Quantum field theory on curved backgrounds and Hadamard states

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Outline of the Talk

- Motivations, *i.e.*, trivia about cosmological models
- QFT on curved backgrounds: ABC
- On the geometry of the background and on the cosmological horizon
- On the underlying field theory: from the bulk to the horizon.
- Constructing distinguished states
- On the Hadamard property of these distinguished states

Based on

- C. D., N. Pinamonti, V. Moretti, J. Math. Phys. **50** (2009) 062304
- C. D., N. Pinamonti, V. Moretti, Commun. Math. Phys. **285** (2009) 1129

Motivations - Part I (The Cosmos)

- The description of the Universe is based on the **Cosmological Principle**
- It yields that topologically: the background $M \sim I \times \Sigma$
  - I is an open interval of $\mathbb{R}$ (*"cosmological time"*)
  - Σ are homegeneous $3D$ manifolds, topologically either a sphere, or a plane or a paraboloid.
- The geometry is given by a Friedmann-Robertson-Walker type of metric

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right]. \quad \kappa = 0, \pm 1$$

- Recent observations suggest (no definitive statement!)
  - $\kappa = 0$.
  - the cosmological constant is slightly negative, *i.e.*, $a(t) = e^{Ht}$

Motivations - Part II (Early Universe)

The standard Cosmological model is plagued by many problems such as

1. homogeneity problem (ad hoc initial conditions)
2. initial singularity, etc...
3. the Universe appears to be in an accelerated phase (dark energy?)
4. there exists a large amount of non-visible cold matter (CDM)

Solutions? Several, mostly models, one common idea:

Matter is best described by quantum fields!

Motivations - Part III

Independently from the chosen model, it is agreed

- that quantum fields play a key role,
- hence even semiclassical Einstein's equations may enter the fray.

Our goal:

1. understand how to rigorously quantize field theories on curved backgrounds[1],
2. understand how to characterize physically acceptable quantum states,
3. understand how to construct distinguished quantum states!

[1] as studied by Haag, Kastler, Dimock, Wald, Fredenhagen, Brunetti, Hollands, Pinamonti, Moretti, Fewster, Verch...

ABC of (A)QFT on curved spacetimes - where

- (A)QFT on curved spacetimes is the **quantization** of a classical field theory

On which backgrounds does a classical field theory make sense?

- We consider only globally hyperbolic spacetimes, that is
  - M is a $4D$ differentiable connected Hausdorff spacetime.
  - it is endowed with a Lorentzian metric g and an isometry $\psi : M \to \mathbb{R} \times \Sigma$
    1. Σ is an embedded $3D$ submanifold,
    2. $\psi^* g$ can be written as $-\beta d\mathcal{T}^2 + h$,
    3. $\beta \in C^\infty(\mathbb{R} \times \Sigma; \mathbb{R}^+)$, $\mathcal{T}$ is a temporal function,
    4. h is a smooth Riemannian metric for each value of $\mathcal{T}$.

ABC of (A)QFT on curved spacetimes - what

- Which classical field theories do we consider?

In principle we consider all possible field theories provided that

- the dynamics is ruled by a Cauchy problem

On practical grounds we work with free field theories

1. their dynamics is ruled by a linear PDE,
2. the principal symbol of the underlying operator is normally hyperbolic,
3. interactions are treated perturbatively.

$$\text{Example: } P = g^{\mu\nu}(x)\partial_\mu\partial_\nu + v^\mu(x)\partial_\mu + B(x)$$

ABC of (A)QFT on curved spacetimes - how

What is AQFT? A two step procedure:

1. associate to a classical field theory a $(C)^*$-algebra of observables $\mathcal{A}$,
2. identify a quantum state, that is $\omega : \mathcal{A} \to \mathbb{C}$ linear and positive:

$$\omega(e) = 1, \quad \omega(a^* a) \geq 0, \; \forall a \in \mathcal{A}$$

The first step is "easy"... the second... not!

We shall explain them via scalar fields

Why $*$-algebras?

