Introduction to multifractal spacetimes
I – *Single- and multiscale geometry*

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Outline

1. Motivation and introduction
   - Some numerology
   - Field theory on multifractal spacetimes

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- HL gravity (effective QFT): UV $d_S = 2$ [Hořava 2008,2009].

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Dimensional reduction *or* Dimensional flow

Changing behaviour of correlation functions (as across a phase transition), spacetime with scale-dependent “dimension.”

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Universal feature in quantum gravity related to UV finiteness ['t Hooft 1993; Carlip 2009,2010].

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Including Planck's constant $\hbar$, the electron charge e, Newton's constant G, and the speed of light c, one can construct a dimensionless constant in a spacetime of Hausdorff dimension $d_{\rm H}$ as

$$C = \ell_{\rm Pl}^{2(3-d_{\rm H})} e^{d_{\rm H}-2} G^{\frac{d_{\rm H}}{2}-1} c^{2(2-d_{\rm H})}, \qquad \ell_{\rm Pl} = \sqrt{\frac{\hbar G}{c^3}}.$$

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Remarkably, in $d_{\rm H} = 2$ the fundamental constant coincides with (the square of) the Planck length, $C = \ell_{\rm Pl}^2$, while all the other couplings disappear.

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- Regarded as fundamental: (power-counting) renormalizable.
- Invariant under some symmetry group [Collins et al. 2004,2006]. Lorentz invariant at large scales.

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- Insight into RG flow.

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- In certain regimes, calculus on fractals is approximated by continuous fractional calculus [Ren et al. 1996–2003; Nigmatullin et al. 1992–2010], natural to consider fractional integrals over a space with (scale-dependent) fractional dimension.
- Dimensional flow is only one among the many very interesting properties of these models.

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- *Main idea:* G.C., Phys. Rev. Lett. 2010 [arXiv:0912.3142]; JHEP 2010 [arXiv:1001.0571]; Phys. Lett. B 2011 [arXiv:1012.1244].

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- *Applications:* Arzano, G.C., Oriti & Scalisi, *Fractional and noncommutative spacetimes*, Phys. Rev. D 2011 [arXiv:1107.5308]; G.C., Nardelli & Scalisi, *QM in fractional and other anomalous spacetimes*, J. Math. Phys. 2012 [arXiv:1207.4473]; G.C., *Multi-fractional spacetimes, asymptotic safety and Hořava-Lifshitz gravity*, arXiv:1209.4376; G.C., Eichhorn & Saueressig, in progress.

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08/35– Questions and caveats

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In what sense do these models live on a “fractal”?

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A1

Dimensional flow is smooth, thus implying transitions through states with noninteger d_{H} and/or d_{S}.

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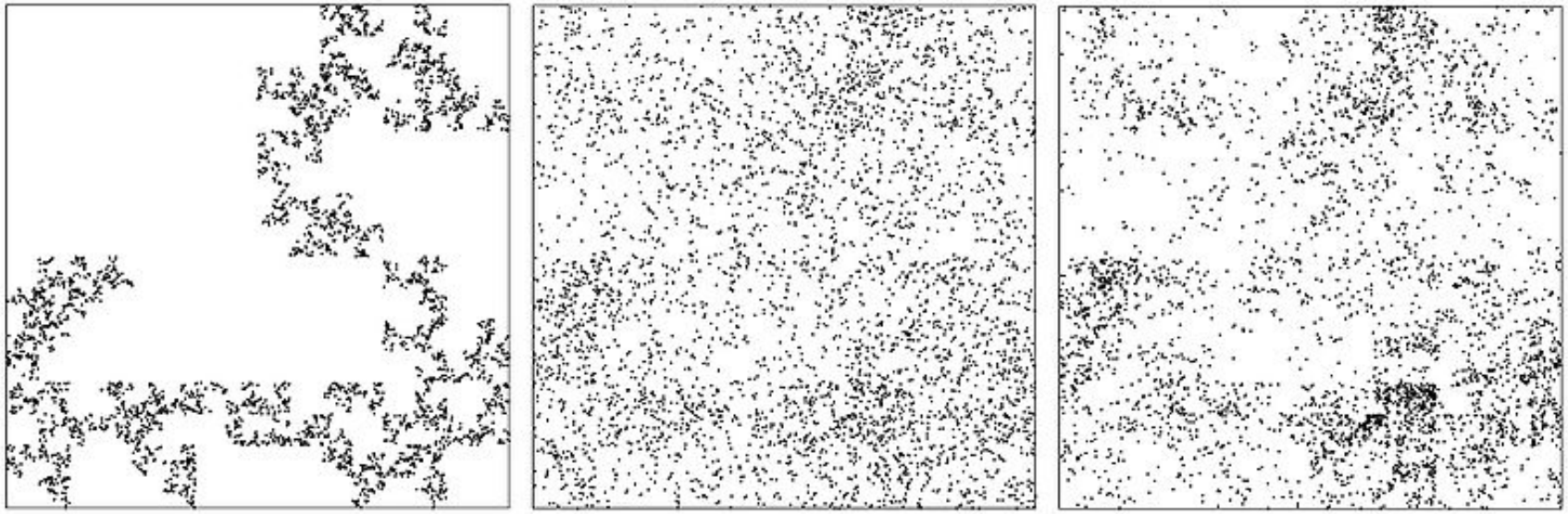

