Effective relational dynamics of a non-integrable cosmological model

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Motivation

Plethora of quantum cosmology problems - we address four of them

- The Hilbert space problem
- The global time problem
  - The multiple choice problem
- The observable problem -

in the problem of 'Effective relational dynamics of the closed FRW model universe minimally coupled to a massive scalar field.' [P. Hoehn, EK, Artur Tsobanjan, Phys. Rev. D **86**, 065014 (2012), arxiv:1111.5193]

Motivation

- Why effective relational framework?
  - the Hilbert space problem is avoided altogether,
  - first order quantum corrections to evolution of the system leave classical solutions untouched,
  - ability to switch between different 'clocks' and thus yield consistent local time evolution of system provided semiclassicality conditions hold.

Motivation

Why closed FRW?

- Study of relational dynamics in more general setting
  - non-trivial coupling of relational clocks to evolving degrees of freedom,
  - no temporally global clock variable exists,
  - non-integrability of the system.
- Simple cosmology which generically produces inflation.
- While classical dynamics is understood in detail, complete and consistent quantisation is still pending.

Effective framework - system description

- Natural phase space structure defined by the Poisson bracket

$$\{\langle \hat{x}_i \rangle, \langle \hat{x}_j \rangle\} = \frac{\langle [\hat{x}_i, \hat{x}_j] \rangle}{i\hbar}, \tag{1}$$

- phase space coordinatized by classical variables $q_i = \langle \hat{q}_i \rangle, p_i = \langle \hat{p}_i \rangle$ associated with expectation values of quantum operators.

Introducing the effective framework

- infinitely many quantum variables - moments

$$\begin{aligned}\Delta(q_1^a p_1^b q_2^c p_2^d) &:= \langle(\hat{q}_1 - \langle\hat{q}_1\rangle)^a(\hat{p}_1 - \langle\hat{p}_1\rangle)^b \\ &\times (\hat{q}_2 - \langle\hat{q}_2\rangle)^c(\hat{p}_2 - \langle\hat{p}_2\rangle)^d\rangle_{\mathrm{Weyl}}\,,\end{aligned} \tag{2}$$

  defined for $(a+b+c+d) \geq 2$.

- Semiclassical approximation
  - assume $\Delta(q_i^a p_j^b) = O(\hbar^{(a+b)/2})$ and truncate the system at the order of $\hbar$ by neglecting all terms of higher order
  $\longrightarrow$ moment of two types: spreads $(\Delta x_i)^2$ and covariances $\Delta(x_i x_j)$. Truncation leads to degenerate Poisson structure, the usual counting of degrees of freedom does not apply here anymore.

Introducing the effective framework

- Following Dirac's constraint quantization condition we demand that physical states satisfy $\hat{C}|\psi\rangle = 0$. The analogue of this condition has been formulated directly on the expectation values as

$$C_{pol} = \langle \widehat{pol}\hat{C} \rangle = 0 \tag{3}$$

for all polynomials $\widehat{pol}$ in the four basic variables.

- Quantum Hamiltonian constraint is a linear combination of quantized classical constraint $\hat{C}$ and polynomial constraints $\hat{C}_{pol}$

$$\hat{C}_H := \alpha\hat{C} + \beta\hat{C}_{p_1} + \gamma\hat{C}_{p_2} + \delta\hat{C}_{q_2}\,. \tag{4}$$

Hamiltonian constraint

- In the present work, attention will be paid to systems governed by classical Hamiltonian constraints of the form

  $$C = p_1^2 - p_2^2 - V(q_1, q_2)\,.$$

  with $V(q_1, q_2)$ polynomial.
  Since no terms involve products of non-commuting variables, we take the corresponding constraint operator to be