We call $\mathcal{A}$ a C*-algebra if

- there exists a map $* : \mathcal{A} \to \mathcal{A}$ such that $* \circ * = id$,
- there exists a map $\| \cdot \| : \mathcal{A} \to \mathbb{R}$ for which $\mathcal{A}$ is a Banach algebra,
- $\| a^* a \| = \| a \|^2$ for all $a \in \mathcal{A}$.

GNS (reconstruction) Theorem:

Let $\mathcal{A}$ be a C*-algebra with a unit element and $\omega : \mathcal{A} \to \mathbb{C}$ a state. Then the following triple is unique up to unitary equivalence:

- $\mathcal{D}_\omega$, a dense subspace of an Hilbert space $(\mathcal{H}_\omega, (,)_\omega)$
- a representation $\pi_\omega : \mathcal{A} \to \mathcal{BL}(\mathcal{D})$
- a cyclic vector $\Omega \in \mathcal{D}_\omega$ such that $\| \Omega \|_{\mathcal{H}_\omega} = 1$ and $\omega(a) = (\Omega, \pi_\omega(a)\Omega)_\omega$

It holds $\mathcal{H}_\omega = \overline{\{\pi_\omega(a)\Omega;\ \forall a \in \mathcal{A}\}}$.

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Axiomatic viewpoint

In 1980 Dimock revised the Haag-Kastler axioms for curved backgrounds:

- $\forall$ relatively compact open subset $\mathcal{O} \subset M$, there exists an assignment of a C^*-algebra $\mathcal{A}(\mathcal{O})$
- **Isotony** holds, that is $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')$ if $\mathcal{O} \subset \mathcal{O}'$,
- **Causality** holds, that is $[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0$ if $\mathcal{O} \cap \left(J^+(\mathcal{O}') \cup J^-(\mathcal{O}')\right) = \emptyset$,
- **Covariance** holds, that is, for any isometry $\psi : M \to M$, there exists an isomorphism $\alpha_\psi : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\psi(\mathcal{O}))$ and $\alpha_\psi \circ \alpha_{\psi'} = \alpha_{\psi\circ\psi'}$, $\alpha_{id} = id_{\mathcal{A}}$

The **algebra of observables** is $\mathcal{A}(M) \doteq \bigcup\limits_{\mathcal{O}\subset M} \mathcal{A}(\mathcal{O})$.

The algebra of observables - Part I

Let $\phi : M \to \mathbb{R}$ fulfill

$$P\phi = \left(-\Box + \xi R + m^2\right)\phi = 0, \quad \xi \in \mathbb{R}, \quad m^2 > 0.$$

The following holds:

1. For any $f \in C_0^\infty(M)$, there exists $\phi_f \in C^\infty(M)$ such that
$$\phi_f = E(f).$$
2. $E \doteq E^+ - E^-$ is the causal propagator, that is
$$PE^\pm = E^\pm P = id, \quad supp\left(E^\pm(f)\right) \subseteq J^\mp(supp(f)),$$

The algebra of observables - Part II

Theorem:

Let

$$\mathcal{S}(M) \doteq \{\phi \in C^\infty(M) \mid \exists f \in C_0^\infty(M),\ \phi = E(f)\}$$

This is a symplectic space if endowed with the weakly non-degenerate

$$\sigma(\phi,\phi') = \int_M E(f)f'\sqrt{|g|}d^4x = \int_\Sigma d\mu(\Sigma)\left(\phi\nabla_n\phi' - \phi'\nabla_n\phi\right).$$

Weyl Algebra:

To each symplectic space $(\mathcal{S}(M),\sigma)$, one can associate a *unique* (up to $*$-isometries) C*-algebra $\mathcal{W}(M)$ whose generators are $W(\phi)$, $\phi \in \mathcal{S}(M)$ and

$$W(\phi)^* = W(-\phi), \quad W(\phi)W(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')}W(\phi+\phi').$$

Hadamard states

The **big** problem of AQFT is the identification of a state!

