Boundary Conditions in Quantum Cosmology

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Predictions in quantum cosmology

Anthropic interpretation We find ourselves in a decohered branch of the wave function that is suitable for life (cf. 'landscape' picture)

Peak in the wave function If the wave function is peaked around particular values of $a, \phi, \ldots$, this corresponds to the prediction that these values occur with high probability; if the wave function vanishes, the corresponding values are not allowed (relevant e.g. for singularity avoidance)

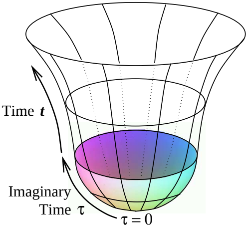
Semiclassical interpretation The wave function can only be interpreted in the semiclassical regime, where an approximate 'WKB time' emerges from the timeless Wheeler–DeWitt equation.

Interpret here a sharp peak in the wave function as a prediction: inflation, for example, occurs 'naturally' if Ψ has a peak at a sufficiently large value of the inflaton field ϕ.

DeWitt's boundary proposal (1967)

- The wave function should obey $\Psi\left[{}^{(3)}\mathcal{G}\right] = 0$ for all singular three-geometries ${}^{(3)}\mathcal{G}$. (This can also include large three-geometries.)
- The danger with this condition is that only the trivial function $\Psi \equiv 0$ may survive as a solution to the Wheeler–DeWitt equation.
- DeWitt expressed the hope that a unique solution is found after this boundary conditon is imposed.

No-boundary proposal



S. W. Hawking, Vatican conference 1982:

There ought to be something very special about the boundary conditions of the universe and what can be more special than the condition that there is no boundary.

$$\Psi[h_{ab}, \Phi, \Sigma] = \sum_{\mathcal{M}} \nu(\mathcal{M}) \int_{\mathcal{M}} \mathcal{D}g\mathcal{D}\Phi \ \mathrm{e}^{-S_{\mathrm{E}}[g_{\mu\nu},\Phi]}$$

A particular example

In minisuperspace, one has

$$\psi(a,\phi) = \int \mathrm{d}N \int \mathcal{D}a\mathcal{D}\phi\, \mathrm{e}^{-I[a(\tau),\phi(\tau),N]},$$

where I denotes here the Euclidean action for a Friedmann Universe with a massive scalar field. Implement the no-boundary condition by demanding $a(0) = 0$ for the Euclidean paths to be summed over in the path integral. In a saddle-point approximation, one obtains

$$\psi_{\mathrm{NB}} \propto \left(a^2 V(\phi) - 1\right)^{-1/4} \exp\left(\frac{1}{3V(\phi)}\right) \cos\left(\frac{(a^2 V(\phi) - 1)^{3/2}}{3V(\phi)} - \frac{\pi}{4}\right)$$

The no-boundary wave function is real; the various complex semiclassical components are separated by decoherence (see morning talk)

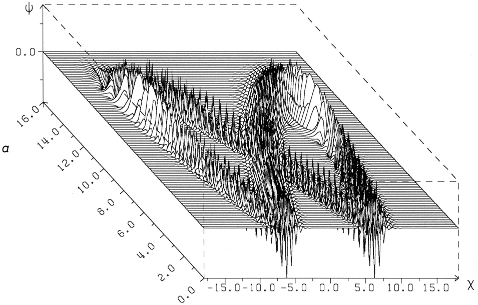
Problems with the no-boundary proposal

- Four-manifolds are not classifiable;
- problems with Euclidean gravitational action
  $\longrightarrow$ evaluation for general complex metrics;
- many solutions in minisuperspace;
- only applicable in a semiclassical approximation;
- solutions do in general not correspond to classical solutions
  (e.g. increase exponentially for large a)

Problem with the classical limit

Indefinite Oscillator

$$\hat{H}\psi(a,\chi) \equiv (-H_a + H_\chi)\psi \equiv \left(\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial \chi^2} - a^2 + \chi^2 \right) \psi = 0$$



C.K. (1990)

No-boundary wave function for the indefinite oscillator

The no-boundary condition does not give solutions that can be used to construct wave packets (C.K. 1991). It yields the solutions

$$K_0\left(\frac{|\chi^2 - a^2|}{2}\right)$$

and

$$I_0\left(\frac{a^2 - \chi^2}{2}\right),$$

which diverge for either $|a| = |\chi|$ or for large a.

Tunnelling condition

There are only *outgoing* modes near singular boundaries of superspace (Vilenkin 1982 and others):

$$j = \frac{\mathrm{i}}{2}(\psi^* \nabla \psi - \psi \nabla \psi^*) \ , \quad \nabla j = 0$$

A WKB solution of the form $\psi \approx C \exp(\mathrm{i}S)$ leads to

$$j \approx -|C|^2 \nabla S$$

For the above minisuperspace model, the result is

$$\psi_{\mathrm{tunnel}} \propto (a^2 V(\phi) - 1)^{-1/4} \exp\left(-\frac{1}{3V(\phi)}\right) \exp\left(-\frac{\mathrm{i}}{3V(\phi)}(a^2 V(\phi) - 1)^{3/2}\right)$$

While the no-boundary state is *real*, the tunneling state is *complex* (distinguishes a direction in superspace). However, without the reference phase $\exp(-\mathrm{i}Et/\hbar)$, the sign of the imaginary unit i has no intrinsic meaning

Symmetric initial condition

SIC!: Demand normalizability for $a \to 0$ through introduction of a ‘Planck potential’ and assume a completely symmetric initial wave function (Conradi and Zeh 1991)