  $$\hat{C} = \hat{p}_1^2 - \hat{p}_2^2 - V(\hat{q}_1, \hat{q}_2)\,. \tag{5}$$

Constraint system

$$
\begin{aligned}
C := \quad \langle \hat{C} \rangle \quad &= \quad p_1^2 - p_2^2 + (\Delta p_1)^2 - (\Delta p_2)^2 - V - \tfrac{1}{2}\ddot{V}(\Delta q_1)^2 - \tfrac{1}{2}V''(\Delta q_2)^2 - \dot{V}'\Delta(q_1 q_2), \\
C_{q_1} := \langle (\hat{q}_1 - q_1)\hat{C} \rangle \quad &= \quad 2p_1\Delta(q_1 p_1) + i\hbar p_1 - 2p_2\Delta(q_1 p_2) - \dot{V}(\Delta q_1)^2 - V'\Delta(q_1 q_2), \\
C_{p_1} := \langle (\hat{p}_1 - p_1)\hat{C} \rangle \quad &= \quad 2p_1(\Delta p_1)^2 - 2p_2\Delta(p_1 p_2) - \dot{V}(\Delta(q_1 p_1) - \tfrac{1}{2}i\hbar) - V'\Delta(p_1 q_2), \qquad (6) \\
C_{q_2} := \langle (\hat{q}_2 - q_2)\hat{C} \rangle \quad &= \quad 2p_1\Delta(p_1 q_2) - 2p_2\Delta(q_2 p_2) - i\hbar p_2 - \dot{V}\Delta(q_1 q_2) - V'(\Delta q_2)^2, \\
C_{p_2} := \langle (\hat{p}_2 - p_2)\hat{C} \rangle \quad &= \quad 2p_1\Delta(p_1 p_2) - 2p_2(\Delta p_2)^2 - \dot{V}\Delta(q_1 p_2) - V'(\Delta(q_2 p_2) - \tfrac{1}{2}i\hbar)\,.
\end{aligned}
$$

These five constraints generate only four independent flows (degenerate Poisson structure due to truncation), it is convenient to fix three of them and be left with one, Hamiltonian flow and interpret it as dynamics of the system.

The *Zeitgeist*

- Choosing q_1 as the clock, we impose three 'q_1–gauge' conditions,

$$\phi_1 := (\Delta q_1)^2 = 0, \quad \phi_2 := \Delta(q_1 q_2) = 0, \quad \phi_3 := \Delta(q_1 p_2) = 0. \quad (7)$$

- we require that the values of these variables satisfy *positivity* conditions

$$\begin{aligned} & q_2, p_2, (\Delta q_2)^2, (\Delta p_2)^2, \Delta(q_2 p_2) \in \mathbb{R} \\ & (\Delta p_2)^2, (\Delta q_2)^2 \geq 0 \\ & (\Delta q_2)^2 (\Delta p_2)^2 - (\Delta(q_2 p_2))^2 \geq \frac{1}{4}\hbar^2 \quad . \end{aligned} \quad (8)$$

- This choice of time and clock is no more than a gauge transformation, we will refer to this choice as a *Zeitgeist*.

- Apply gauge to obtain Hamiltonian constraint

$$C_H := C - \frac{1}{2p_1} C_{p_1} - \frac{p_2}{2p_1^2} C_{p_2} - \frac{V'}{4p_1^2} C_{q_2} \,. \tag{9}$$

- Observables computed in the chosen *Zeitgeist* are of transient nature, they are valid only as long as the *Zeitgeist* is - we call them *fashionables*.

- Evolution of the system in chosen *Zeitgeist* through corresponding *fashionables* is given by Hamiltonian equations of motion $\dot{x} = \{x, C_H\}$, where x denotes both canonical variables and moments.
- Consistent solution of the constraints and equations of motion requires that the expectation value of the clock picks up a specific imaginary contribution

$$\Im[q_1] = -\frac{\hbar}{2p_1}\,. \tag{10}$$

Closed FRW universe

- The action of a homogenous massive scalar field $\phi(t)$ minimally coupled to a (homogeneous and isotropic) closed Friedman–Robertson–Walker spacetime, of topology $\mathbb{R} \times \mathbb{S}^3$ and described by the metric

$$ds^2 = -N^2(t)\, dt^2 + a^2(t)\, d\Omega^2 \tag{11}$$

(where $d\Omega^2$ is the line element on a unit $\mathbb{S}^3$), is given by

$$S[a, \phi] = \frac{1}{2} \int dt\, Na^3 \left(-\left(\frac{1}{aN} \frac{da}{dt} \right)^2 + \frac{1}{a^2} + \left(\frac{1}{N} \frac{d\phi}{dt} \right)^2 - m^2 \phi^2 \right) \tag{12}$$

- the Hamiltonian constraint corresponding to the system

$$C_H = p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2 \phi^2 e^{6\alpha} = 0 \quad , \tag{13}$$

is (9) with potential $V(\alpha, \phi) = e^{4\alpha} - m^2 \phi^2 e^{6\alpha}$ in (6).