- In Minkowski there exists a distinguished state: the Poincaré vacuum!
- In a curved background there are fewer isometries, hence

What is a physically sound state?

Minimal requirements:

1. same UV behaviour of the Poincaré vacuum
2. quantum fluctuations of observables are bounded

Answer: Hadamard states

Glimpses of Hadamard(ology)

Note: A quasi-free state ω is fully characterized by its two-point function

Local description: A two-point function $\omega(x, y)$ of a state ω is **Hadamard** if for any normal neighbourhood $\mathcal{O}_p$

$$\omega(x, y) = \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \ln \frac{\sigma_\epsilon(x, y)}{\lambda} + W(x, y)$$

Global description: using **microlocal analysis**, a state ω of a real smooth K.-G. field is of Hadamard form if and only if the Schwartz kernel of the two-point function satisfies

$$WF(\omega) = \left\{((x, k_x), (y, -k_y)) \in (T^*M)^2 \setminus 0 \mid (x, k_x) \sim (y, k_y), k_x \triangleright 0\right\}.$$

Why Hadamard?

In order to treat *interactions* as perturbations, we need

Wick powers

- we can define
$$\omega(:\phi^2(f):) \doteq \left(\lim_{x\to y}\omega_2(x,y) - H(x,y)\right)(f).$$
- The definition is covariant, that is $:\phi^2(x):$ depends only on the geometry,
- The definition extends naturally to derivatives and higher powers,
- The Wick theorem and the time-ordered product can be defined,
- the Wick polynomials can be endowed with the structure of a $*$-algebra.

A distinguished class of "cosmological spacetimes" - I

Hyp. 1) Flat Spatial Sections $\Longrightarrow$

$$g_{FRW} = -dt^2 + a^2(t)\left[dr^2 + r^2 dS^2(\theta,\varphi)\right], \quad M \sim I \times \mathbb{R}^3$$

and $a(t) \in C^\infty(I, R^+)$.

Immediate consequences:

1. Consider a co-moving observer as the integral line $\gamma(t)$ of ∂_t. If $M \setminus J^-(\gamma) \neq \emptyset$, then causal signal departing from each $x \in M \setminus J^-(\gamma)$ never reach $\gamma(t)$. Then we call $\partial J^-(\gamma)$ the (future) **cosmological horizon**
2. Let $d\tau = \frac{dt}{a(t)}$ such that τ ranges in $(\alpha, \beta) \subset \mathbb{R}$.

$$g_{FRW} = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2 dS^2(\theta,\varphi)\right],$$

Sufficient condition for an horizon is $\alpha > -\infty$ and/or $\beta < \infty$.

A distinguished class of "cosmological spacetimes" - I

Hyp. 2) We restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right) ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right) , \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right) .$$

- Here H is chosen as *positive* and the interval $I \doteq (-\infty, 0)$.

Consequences and Properties - I

1. If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence

$$ds^2 = -dt^2 + e^{-2Ht}(dr^2 + r^2 d\mathbb{S}^2(\theta, \varphi)), \quad t \in (-\infty, \infty).$$

   This is the cosmological de-Sitter spacetime.

2. For our choice of $a(\tau)$, as $\tau \to -\infty$, the background **"tends to"** de Sitter. Hence we are dealing with an exponential acceleration in the proper time t.
   - Good approx. at early times (**inflation**)
   - Good approx. at late times (**de Sitter phase of acceleration**)

Consequences and Properties - II

- There is always a **Cosmological horizon**. Under the coordinate change

$$U = \tan^{-1}(\tau - r)\ , \qquad V = \tan^{-1}(\tau + r),$$

the metric becomes:

$$g_{FRW} = \frac{a^2(U,V)}{\cos^2 U \cos^2 V}\left[-dUdV + \frac{\sin^2(U-V)}{4} dS^2(\theta,\varphi)\right].$$