The SIC! can be justified, for example, from loop quantum cosmology

Beyond the tree-level approximation

- Barvinsky and Kamenshchik (1990): probability $\rho(\phi) \sim \mathrm{e}^{\pm I - \Gamma_{1-\mathrm{loop}}} \sim \mathrm{e}^{\pm I} \phi^{-Z-2}$: normalizable state for anomalous scaling $Z > -1$
- Barvinsky and Kamenshchik (1998): For non-mimimal coupling, the tunnelling wave function is also at the one-loop order peaked around values suitable for inflation
- more recently: quantum cosmology for non-minimal Higgs inflation

The Higgs field as an inflaton

Application to a cosmological model for which the Lagrangian of the graviton–inflaton sector reads

$$\boldsymbol{L}(g_{\mu\nu}, \Phi) = \frac{1}{2}\left(M_{\rm P}^2 + \xi|\Phi|^2\right) R - \frac{1}{2}|\nabla\Phi|^2 - V(|\Phi|),$$
$$V(|\Phi|) = \frac{\lambda}{4}(|\Phi|^2 - v^2)^2, \quad |\Phi|^2 = \Phi^\dagger\Phi,$$

where Φ is the Standard Model Higgs boson, whose expectation value plays the role of an inflaton and which is assumed here to possess a strong non-minimal curvature coupling with $\xi \gg 1$; $M_{\rm P}^2 = 1/8\pi G$ ($\hbar = 1 = c$).

A. O. Barvinsky, A. Yu. Kamenshchik, C. K., A. A. Starobinsky, and C. Steinwachs, *J. Cosmol. Astropart. Phys.* 12 (2009) 003

Probability of inflation

$$\rho_{\text{tunnel}}(\varphi) = \exp\left(-\frac{24\pi^2 M_P^4}{\hat{V}(\varphi)}\right)$$

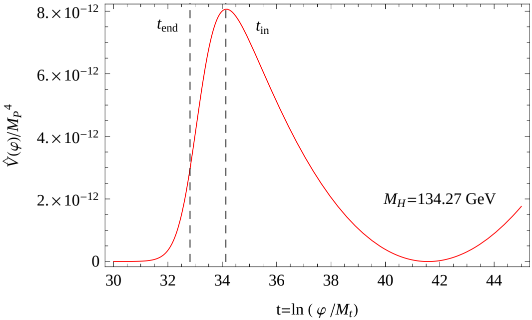


Figure: The effective potential for the instability threshold $M_H^{\text{inst}} = 134.27$ GeV. A false vacuum occurs at $t_{\text{inst}} \simeq 41.6$, $\varphi \sim 80 M_P$. An inflationary domain for a $N = 60$ CMB perturbation is marked by dashed lines.

Boundary conditions and singularity avoidance

No general agreement!

Sufficient criteria in quantum geometrodynamics:

- Vanishing of the wave function at the point of the classical singularity (DeWitt 1967)
- Spreading of wave packets when approaching the region of the classical singularity

concerning the second criterium:
only in the semiclassical regime (narrow wave packets following the classical trajectories) do we have an approximate notion of geodesics $\longrightarrow$ only in this regime can we apply the classical singularity theorems

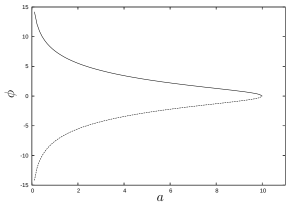
Quantum cosmology with big brake

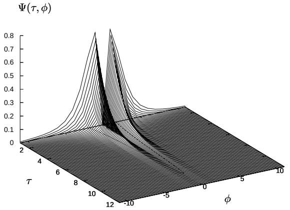
Classical model: Equation of state $p = A/\rho$, $A > 0$, for a Friedmann universe with scale factor $a(t)$ and scalar field $\phi(t)$ with potential ($24\pi G = 1$)

$$V(\phi) = V_0 \left(\sinh(|\phi|) - \frac{1}{\sinh(|\phi|)} \right) ;$$

develops pressure singularity (only $\ddot{a}(t)$ becomes singular)

Quantum model: Normalizable solutions of the Wheeler–DeWitt equation vanish at the classical singularity





(Kamenshchik, C. K., Sandhöfer 2007)

Supersymmetric quantum cosmological billiards

$D = 11$ supergravity: near a spacelike singularity, the cosmological billiard description is based on the Kac–Moody group $E_{10} \longrightarrow$ discussion of the Wheeler–DeWitt equation

- $\Psi \to 0$ near the singularity
- Ψ is generically complex and oscillating

(Kleinschmidt, Koehn, Nicolai 2009)

Quantum phantom cosmology

Classical model: Friedmann universe with scale factor $a(t)$ containing a scalar field with negative kinetic term ('phantom')
$\longrightarrow$ develops a big-rip singularity
(ρ and p diverge as a goes to infinity at a *finite time*)

Quantum model: Wave-packet solutions of the Wheeler–DeWitt equation disperse in the region of the classical big-rip singularity
$\longrightarrow$ time and the classical evolution come to an end; only a stationary quantum state is left

Exhibition of quantum effects at large scales!

(Dąbrowski, C. K., Sandhöfer 2006)

Loop quantum cosmology

M. Bojwald (2001): Dynamical initial conditions;
the difference equation for the quantum state leads to a consistency condition for the initial data

Unification of dynamical law and initial condition?
(was also the original hope with the no-boundary condition)