Classical solutions to FRW universe

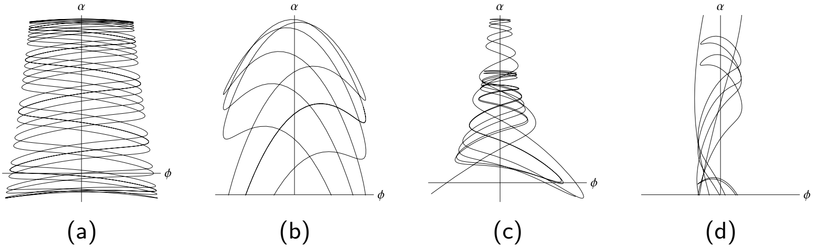


Figure: Two typical classical solutions to the closed FRW spacetime—both ϕ and a generically fail to be globally valid internal clock functions in this model. Here we used $\alpha = \ln(a)$ as appropriate for the canonical discussion following (13). (a) and (c) show extended segments of (both the expanding and re–contracting branch of) relational evolution up to the point of maximal expansion $\alpha_{max} = \ln(a_{max})$. The (new) scale factor α oscillates between points of regular (non–global) maxima $\alpha_{max,k} = \ln(a_{max,k})$ and (non–global) minima $\alpha_{min,k} = \ln(a_{min,k})$; (b) shows a close–up of the same configuration space trajectory as (a) near α_{max}, displaying the non–global extrema in a greater detail, while (d) depicts a close–up on an intermediate section of the trajectory in(c).

Constraint system for closed FRW universe

$$
\begin{aligned}
C &= p_\phi^2 + (\Delta p_\phi)^2 - p_\alpha^2 - (\Delta p_\alpha)^2 - e^{4\alpha} - 8e^{4\alpha}(\Delta\alpha)^2 + m^2\phi^2 e^{6\alpha} + m^2 e^{6\alpha}(\Delta\phi)^2,\\
C_\alpha &= 2p_\phi\Delta(\alpha p_\phi) - 2p_\alpha\Delta(\alpha p_\alpha) - i\hbar\, p_\alpha + 2m^2\phi e^{6\alpha}\Delta(\alpha\phi) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})(\Delta\alpha)^2,\\
C_\phi &= 2p_\phi\Delta(\phi p_\phi) + i\hbar\, p_\phi - 2p_\alpha\Delta(\phi p_\alpha) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha\phi) + 2m^2\phi e^{6\alpha}(\Delta\phi)^2,\\
C_{p_\alpha} &= 2p_\phi\Delta(p_\alpha p_\phi) - 2p_\alpha(\Delta p_\alpha)^2 + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\alpha) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\alpha)\\
&\quad - i\hbar\,(3m^2\phi^2 e^{6\alpha} - 2\, e^{4\alpha}),\\
C_{p_\phi} &= 2p_\phi(\Delta p_\phi)^2 - 2p_\alpha\Delta(p_\alpha p_\phi) + (6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\phi) + 2m^2\phi e^{6\alpha}\Delta(\phi p_\phi)\\
&\quad - i\hbar\, m^2\phi e^{6\alpha}.
\end{aligned}
$$

What time is it?

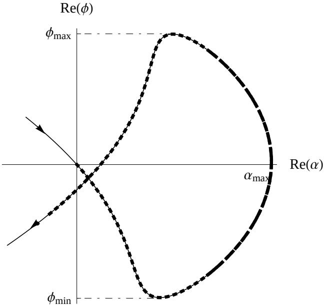
- two (gauge) choices of relational clocks:
  - α–*Zeitgeist*,
$$(\Delta\alpha)^2 = \Delta(\phi\alpha) = \Delta(\alpha p_\phi) = 0, \tag{14}$$
  - ϕ–*Zeitgeist*,
$$(\Delta\phi)^2 = \Delta(\alpha\phi) = \Delta(\phi p_\alpha) = 0, \tag{15}$$
- and corresponding Hamiltonian constraints
  -
$$\begin{aligned} C_H &= p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} + \left[1 - \frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha} p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) \\ &+ \left[m^2 e^{6\alpha} - \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2 + i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha}. \end{aligned} \tag{16}$$
  -
$$\begin{aligned} C_H &= p_\phi^2 - p_\alpha^2 - e^{4\alpha} + m^2\phi^2 e^{6\alpha} - \left[1 - \frac{p_\alpha^2}{p_\phi^2}\right](\Delta p_\alpha)^2 - \frac{p_\alpha}{p_\phi^2}(6m^2\phi^2 e^{6\alpha} - 4e^{4\alpha})\Delta(\alpha p_\alpha) \\ &+ \left[18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha} + \frac{(3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha})^2}{p_\phi^2}\right](\Delta\alpha)^2 + i\hbar\,\frac{m^2\phi e^{6\alpha}}{p_\phi} \end{aligned} \tag{17}$$