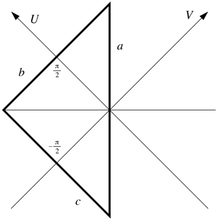
Theorem:

Under the previous assumptions the spacetime (M, g_{FRW}) can be extended to a larger spacetime $(\widehat{M}, \widehat{g})$ which is a conformal completion of the asymptotically flat spacetime at past (or future) null infinity $(M, a^{-2}g_{FRW})$, *i.e.*, "a" plays the role of the conformal factor

Consequences and Properties - III

- Conformal null infinity $\Im^-$ corresponds to the horizon *(region c)* and it is a null degenerate manifold with

$$g|_{\Im^-} = 0 \cdot dl^2 + H^{-2}\left(d\mathbb{S}^2(\theta,\varphi)\right),$$



1. the vector field ∂_τ is a conformal Killing vector for $\widehat{g}$ in M,
2. the vector ∂_τ becomes tangent to $\Im^\pm$ and coincides with $-H^{-1}\widehat{\nabla}^b a$,

Aim of the analysis:

We want to model a scalar QFT on a cosmological spacetime and we want to find a distinguished ground state

More on classical solutions

Next Problem :

We want to better characterize the space of solutions $S(M)$

Any $\Phi \in \mathcal{S}(M)$ can be decomposed in modes ($\mathbf{k} \in \mathbb{R}^3$, $k = |\mathbf{k}|$,)

$$\Phi(\tau, \vec{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\phi_{\mathbf{k}}(\tau, \vec{x})\widetilde{\Phi}(\mathbf{k}) + \overline{\phi_{\mathbf{k}}(\tau, \vec{x})\widetilde{\Phi}(\mathbf{k})}\right],$$

with respect to the functions

$$\phi_{\mathbf{k}}(\tau, \vec{x}) = \frac{1}{a(\tau)} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau),$$

$\chi_{\mathbf{k}}(\tau)$, is solution of the differential equation

$$\frac{d^2}{d\tau^2}\chi_{\mathbf{k}} + (V_0(\mathbf{k}, \tau) + V(\tau))\chi_{\mathbf{k}} = 0,$$

$$V_0(\mathbf{k}, \tau) := k^2 + \left(\frac{1}{H\tau}\right)^2 \left[m^2 + 2H^2\left(\xi - \frac{1}{6}\right)\right], \quad V(\tau) = O(1/\tau^3).$$

- Furthermore it holds the normalization

$$\frac{d\overline{\chi_{\mathbf{k}}(\tau)}}{d\tau}\chi_{\mathbf{k}}(\tau) - \overline{\chi_{\mathbf{k}}(\tau)}\frac{d\chi_{\mathbf{k}}(\tau)}{d\tau} = i. \quad \forall \tau \in (-\infty, 0)$$

Idea: Construct a general solution treating $V(\tau)$ as a perturbation potential over solutions with $V = 0$, that is as in purely de-Sitter background.

Thus for $V(\tau) = 0$

$$\chi^0_{\mathbf{k}}(\tau) = \frac{\sqrt{-\pi\tau}}{2} e^{\frac{i\pi\nu}{2}} \overline{H^{(2)}_\nu(-k\tau)},$$

with

$$\nu = \sqrt{\frac{9}{4} - \left(\frac{m^2}{H^2} + 12\xi\right)},$$

where $H^{(2)}_\nu$ is the Hankel function of second kind.

Perturbative solutions in the general case

- Let us consider the retarded fundamental solutions $S_{\mathbf{k}}$ of

$$\frac{d^2}{d\tau^2}\chi^0_k(\tau) + (V_0(k,\tau))\chi^0_k(\tau) = 0$$

- Then the general solutions $\chi_{\mathbf{k}}$ can be constructed

$$\chi_{\mathbf{k}}(\tau) = \chi^0_{\mathbf{k}}(\tau)$$

$$+(-1)^n \sum_{n=1}^{+\infty} \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n S_{\mathbf{k}}(\tau, t_1) S_{\mathbf{k}}(t_1, t_2) \cdots$$
$$S_{\mathbf{k}}(t_{n-1}, t_n) V(t_1) V(t_2) \cdots V(t_n) \chi_{\mathbf{k}}(t_n),$$

The series is convergent

if $|Re\nu| < 1/2$ and $V = O(\tau^{-3})$
if $\frac{1}{2} \leq |Re\nu| < 3/2$ and $V = O(\tau^{-5})$

From the bulk to the horizon ...