Figure : Multiplicative cascade ($d_{\mathrm{H}} \in]0, 2[$)

09/35– Questions and caveats

Q2

Are there fractals with integer d_{H} and/or d_{S}?

09/35– Questions and caveats

Q2

Are there fractals with integer d_{H} and/or d_{S}?

A2

Yes. In particular, diamond fractals can have $d_{\mathrm{H}} = d_{\mathrm{S}} = 2$.

Skip examples

09/35– Fractals with integer Hausdorff dimension

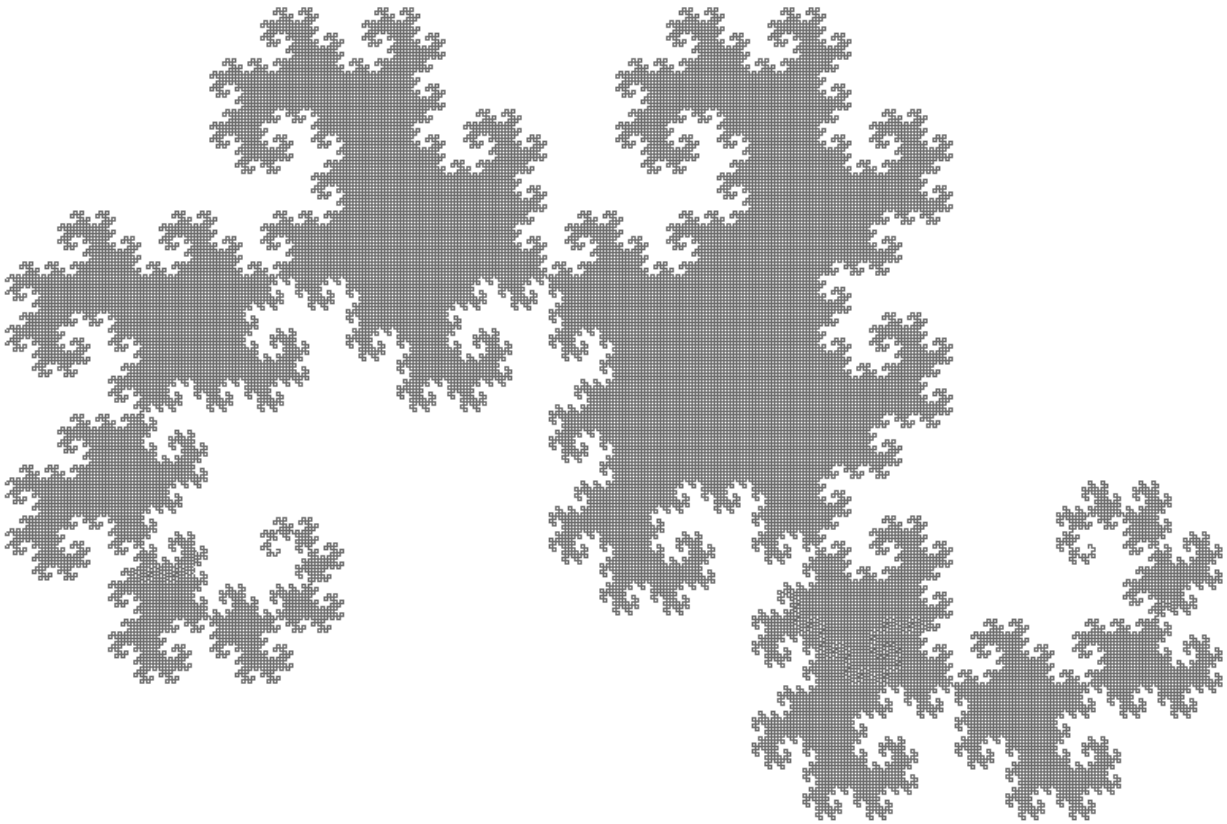

Figure : Dragon curve ($d_{\mathrm{H}} = 2$)

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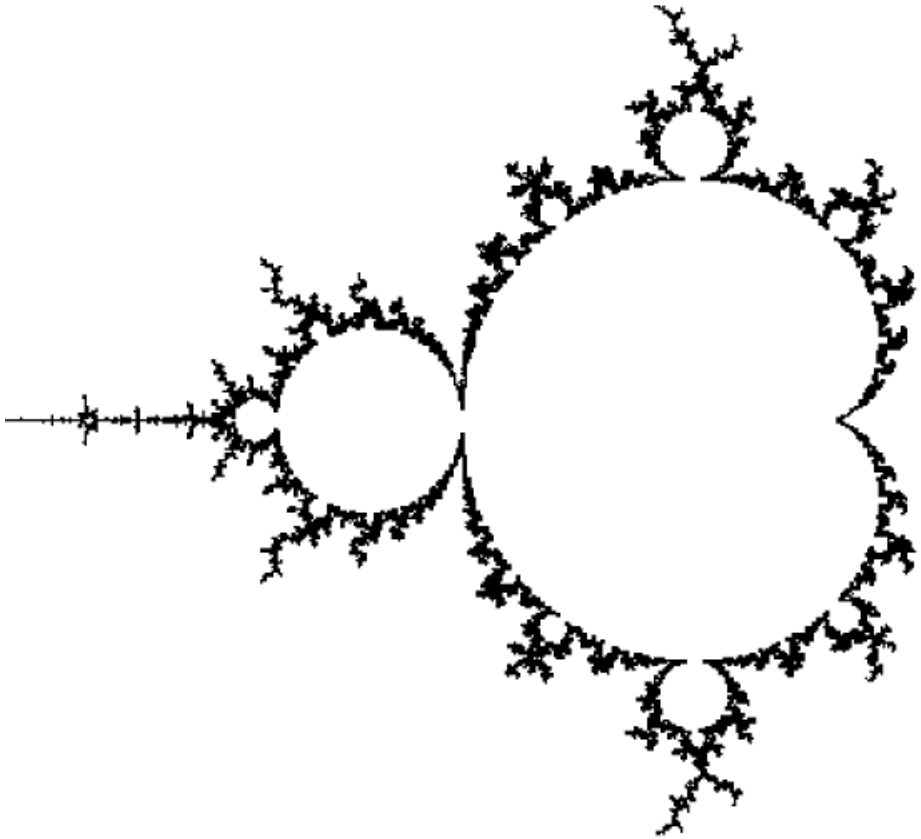

Figure : Mandelbrot set and its boundary ($d_{\mathrm{H}} = 2$)

