Stable cosmological models and the semiclassical Eintein's equations

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Stockholm, 16th of November 2012

Outline of the Talk

- Recap. about free fields on curved spacetimes,
- Distinguished states in FRW spacetimes and $\langle : T_{\mu\nu} :\rangle_\omega$,
- Semiclassical Einstein's equations in flat FRW,
- Theory and experiments.

Based on

- C. D., K. Fredenhagen, N. Pinamonti: PRD**77** (2008) 104015.
- C. D., T. -P. Hack, J. Möller, N. Pinamonti, [arXiv:1007.5009 [astro-ph.CO]].
- C. D., T. -P. Hack, N. Pinamonti, Annales Henri Poincare **12** (2011) 1449.

Motivations - Part I - Geometry of the Universe

- The description of the Universe is based on the **Cosmological Principle**
- It yields that the background $M \sim I \times \Sigma$
  - I is an open interval of $\mathbb{R}$ (*"cosmological time"*)
  - Σ are homogeneous $3D$ manifolds, topologically either a sphere, or a plane or a hyperboloid.
- The geometry is described by a Friedmann-Robertson-Walker metric

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right]. \quad \kappa = 0, \pm 1$$

- Also in view of observations,
  - we select $\kappa = 0$.

Motivations - Part II - The matter content

The dynamics of $a(t)$ is ruled by Einstein's equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Which $T_{\mu\nu}$?

- In textbooks, the stress-energy tensor is that of a classical perfect fluid.
- Yet we know:
  - The matter content should be described by suitable fields,
  - These should be analyzed within the framework of QFT on curved backgrounds,
  - In this perspective, $T_{\mu\nu}$ is up to regularization a good observable... hence

$$G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} : \rangle_\omega$$

The Goal(s)

We want to address the following questions:

1. Can we find a "natural state" ω to be used in cosmological spacetimes?
2. Can we find solutions to the semiclassical Einstein's equations?
3. Is the late time cosmological evolution arising from these solutions?
4. Can we detect a contribution of quantum free fields to ΛCDM?
5. Are these contributions experimentally accessibile?

We answered yes to 1 and we shall address 2-5!

Free Fields: kinematical and dynamical configurations

In principle we consider any free field on any globally hyperbolic (M, g)!
Here for simplicity we consider:

- A scalar field $\phi : M \to \mathbb{R}$, such that

$$P\phi \doteq (\Box - m^2 - \zeta R)\phi = 0. \qquad \zeta \in \mathbb{R}.$$

- The space of solutions can be written as

$$\mathcal{S}(M) = \{\phi \in C^\infty(M) \mid \exists f \in C_0^\infty(M), \text{ such that } \phi = E(f) = E^+(f) - E^-(f)\}.$$

Here $E^\pm$ are the advanced/retarded Green functions such that

$$PE^\pm = E^\pm P = id, \qquad supp(E^\pm(f)) \subseteq J^\pm(supp(f)).$$

Free Fields: the algebra of observables

To quantize a free field theory, we need an algebra of observables.

- There is a canonical way to do it for free fields:
  - construct the algebra generated via the linear map $f \in C_0^\infty(M;\mathbb{C}) \to \phi(f)$

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} [C_0^\infty(M;\mathbb{C})]^{\otimes n},$$

    together with an involution $* : \mathcal{A} \to \mathcal{A}$ given by complex conjugation
  - to encode dynamics in the algebra we single out the ideal $\mathcal{I}$ generated by elements abc with $a, c \in \mathcal{A}$ while b is of the following forms ($\lambda, \mu \in \mathbb{C}$):
    1. $\phi(\lambda f + \mu f') - \lambda\phi(f) - \mu\phi(f')$,
    2. $\phi(f)^* - \phi(\bar{f})$,
    3. $\phi(Pf)$,
    4. $[\phi(f), \phi(f')] - iE(f, f')\mathbb{I}$.

