Models of Quantum Spacetime, and Quantum Geometry

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Outline

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Relevant scale: $\lambda_C(m) \sim \lambda_S(m) \Rightarrow m \sim m_P$, in which case scale$\sim \lambda_P \sim 10^{-33}$cm.

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One possible strategy: reason about possibly realistic, intermediate models (semiclassical quantisation) and get inspired by them.

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In particular, we are NOT aiming at some “(more) noncommutative quantum mechanics”!

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In view of future generalisations: Approach with coordinates not in contradiction with GR. Even in classical GR, coordinates describe the localisation of events; clearly, this makes sense even if the coordinates themselves are not observable quantities.

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The Amati–Ciafaloni–Veneziano relations are not of this kind (they contain an absolute bound).

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With $a = \min_j \Delta x^j$, $b = \max \Delta x^j$, $\tau = \Delta x^0$,

- Energy $\sim 1/\tau$ localised in box of sides Δx^j generates gravitational potential

$$|V| \lessapprox \frac{1}{b \min(a, \tau)};$$

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More detailed analysis with localised states construed with free fields on classical spacetime $\Psi = e^{i\phi(f)}\Omega$ localised in box of sides Δx^μ; estimating the corresponding energy tensor and linearising Einstein equations leads to weaker set of relations:

$$\Delta x^0(\Delta x^1 + \Delta x^2 + \Delta x^3) \gtrsim \lambda_P^2,$$
$$\Delta x^1 \Delta x^2 + \Delta x^2 \Delta x^3 + \Delta x^3 \Delta x^1) \gtrsim \lambda_P^2.$$

The Relations

$$Q^{\mu\nu} = -i\lambda_P^2[q^\mu, q^\nu] \quad \text{definition of } Q^{\mu\nu},$$
$$[q^\mu, Q^{\nu\mu}] = 0 \quad \text{(ansatz for simplicity)},$$
$$Q^{\mu\nu}Q_{\mu\nu} = 0,$$
$$Q^{\mu\nu}(*Q)_{\mu\nu} = \pm 4I.$$

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The Uncertainty Relations (now a mathematical consequence of commutation relations):

$$\Delta(q^0)(\Delta(q^1) + \Delta(q^2) + \Delta(q^3)) \gtrsim \lambda_P^2,$$
$$\Delta(q^1)\Delta(q^2) + \Delta(q^2)\Delta(q^3) + \Delta(q^3)\Delta(q^1) \gtrsim \lambda_P^2.$$

Weaker than those arising from heuristic analysis.

Note: $\Delta(\cdot)$ is not linear, hence $\Delta(q^\mu)$ is not a 4-vector. The uncertainty relations are true in any reference frame.

No Relations whithout Representations!

(Lorentz covariant coordinates only, for simplicity; fully Poincaré covariant coordinates may be constructed as well)

- Hilbert Space:

$$\mathfrak{H} = L^2(\mathscr{L}, d\Lambda) \otimes L^2(\mathbb{R}^2, ds_1 ds_2),$$

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- normalisation:

$$\begin{aligned}\{\langle\Lambda|\langle s_1, s_2|\}\{|\Lambda'\rangle|s_1', s_2'\rangle\} &= \langle\Lambda|\Lambda'\rangle\langle s_1, s_2|s_1', s_2'\rangle = \\ &= \delta_I(\Lambda^{-1}\Lambda')\delta(s_1 - s_1')\delta(s_2 - s_2'),\end{aligned}$$

  where integrals are taken with the measure $d\Lambda ds_1 ds_2$.

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- Commutators

$$Q^{\mu\nu}|\Lambda\rangle|\xi\rangle = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\sigma_0{}^{\mu'\nu'}|\Lambda\rangle|\xi\rangle,$$

where $[X^\mu, X^\nu] = i\sigma_0{}^{\mu\nu}$ They have joint spectrum (=set of common generalised eigenvalues)

$$\Sigma = \{\sigma = -\sigma^t : \sigma^{\mu\nu}\sigma_{\mu\nu} = 0, \pm(*\sigma)^{\mu\nu}\sigma_{\mu\nu} = \pm 4\}.$$