09/35– Fractals with integer Hausdorff dimension

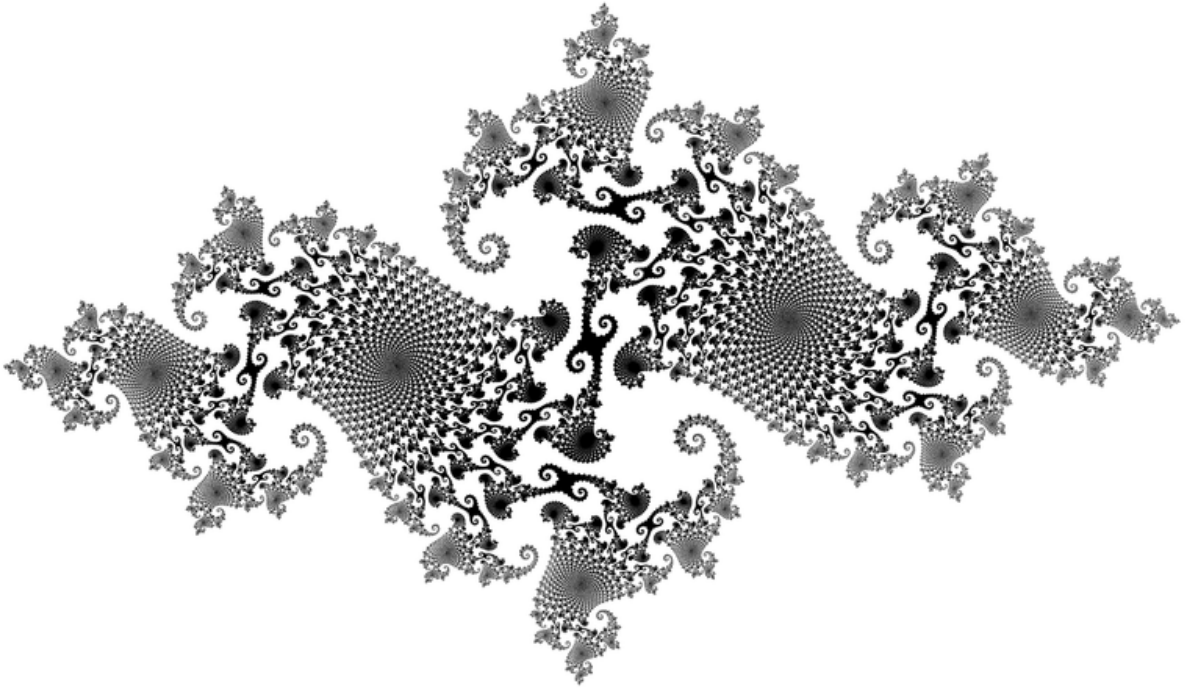

Figure : Julia sets (for some c, $d_{\mathrm{H}} = 2$)

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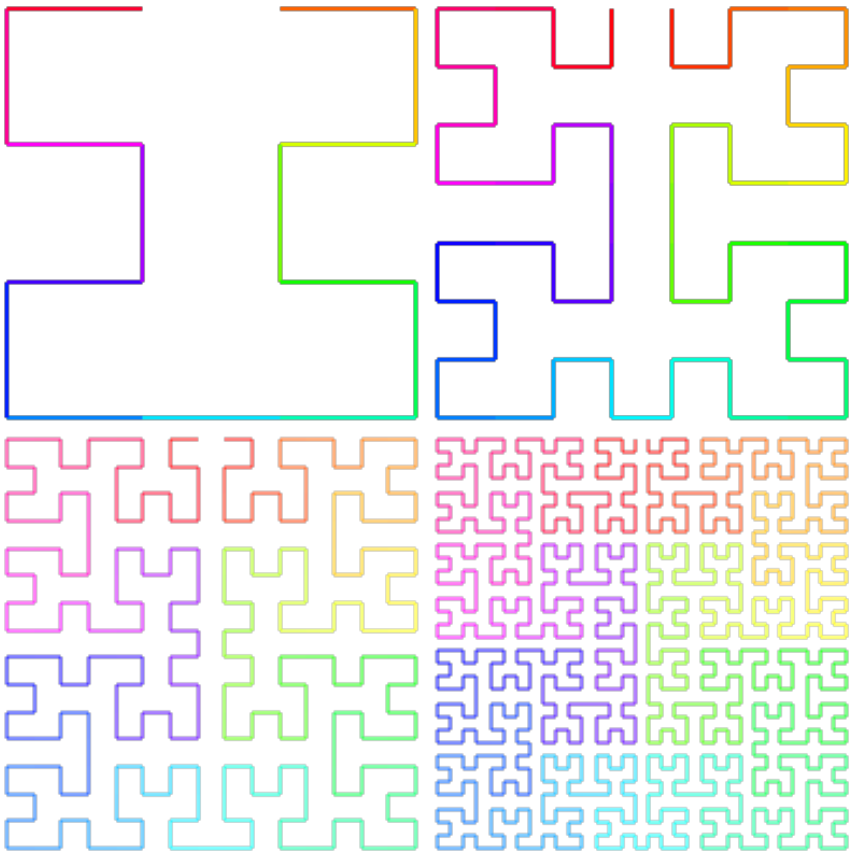

Figure : Moore curve ($d_{\mathrm{H}} = 2$)