Free Fields: Hadamard States

A quantum state is defined as a continuous linear functional

$$\omega : \frac{\mathcal{A}}{\mathcal{I}} \to \mathbb{C}, \qquad \omega(a^* a) \geq 0,\ \forall a \in \frac{\mathcal{A}}{\mathcal{I}}.$$

- In physical scenarios we are often interested in *Gaussian states*,completely determined from

$$\omega(\phi(f)\phi(g)) = \int_M d\mu(x) d\mu(y) \omega(x,y) f(x) g(y).$$

- There exists a plethora of possible states on a curved spacetime but:
  - The UV behaviour must mimic that of Minkowski vacuum,
  - The quantum fluctuations of all observables are bounded!

Distinguished states in FRW spacetimes - I

We consider FRW spacetimes such that

$$a(\tau) \propto \tau^{-1} + O\left(\tau^{-2}\right), \; \frac{da(\tau)}{d\tau} \propto \tau^{-2} + O\left(\tau^{-3}\right)$$

- The FRW Universe is asymptotically de Sitter
- The FRW possesses a conformal boundary $\Im^+$

Distinguished states in FRW spacetimes - II

- As seen in the previous talk, there exists a distinguished Hadamard state

$$\omega(\phi,\phi') = \int\limits_{\mathbb{R}\times\mathbb{S}^2} dk d\mathbb{S}^2(\theta,\varphi)\,\frac{2k}{1-e^{\beta k}}\Theta(k)\overline{\widehat{\Gamma\phi}(k,\theta,\varphi)}\widehat{\Gamma\phi'}(k,\theta,\varphi),$$

Further mathematical analyses yield:

- In de Sitter spacetime, ω is the Bunch-Davies state if $\beta = 0$,
- it is invariant under the natural action of any bulk isometry,
- the state is of **Hadamard form** and it is thermal for bulk massless, conformally coupled scalar field theories,
- it is possible to construct out of ω a state which is both Hadamard and with an "approximate thermal interpretation" at a certain $a_0 > 0$.

Normal Ordering

Hadamard states allow to define normal ordering on curved backgrounds!!!

If a state ω is Hadamard

$$\omega(x,y) = H(x,y) + W(x,y),$$

where $H(x,y)$ is the singular part which depends **only** on the geometry.

$$:\phi(x)^2: \doteq \lim_{y\to x} \left[\phi(x)\phi(y) - H(x,y)\right].$$

Notice that

- the definition is well-posed from a microlocal point of view (Hollands-Wald 2001)
- it is unique up to local geometric terms
- It extends to all derivatives of the fields in a natural way (Moretti 2003)

Computing $\langle : T_{\mu\nu} : \rangle_\omega$ - I

We have all the ingredients to compute the expectation value of $T_{\mu\nu}$! Yet:

- $\langle : T_{\mu\nu} : \rangle_\omega$ is defined up to local curvature terms, that is we have the freedom

$$\langle : T_{\mu\nu} : \rangle_\omega \to \langle : T_{\mu\nu} : \rangle_\omega + \alpha_1 m^4 g_{\mu\nu} + \alpha_2 m^2 G_{\mu\nu} + \alpha_3 I_{\mu\nu} + \alpha_4 J_{\mu\nu},$$

  where

$$I_{\mu\nu} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M d^4x \sqrt{|g|} R^2,$$
$$J_{\mu\nu} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M d^4x \sqrt{|g|} R_{\mu\nu} R^{\mu\nu}.$$

- a direct computation shows that $\nabla^\mu \langle : T_{\mu\nu} : \rangle_\omega \neq 0$ which is inconsistent with the semiclassical Einstein's equations where $\nabla^\mu G_{\mu\nu} = 0$.