Bulk) A Weyl C^*-algebra $\mathcal{W}(M)$ can be associated to $(S(M), \sigma)$. This is, up to $*$-isomorphisms, unique and its non vanishing generators $W_M(\phi)$ satisfy:

$$W_M(-\phi) = W_M(\phi)^*, \quad W_M(\phi)W_M(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')}W_M(\phi+\phi'),$$

Horizon) The symplectic space of real wavefunctions is:

$$\mathcal{S}(\Im^-) = \left\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2) \mid \psi \in L^\infty, \partial_\ell\psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty\right\},$$

$$\sigma_{\Im^-}(\psi,\psi') = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi\frac{\partial\psi'}{\partial\ell} - \psi'\frac{\partial\psi}{\partial\ell}\right). \quad \forall\psi,\psi' \in \mathcal{S}(\Im^-)$$

Algebra) Since $\sigma_{\Im^-}$ is nondegenerate, we can construct a Weyl C^*-algebra $\mathcal{W}(\Im^-)$ as

$$W_{\Im^-}(\psi) = W^*_{\Im^-}(-\psi), \qquad W_{\Im^-}(\psi)W_{\Im^-}(\psi') = e^{\frac{i}{2}\sigma_{\Im^-}(\psi,\psi')}W_{\Im^-}(\psi+\psi').$$

Distinguished state on $\Im^-$

A *d*istinguished state $\lambda : \mathcal{W}(\Im^-) \to \mathbb{C}$ is unambiguously defined as

$$\lambda\left(W(\psi)\right) = e^{-\frac{\mu(\psi,\psi)}{2}}, \quad \forall W(\psi) \in \mathcal{W}(\Im^-)$$

where $\forall \psi, \psi' \in \mathcal{S}(\Im^-)$

$$\mu(\psi,\psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

being $\psi(k), \psi'(k)$ the Fourier-Plancherel transform

$$\psi(k) = \int\limits_{\mathbb{R}} dl\, \frac{e^{ikl}}{\sqrt{2\pi}}\psi(l,\theta,\varphi).$$

The state λ enjoys the following (almost straightforward) properties:

- it is quasifree and pure,
- referring to its GNS triple $(\mathcal{H}, \Pi, \Upsilon)$ it is invariant under the left action of the horizon symmetry group.

Properties of λ

Let us consider the timelike future directed vector field ∂_τ whose projection on the horizon is $\widetilde{Y} \propto \partial_l$ (also a generator of the algebra of horizon simmetries). Then

- then λ is the **unique** quasifree pure state on $\mathcal{W}(\Im^-)$ which is invariant under $\alpha_{\exp(t\partial_l)}$ $(t \in \mathbb{R})$ and the unitary group implementing such representation leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator,
- Each folium of states on $\mathcal{W}(\Im^-)$ contains at most one pure state which is invariant under $\alpha_{\exp(t\partial_l)}$.
- and much more...

Back to the bulk

Notice: $\phi \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\phi \doteq \phi|_{\Im^-} \in C^\infty(\Im^-)$.

Theorem 1

If $\Phi \in S(M)$ and $0 < \epsilon < \frac{3}{2} - \nu$, then

- $\Gamma\phi$ decays faster than $1/l^\epsilon$ whereas $\partial_l \Gamma\phi$ faster than $1/l^{1+\epsilon}$,
- $\Gamma\phi \in \mathcal{S}(\Im^-)$,
- $\sigma_{\Im^-}(\Gamma\phi, \Gamma\phi') = H^2\sigma(\phi, \phi')$.