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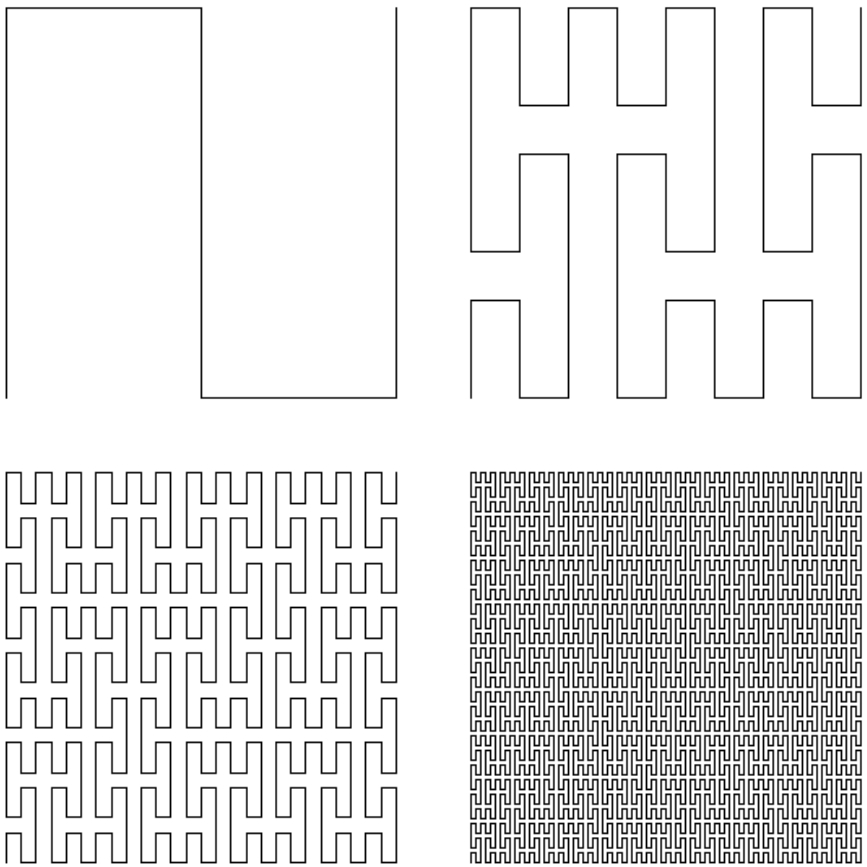

Figure : Peano curve ($d_{\mathrm{H}} = 2$)

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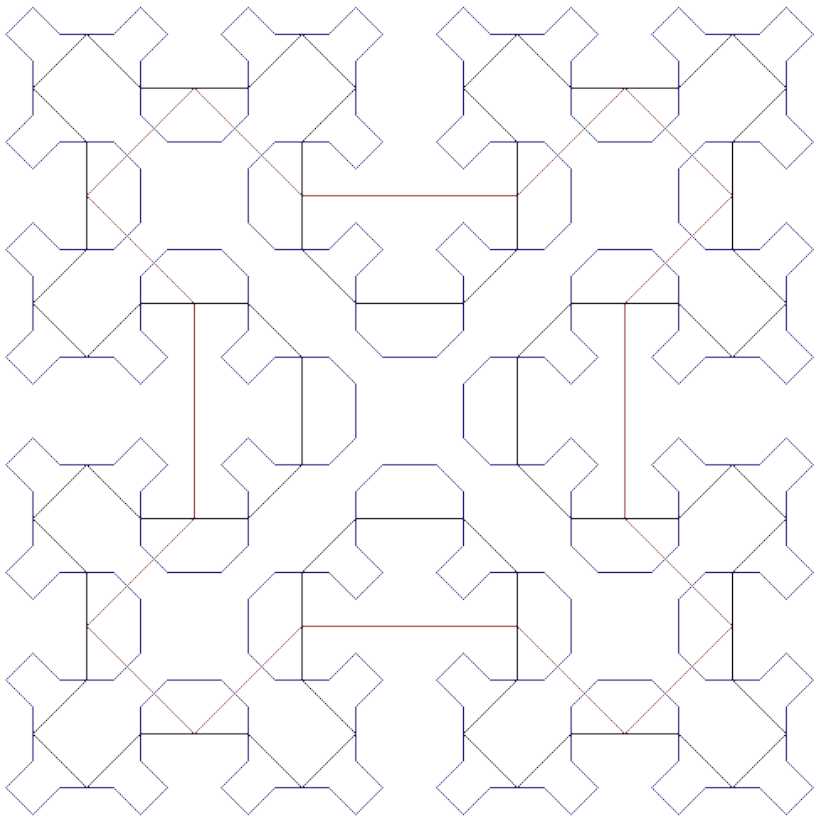