Equations of motion in α–*Zeitgeist*

Equations of motion in α–*Zeitgeist*

$$\begin{aligned}
\dot{\alpha} &= -2p_\alpha + \frac{2p_\phi^2}{p_\alpha^3}(\Delta p_\phi)^2 + \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^3}\Delta(\phi p_\phi) + \frac{2m^4\phi^2 e^{12\alpha}}{p_\alpha^3}(\Delta\phi)^2 - i\hbar\,\frac{3m^2\phi^2 e^{6\alpha} - 2e^{4\alpha}}{p_\alpha^2}, \\
\dot{p_\alpha} &= 4e^{4\alpha} - 6m^2\phi^2 e^{6\alpha} + \frac{12m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) - \left[6m^2 e^{6\alpha} - \frac{12m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right](\Delta\phi)^2 \\
&\quad - i\hbar\,\frac{18m^2\phi^2 e^{6\alpha} - 8e^{4\alpha}}{p_\alpha}, \\
\dot{\phi} &= 2p_\phi - \frac{2p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 - \frac{2m^2\phi e^{6\alpha}}{p_\alpha^2}\Delta(\phi p_\phi), \\
\dot{p_\phi} &= -2m^2\phi e^{6\alpha} + \frac{2m^2 e^{6\alpha}p_\phi}{p_\alpha^2}\Delta(\phi p_\phi) + \frac{2m^4\phi e^{12\alpha}}{p_\alpha^2}(\Delta\phi)^2 - i\hbar\,\frac{6m^2\phi e^{6\alpha}}{p_\alpha}, \\
(\Delta\dot{\phi})^2 &= 4\left[1 - \frac{p_\phi^2}{p_\alpha^2}\right]\Delta(\phi p_\phi) - \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta\phi)^2, \\
\Delta(\dot{\phi p_\phi}) &= 2\left[1 - \frac{p_\phi^2}{p_\alpha^2}\right](\Delta p_\phi)^2 + 2\left[\frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2} - m^2 e^{6\alpha}\right](\Delta\phi)^2, \\
(\Delta\dot{p_\phi})^2 &= \frac{4m^2\phi e^{6\alpha}p_\phi}{p_\alpha^2}(\Delta p_\phi)^2 + 4\left[-m^2 e^{6\alpha} + \frac{m^4\phi^2 e^{12\alpha}}{p_\alpha^2}\right]\Delta(\phi p_\phi).
\end{aligned} \tag{18}$$

Numerical results



(a)

Figure: Classical trajectory (dotted) and effective relational trajectory in the configuration space patched together by first evolving it using α as a clock (solid), followed by transforming to the ϕ–*Zeitgeist* (dashed) between the extremal points $\phi = \phi_{min}$ and $\alpha = \alpha_{max}$, finally switching back to the α–*Zeitgeist* (solid) after $\alpha = \alpha_{max}$, but before $\phi = \phi_{max}$.

Numerical results

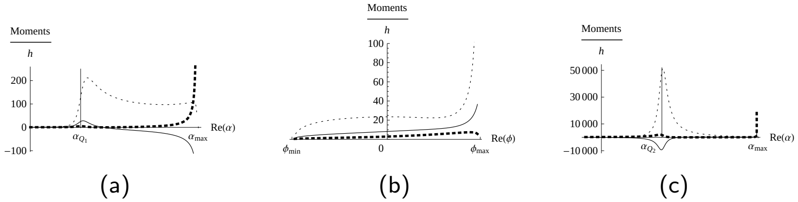


Figure: (a) Moments in α–gauge on the incoming branch: $(\Delta\phi)^2$ (thick, dashed), $(\Delta p_\phi)^2$ (thin, dashed), $\Delta(\phi p_\phi)$ (solid). α_{Q_1} is the quasi–turning point of α on the incoming branch where the clock becomes 'slow'. (b) Moments in ϕ–gauge: $(\Delta\alpha)^2$ (thick, dashed), $(\Delta p_\alpha)^2$ (thin, dashed), $\Delta(\alpha p_\alpha)$ (solid). (c) Moments in α–gauge on the outgoing branch: $(\Delta\phi)^2$ (thick, dashed), $(\Delta p_\phi)^2$ (thin, dashed), $\Delta(\phi p_\phi)$ (solid). α_{Q_2} is the quasi–turning point of α on the outgoing branch where the clock becomes 'slow'.

Conclusions and outlook

- Effective approach sheds new light on the evolution of more interesting, non-integrable systems.
- Although generic trajectories can not be resolved within the effective framework, we are at least able to make out behaviour of some more benigne trajectories.
- It would appear, that the flat universe with cosmological constant is favoured (see 7 years of WMAP paper), it makes sense to study such cosmology.

Thank you!