There exists an isometric $*$-homomorphism:

$$\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-).$$

Back to the bulk II

- Any state $\widetilde{\lambda}:\mathcal{W}(\Im^-)\to\mathbb{C}$ can be pulled back to $\imath^*(\widetilde{\lambda}):\mathcal{W}(M)\to\mathbb{C}$.
- Particularly the preferred state
$$\lambda_M(a):=\lambda(\imath(a)).\quad \forall a\in\mathcal{W}(M)$$
- In the de Sitter spacetime, λ_M is the Bunch-Davies state,
- λ_M is considered by cosmologist as the "ground (vacuum) state" in the study of linear perturbations,
- it is invariant under the natural action of any bulk isometry on the algebra. Let $Y\doteq\partial_t$, the one-parameter U^Y_t group implementing such an action leaves fixed the cyclic vector in the GNS representation of λ_M,
- since Y is everywhere timelike and future-directed in M, then the 1-parameter group U^Y_t has positive self-adjoint operator.

Is λ_M Hadamard?

To investigate λ_M, we first write its Schwarz kernel as the quadratic form

$$\lambda_M(f,f') = \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}_f(k,\theta,\varphi)}\widehat{\psi}_{f'}(k,\theta,\varphi)dkd\mathbb{S}^2(\theta,\varphi),$$

where $\psi_f = \Gamma\left(E\left(f\right)\right)$ and $\psi_{f'} = \Gamma\left(E\left(f'\right)\right)$.

Theorem

λ_M inviduates a distribution on $\mathcal{D}'(M\times M)$ such that

$$WF(\lambda_M) = \mathcal{V} =$$

$$= \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

thus it is Hadamard.

On the inclusion $\supset$

Since it holds

$$\lambda_M(f \otimes Pf') = \lambda_M(Pf \otimes f') = 0, \qquad \lambda_M(f \otimes f') - \lambda_M(f' \otimes f) = E(f \otimes f'),$$

then the inclusion $\supset$ descends from $\subset$ by means of the theorem of propagation of sigularities proved by Hörmander (see Radzikowski and many others).

Sketch of the proof. $\subset$

- Let us read λ_M as follows: introduce

$$K = (T \otimes I)(\Gamma E \otimes \Gamma E) \in \mathcal{D}'((\Im^- \times \Im^-) \times (M \times M))/$$

- introduce a sequence of cut-off functions $\chi_n \in C_0^\infty(\Im^-; \mathbb{C})$ and

$$\lambda_n \mathcal{K}(\chi_n \otimes \chi_n) \longrightarrow \lambda_M,$$

where the convergence is with respect to Hörmander pseudo-topology. Here $\mathcal{K} : C_0^\infty(\Im^- \times \Im^-) \to \mathcal{D}'(M \times M)$ is the map associated with the kernel ${}^tK$ in view of Schwarz kernel theorem.

Big final fat Theorem:

The sequence λ_n are such that:

1. $WF(\lambda_n) \subset \mathcal{V}$
2. $\lambda_n \to \lambda_M$ in the weak sense in $\mathcal{D}'(M \times M)$
3. $\sup_n \sup_{k \in V} |k|^N |\widehat{h\lambda_n}| < \infty$ for all $N \geq 1$ and for all $h \in C_0^\infty(M \times M; \mathbb{C})$ where V is any cone closed in $(T^*M)^2 \setminus 0$ lying in the complement of $\mathcal{V}$.

Hence λ_M satisfies $\subset$ and its of Hadamard form.

What lies in front of us?

Summary:

- A distinguished Hadamard state for a scalar field theory exists in a large class of FRW backgrounds.
- These backgrounds are of cosmological relevance
- It has interesting properties:

Open Questions:

- How can we connect this results to present observations?
- Can we obtain concrete results (*a.k.a* numbers?)[2]
- Can we repeat the construction for non scalar fields?
- Is there a natural notion of "temperature"?

[2] C.D., Klaus Fredenhagen and Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015