Figure : Sierpiński curve ($d_{\mathrm{H}} = 2$)

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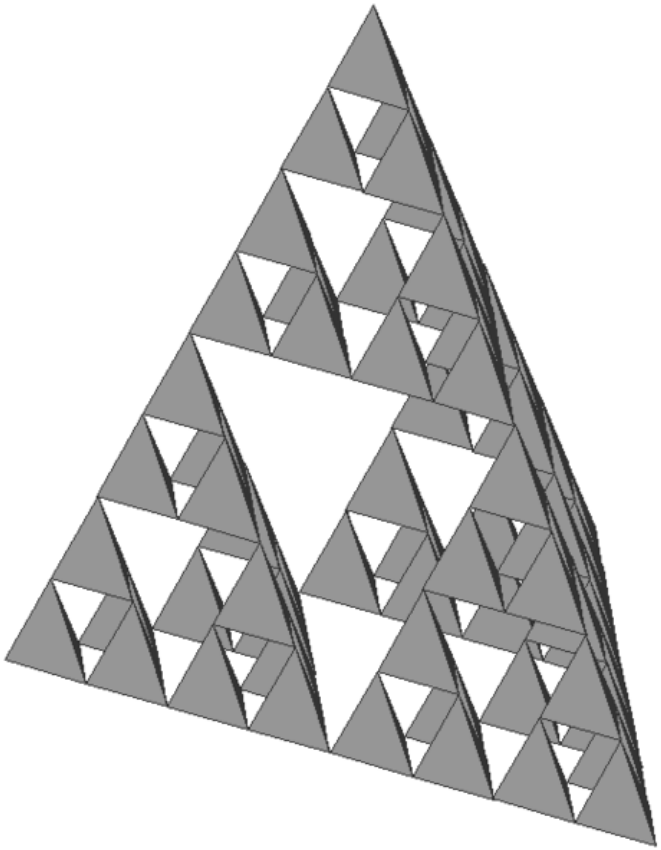

Figure : Sierpiński tetrahedron ($d_{\mathrm{H}} = 2$)

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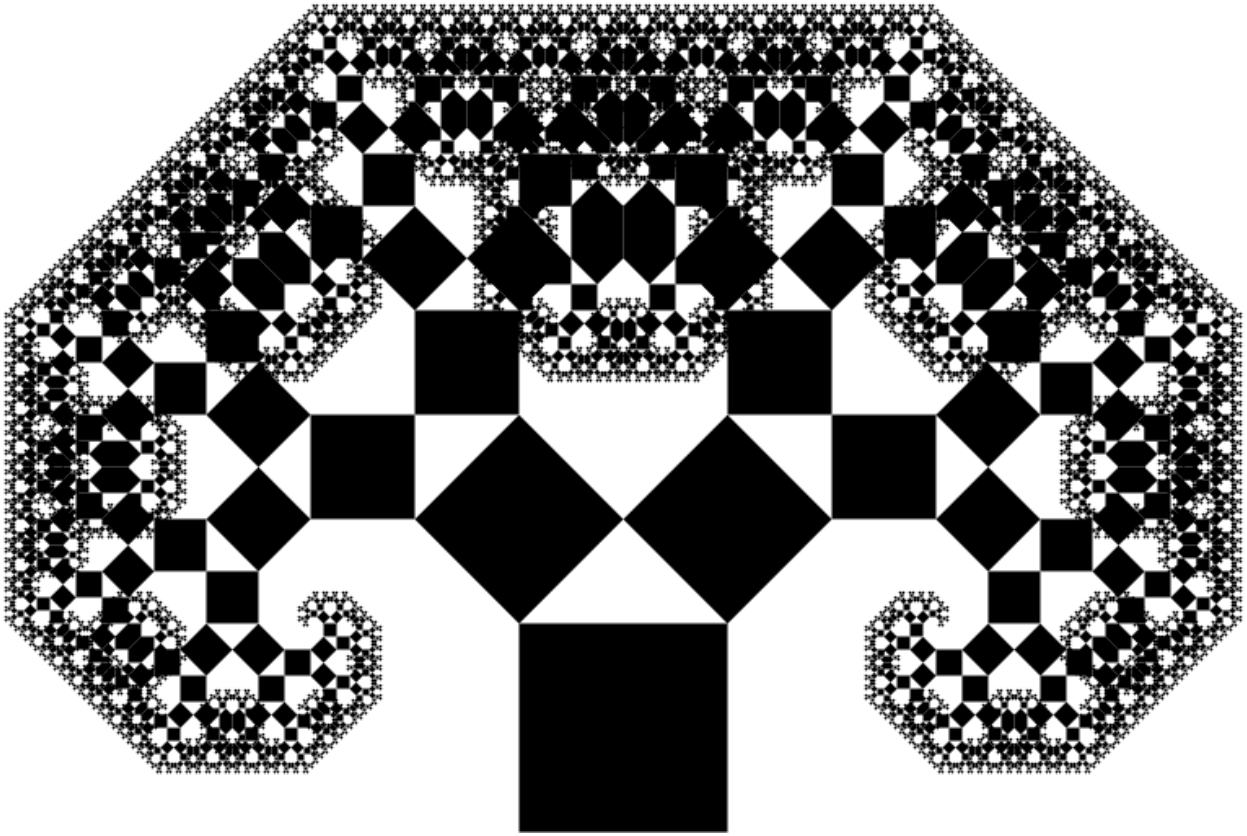