Computing $\langle : T_{\mu\nu} : \rangle_\omega$ - II

We can cure the problem with

$$T_{\mu\nu} \longrightarrow T_{\mu\nu} + \frac{1}{6} g_{\mu\nu}(\phi(P\phi) + (P\phi)\phi).$$

- This new term is on shell both traceless and conserved!
  - with this modification $\nabla^\mu \langle : T_{\mu\nu} : \rangle_\omega = 0$ even for thermal states
  - yet, even though $T = g^{\mu\nu} T_{\mu\nu} = 0$, we have in FRW with $\kappa = 0$

$$\langle : T : \rangle_\omega = -\frac{1}{240\pi^2}(\dot{H}H^2 + H^4) + \alpha m^4 + \gamma m^2(\dot{H} + 2H^2) + \\ + \delta(\dddot{H} + 6\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2) - m^2 \langle : \phi^2 : \rangle_\omega .$$

Solutions of the semiclassical Einstein's equations-I

- Let us now generalize to a scenario with
  - N_0 free conformally coupled scalars of mass m,
  - $N_{1/2}$ free Dirac fields ψ of mass m,
  - N_1 massless free spin 1 fields A.

$$\langle : T :\rangle_\omega = -12f(\dot{H}H^2 + H^4) + \alpha m^4 + \gamma m^2(\dot{H} + H^2) +$$
$$+\delta(\dddot{H} + 6\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2) - m^2 N_0\langle : \phi^2 :\rangle_\omega - N_{\frac{1}{2}} m\langle : \overline{\psi}\psi :\rangle_\omega,$$

with $f = \frac{1}{2880\pi^2}\left(N_0 + \frac{11}{2}N_{1/2} + 62N_1\right)$.

We look at

$$G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} :\rangle_\omega, \quad \nabla^\mu\langle : T_{\mu\nu} :\rangle_\omega = 0 \Longrightarrow H^2 = \frac{8\pi}{3}\rho_Q,$$

where $\rho_Q = \langle : T_{00} :\rangle_\omega$.

Solutions of the semiclassical Einstein's equations-II

The dynamics is fully determined by the effective Friedmann's equation:

$$\frac{\dot{\rho}_Q}{H} + 4\rho_Q = \langle : T : \rangle_\omega .$$

We use the **thermal state** we constructed and the following approximations:

- $H << m \longrightarrow H(t_0) = (10^{-33} ev) \longrightarrow$ discard terms $O(H^5)$ in ρ_Q,
- $\dot{H} << H^2 \longrightarrow$ slowly varying Hubble parameter $\longrightarrow$ discard terms $O(\dot{H}, \ddot{H}, ...)$ in ρ_Q,
- $T \doteq \beta^{-1} << am \longrightarrow$ cold dark matter.

A general solution of the above ODE reads

$$\rho_Q = c_1 H^4 + c_2 \frac{mT^3}{a^3} + c_3 \frac{T^5}{ma^5} + \frac{c_4}{a^4} + c_5 m^4 + c_6 H^2 + O\left(\frac{T^7}{m^3 a^7}\right).$$

Solutions of the semiclassical Einstein's equations-III

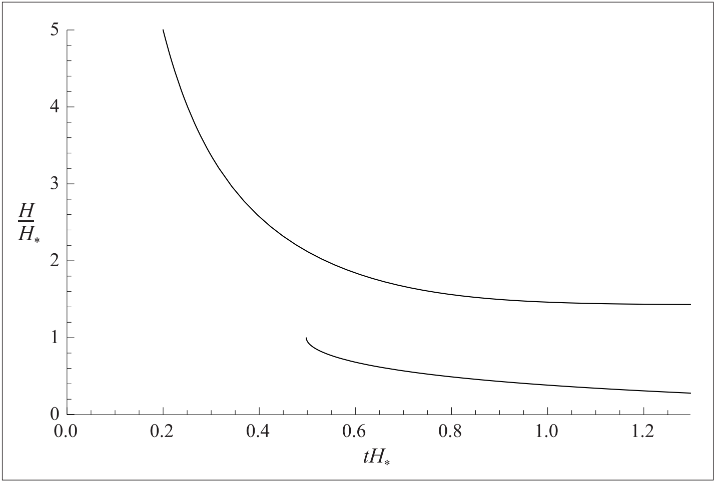
The effective Friedmann's equation yields

$$H^2(a) = H_*^2 \pm \sqrt{H_*^4 - \frac{C_1}{a^4} - \frac{C_2}{a^3} - \frac{C_3}{a^5} - C_4},$$