Weyl quantisation and $\star$-product

Given function f on $\mathbb{R}^4$, define the operator

$$f(q) = \frac{1}{4\pi^2}\int dk\, e^{ikq}\int dx\, f(x)e^{-ikx}.$$

Problem:

$$f(q)g(q) \quad \text{not of the form } h(q) \text{ (some } h\text{).}$$

Need more general symbols, i.e. functions $f = f(\sigma, x)$ of $\Sigma \times \mathbb{R}^4$.
Then DFR generalisation of Weyl quant.:

$$f(\sigma, x)\underbrace{\to f(Q, x)}_{\text{funct. calc.}}\underbrace{\to f(Q, q)}_{\text{Weyl. Quant.}}$$

$\star :=$ pullback of operator product:

$$f(Q, q)g(Q, q) = (f \star g)(Q, q)$$

which gives:

$$(f \star g)(\sigma, \cdot) = f(\sigma, \cdot) \star_\sigma g(\sigma, \cdot)$$

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The resulting algebra is $\mathcal{E} = \mathcal{C}(\Sigma, \mathcal{K})$.

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The only mathematically well defined possibility: states with optimal localisation, namely which minimize $\sum_\mu (\Delta(q^\mu))^2$.

Of coursič, this definition breaks covariance under Lorentz boosts.

Define the orthogonal projection

$$E_0 = \int_{O(\mathbb{R}^3)} dR\,|R\rangle\langle R| \otimes I.$$

We have $[q^\mu, E_0] = 0$, so for every state $|\Psi\rangle$

$$\sum_\mu (q^\mu)^2 E_0|\Psi\rangle = \lambda_P^2 \int dR \sum_{\mu\nu\tau} R^\mu{}_\nu R^\mu{}_\tau \{|R\rangle\langle R| \otimes X^\nu X^\tau\}|\Psi\rangle =$$

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Hence, if $\langle\Psi|q^\mu|\Psi\rangle = 0$ and $\Psi = E_0\Psi$,

$$\sum_\mu \Delta_\Psi(q^\mu)^2 = 2\lambda_P^2 \langle\Psi|I \otimes H_0|\Psi\rangle \geqslant 2\lambda_P^2,$$

saturated by the states which are coherent on the second tensor factor= states with optimal localisation (a frame dependent definition).
Note that the breakdown of covariance "only" means that relatively boosted observers do not agree on the set of states with optimal localisation; the bound stays true for every observer!

Using the states with optimal localisation, the classical limit is

$$\mathbb{R}^4 \times \Sigma_0,$$

where

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Analogously one relates $q^\mu q_\nu$ to with the Hamiltonian of the anharmonic oscillator, which hase spectrum $\mathbb{R}$.

Independent events

We go one step further

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Same relations means same bound:

$$\sum_\mu (q_j^\mu - q_k^\mu)^2 \geq 4\lambda_P^2$$

The Euclidean quantum distance is bounded below

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$$\sum_\mu (q_j^\mu - q_k^\mu)^2 \geq 4\lambda_P^2$$

The Euclidean quantum distance is bounded below

We want now make this a bit more systematic.

Perturbative models

Consider $\phi(x) = \int dk\check{\phi}(k)$ free scalar quantum field of mass m; define "third quantisation" according to Weyl quantisation:

$$\phi(q) = \int dk\check{\phi}(k) \otimes e^{ik_\mu q^\mu}.$$

It is covariant! Evaluation on a localisation state is

$$\langle \omega, \phi(q) \rangle = \int dk\check{\phi}(k)\omega(e^{ikq}) = \phi(f_\omega).$$

if *omega* optimally localised around x and ω_a=translation of ω by a,

$$[\phi(f_\omega), \phi(f_{\omega_a}]$$

falls off exponentially in a in any spacelike direction.

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Perturbative Dyson series with effective non local Hamiltonian: based either on $:\phi^n(x):$ replaced with $:\phi^n(q): = :(\phi \star \cdots \star \phi)(q):$ or on setting $q_j - q_k$ to minimum on $:\phi(q_1) \cdots \phi(q_n):$.

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Apparently the problem is conceptual: we do not know which concept should replace locality in this setting, so to reproduce it in the large scale limit.

Outline

The Universal Calculus of Dubois-Violette)

Given unital algebra $\mathcal{A}$, take

$$\Lambda(A) = \bigoplus_n \Lambda^n(A) = \bigoplus_n A^{n\otimes}$$

with product and differential

$$(a_1 \otimes \cdots a_n) \cdot (b_1 \otimes \ldots \otimes b_m) = a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n b_1 \otimes b_2 \otimes \cdots \otimes b_m,$$
$$da = a \otimes I - I \otimes a,$$

(extended as a graded differential). Define $\Omega(A)$ as the d-stable subalgebra of $\Lambda^n(A)$, generated by A.

