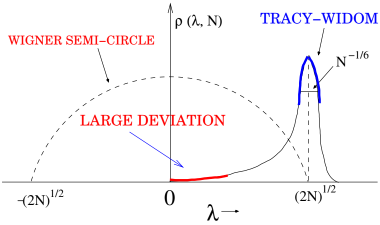

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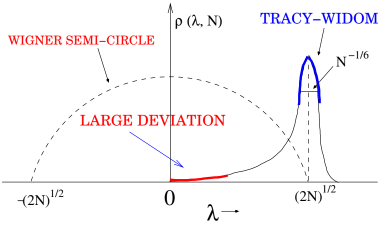

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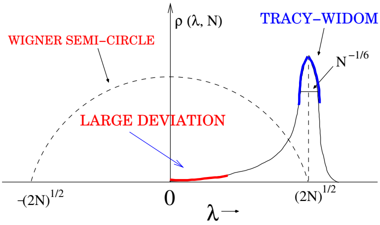

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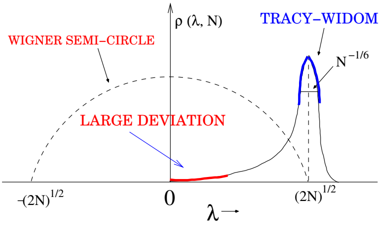

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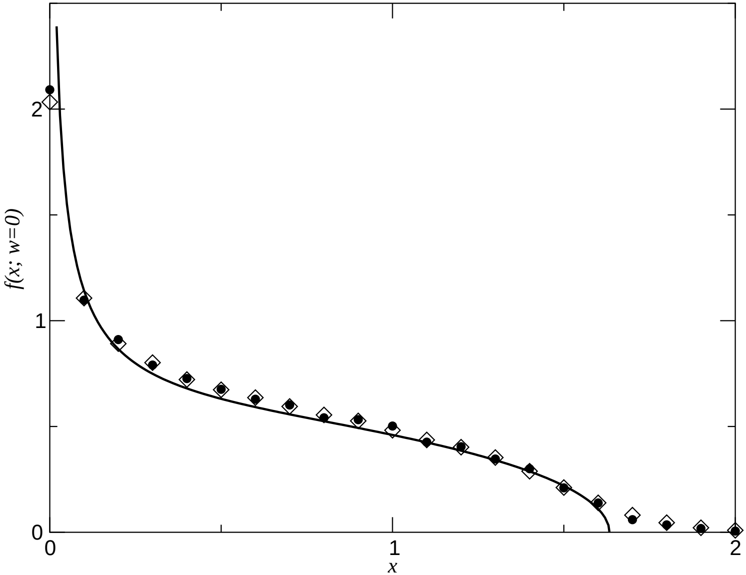
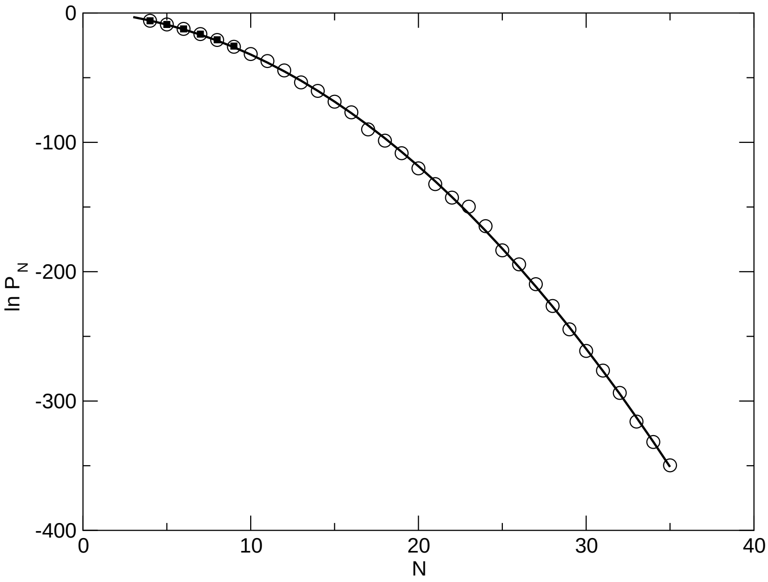
Numerical Results: Charge Density at $w = 0$



Numerical Result for P_N:



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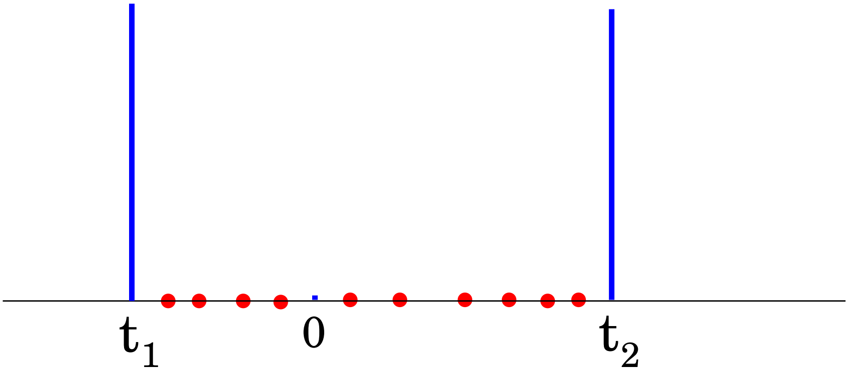