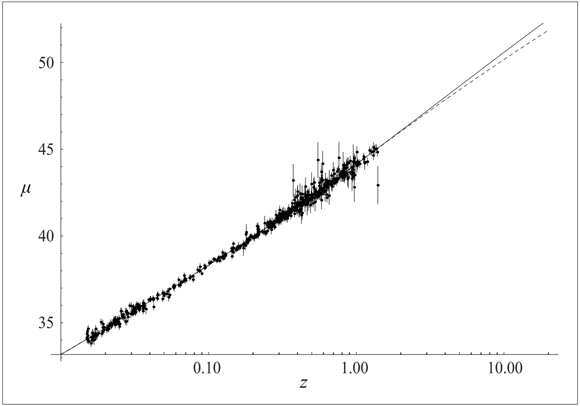
where

- H_* depends on the number of fields and on the renormalization freedom,
- C_2 and C_3 depend on m, T and $a(t_0)$,
- C_4 is a renormalization freedom,
- C_1 stems from an integration constant (depends on the state of the massless fields).

The two branches with $\dot{H} < 0$ and $\ddot{H} > 0$



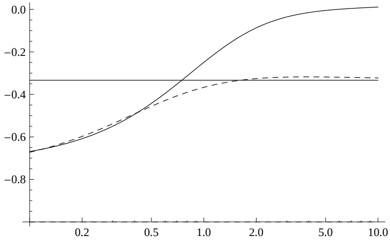
Comparison with Supernovae data



Union2 supernova compilation [Amanullah et al. 2010]

Both branches are compatible with the experimental data

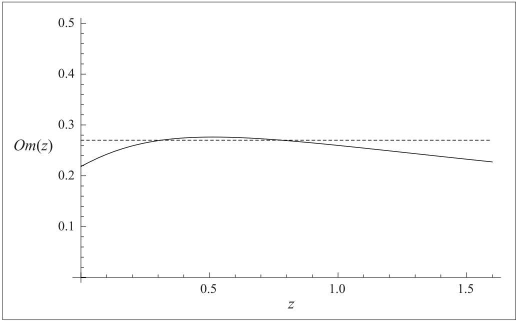
Effective equation of state



$$\omega_Q(z) = \frac{p_Q(z)}{\rho_Q(z)}$$

The upper branch does not have a significant matter-dominated phase.

$Om(z)$-diagnostics



$$Om(z) = \frac{(H(z)/H_0)^2 - 1}{(1+z)^3 - 1}. \qquad \text{Starobinski et al. (2008)}$$

The solid line is the upper-branch, the dashed the lower as well as ΛCDM.

ΛCDM and the lower branch

• The best fit of the lower branch yields values of the parameters such that we can expand in H_*^{-1}

$$H^2(a)_- = K_0 + \frac{K_3}{a^3} + \frac{K_4}{a^4} + \frac{K_5}{a^5} + O(a^{-6}).$$

- The best fit data for K_0 and K_3 are compatible with those of ΛCDM,
- the current data are insensitive to quantum corrections as large as $K_5 = O(10^{-3} K_0)$.

The interpretation

Summary:

- We have a natural choice for a ground and for a thermal state for free fields in FRW spacetime
- There exist explicit solutions of the semiclassical Einstein's equations
- The lower branch displays a behaviour compatible to experiments and similar to ΛCDM
- The cosmological constant appears as a pure renormalization freedom, hence no prediction from QFT on its size!
- Free fields in a thermal state at time t_0 display a component in $H(a)$ mimicking dark matter

Open Questions:

- Can we improve our understanding of the quantum corrections?
- Can we repeat a similar procedure in spherically symmetric and static spacetimes?
- Can we improve our theoretical understanding to include interactions, even perturbatively?