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$$dq^\mu \, dq^\nu = (q^\mu \otimes I - I \otimes q^\mu)(q^\nu \otimes I - I \otimes q^\nu) =$$
$$= q^\mu \otimes q^\nu \otimes I - q^\mu \otimes I \otimes q^\nu - I \otimes q^\mu q^\nu \otimes I + I \otimes q^\mu \otimes q^\nu$$

“lives” in $M(\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E})$.

Volume operators

We use DV Calculus to define the covariant volume operator: e.g.

$$V = dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3 = \epsilon_{\mu\nu\rho\sigma} dq^\mu dq^\nu dq^\rho dq^\sigma$$

(but also area operators $dq^\mu \wedge dq^\nu$, 3-volume operators,...).
In particular V "lives" in

$$\underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{\text{5 factors}}$$

Strength: use the abstract universal differential calculus to define them, but then can compute spectra as operators affiliated to C*-algebras.

Spectrum of the 4-volume

V is a normal operator and has pure point spectrum

$$\mathrm{spec}_{pp}(V) = \lambda_P^4 S$$

where

$$S = \pm 2 + \mathbb{Z} a_+ a_- + i\,(\mathbb{Z} a_+ + \mathbb{Z} a_-)\,.$$

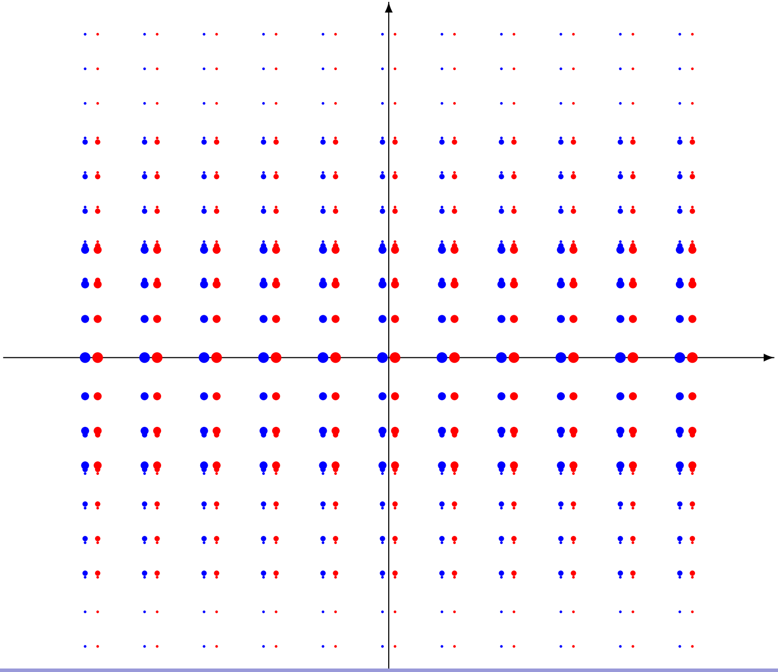
Above,

$$a_\pm = \sqrt{5 \pm 2\sqrt{5}}.$$

Then

$$\mathrm{spec}(V) = \overline{\mathrm{spec}_{pp}(V)} = \lambda_P^4(\pm 2 + \mathbb{Z}\sqrt{5} + i\mathbb{R}).$$

Note that $\mathrm{spec}(V)$ stays away from zero by a constant of order of λ_p^4.



A bound on 3-volume's euclidean length

Computation of spectrum of -volume is too long. To get a flavour, consider 3-volume instead:

$$V_\sigma = \epsilon_{\mu\nu\rho\sigma} dq^\mu dq^\nu dq^\rho = A_\sigma + iB_\sigma$$

where

$$A_\sigma = \frac{1}{6} \det \begin{pmatrix} 1 & q_1^\mu & q_1^\nu & q_1^\rho \\ 1 & q_2^\mu & q_2^\nu & q_2^\rho \\ 1 & q_3^\mu & q_3^\nu & q_3^\rho \\ 1 & q_4^\mu & q_4^\nu & q_4^\rho \end{pmatrix} \epsilon_{\mu\nu\rho\sigma}, \tag{1a}$$

$$\begin{aligned} B_\sigma &= \frac{1}{2} Q^{\mu\nu} (q_1^\rho - q_2^\rho + q_3^\rho - q_4^\rho) \epsilon_{\mu\nu\rho\sigma} = \\ &= \tilde{q}_{1\sigma} - \tilde{q}_{2\sigma} + \tilde{q}_{3\sigma} - \tilde{q}_{4\sigma}. \end{aligned} \tag{1b}$$

where $\tilde{q} = Q^{-1} q$. Then

$$\sum_\sigma V_\sigma^* V_\sigma = \sum_\sigma (A_\sigma^2 + B_\sigma^2) \geqslant \sum_\sigma B_\sigma^2.$$

Since $[\tilde{q}, \tilde{q}] = iQ^{-1}$, we have $[B_\sigma, B_\rho] = i{Q^{-1}}_{\sigma\rho}$ and thus $\sum_\sigma B_\sigma^2 \geqslant 8$.