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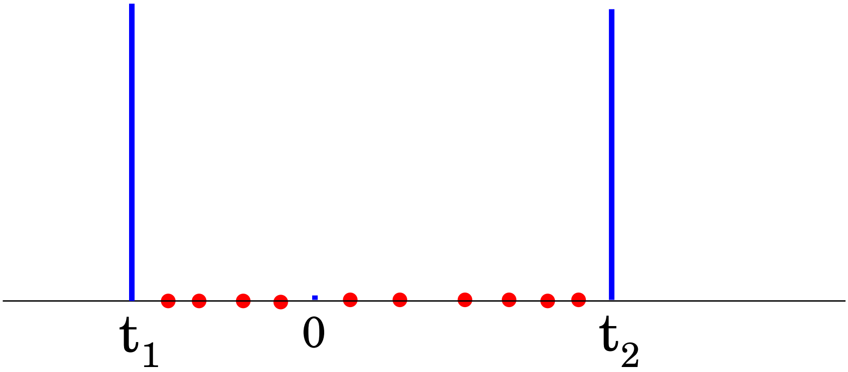

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- $\Psi(z_1, z_2) = U(z_1, z_2 - z_1) - \frac{3+\ln(4)}{8}$

$$\begin{aligned} U(x,y) &= \frac{1}{32}\left[32\,\ln(2) - 16\,\ln(y) + 16\,x^2 + 6\,y^2 \right. \\ &+ \left. 16\,y\,x - 2\,x^2\,y^2 - 2\,y^3\,x - \frac{9}{16}\,y^4\right] \end{aligned}$$

(D.Dean & S.M, 2008)

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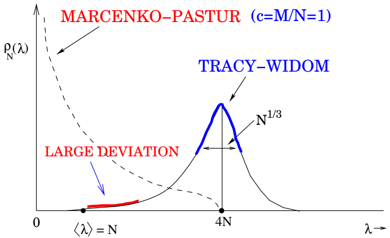
- Marcenko-Pastur law (1967): $f_{\mathrm{MP}}(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_)}$

$x_\pm = (1 \pm \frac{1}{\sqrt{c}})^2$ where $c = N/M \leq 1$

Largest Eigenvalue of the Wishart Matrix



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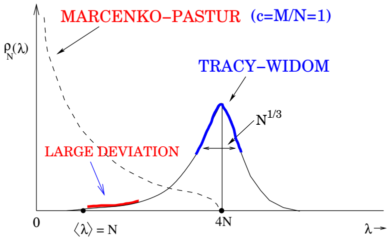

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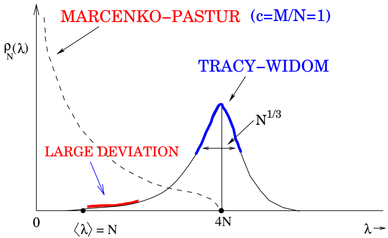

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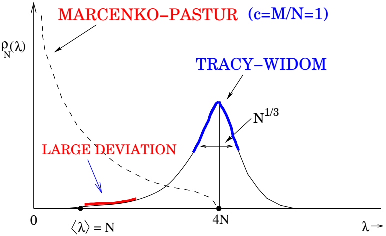

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- for $4N - t \sim O(N)$ (large negative fluctuation):

$$\mathrm{Prob}[\lambda_{\max} \le t, N] \sim \exp\left[-\beta\, N^2\, \Phi_W\left(\frac{4N-t}{N}\right)\right]$$

Large Deviation for Wishart Matrix:

- large deviation function $\Phi_W(y) \to$ computed explicitly for all $c = N/M \leq 1$

(P.Vivo, S.M, & O. Bohigas, 2007)

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$$\boxed{\theta(1) = \log(2) - \frac{33}{64} = 0.177522\ldots}$$

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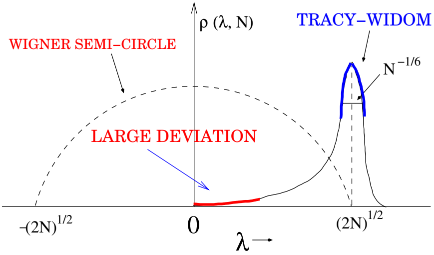
Summary and Conclusions



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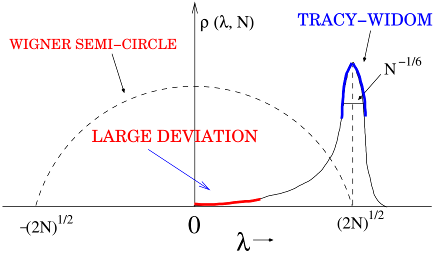

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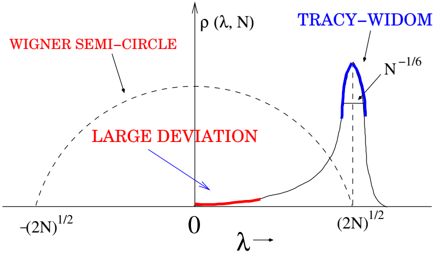

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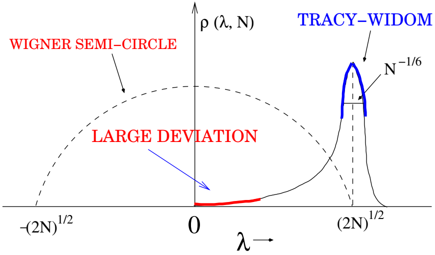

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Conclusions and Related Problems

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$$\boxed{\theta = \frac{1}{4}\ln(3) = 0.274653..} \quad \text{(D. Dean \& S.M, 2006)}$$

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