Back to Calculus

We define a new, A-valued pairing on $\Lambda(A)$, which we name the q-pairing:

$$\langle a_0 \otimes \cdots \otimes a_n, b_0 \otimes \cdots \otimes b_m \rangle := \delta_{n,m}\, a_0 b_0 \ldots a_n b_n.$$

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when restricted to $\Omega(A)$, the q-pairing has some additional properties: for any $a_i, b_i, a, b \in A$, and $\omega, d\psi \in \Omega^n(A)$, and $\phi, \lambda \in \Omega^m(A)$, we have

$$\begin{aligned} \langle da_1 ... da_n, db_1 ... db_n \rangle &= [a_1, b_1]...[a_n, b_n] \\ \langle \omega\phi, (d\psi)\lambda \rangle &= \langle \omega, d\psi \rangle \langle \phi, \lambda \rangle \\ \langle \lambda d\psi, \phi\omega \rangle &= \langle \lambda, \phi \rangle \langle d\psi, \omega \rangle \end{aligned}$$

The A-valued q-pairing can be turned into a $\mathbb{C}$-valued pairing by composition with a trace τ.
Let δ denote the *Hochschild boundary* defined by

$$\delta(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_n + \\ + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

Then the Hochschild boundary is a Hodge dual of the differential for the pairing $\tau(\langle \cdot, \cdot \rangle)$, namely

$$\tau(\langle \delta\omega, \phi \rangle) = \tau(\langle \omega, d\phi \rangle), \omega, \phi \in \Lambda(A).$$

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The associated Laplacian $d^2 + \delta^2 = d\delta + \delta d$ has been studied by Cuntz and Quillen.

Connection and Parallel Transport

If H is a right module over A, set

$$\Lambda(A,H) = H \otimes_A \Lambda(A), \quad \Omega(A,H) = H \otimes_A \Omega(A),$$

and extend the q-pairing:

$$\langle \sigma \otimes a_1 \otimes \cdots \otimes a_n, b_0 \otimes \cdots \otimes b_n \rangle := \sigma b_0 \prod_{i=1}^{n} a_i b_i \ .$$

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$(a_j, b_j \in A, \sigma \in H)$.
A universal connetion on H is a linear map

$$D : H \to \Omega^1(A,H)$$

satisfying the Leibniz rule

$$D(\sigma a) = (D\sigma)a + \sigma da \ .$$

for all $\sigma \in H$, $a \in A$.

D has a unique extension to $\Omega(A, H)$, which is uniquely fixed by the requirement

$$D(\sigma\alpha) := (D\sigma)\alpha + \sigma d\alpha$$

D^2 is a right $\Omega(A)$-module homomorphism, which we call the curvature of the connection.

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D^2 is a right $\Omega(A)$-module homomorphism, which we call the curvature of the connection. We introduce the generalisation of covariant coordinates as

$$L(a)\sigma := \sigma a - \langle D\sigma, da\rangle$$

(which is a right module map). Take for example $H = A = \mathcal{E}$, the DDFR algebra generated by q^μ. Pick a covariant derivative $Da = da + Aa$ where A is 1-form. We find

$$L(q^\mu)(a) = q^\mu a + \langle A, dq^\mu\rangle a.$$

If $A = A_\nu^{(1)} dq^\nu A_\nu^{(2)}$ (in Sweedler's notation), we indeed find,

$$L(q^\mu)(a) = (q^\mu + iQ^{\mu\nu} A_\nu^{(1)} A_\nu^{(2)})\, a \tag{2}$$

.

Note that the generalised covariant coordinates can be written as

$$L(a)\sigma = \sigma a - \langle U\sigma, da\rangle,$$

where

$$U\sigma = D\sigma + \sigma \otimes 1$$

can be interpreted as a parallel transport.

Outline

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- Some expectations in this direction in a recent paper by Doplicher Morsella and Pinamonti.

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