Figure : Pythagoras tree ($d_{\mathrm{H}} = 2$)

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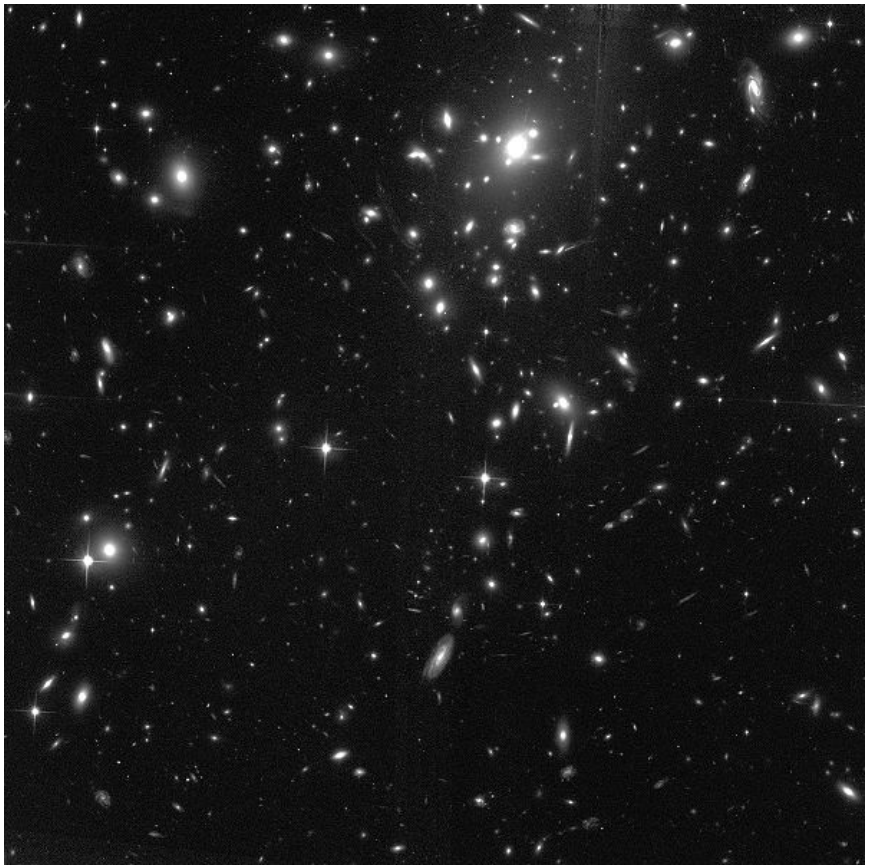

Figure : Galaxy distribution ($d_{\mathrm{H}} \approx 2$) [NATURAL FRACTAL]

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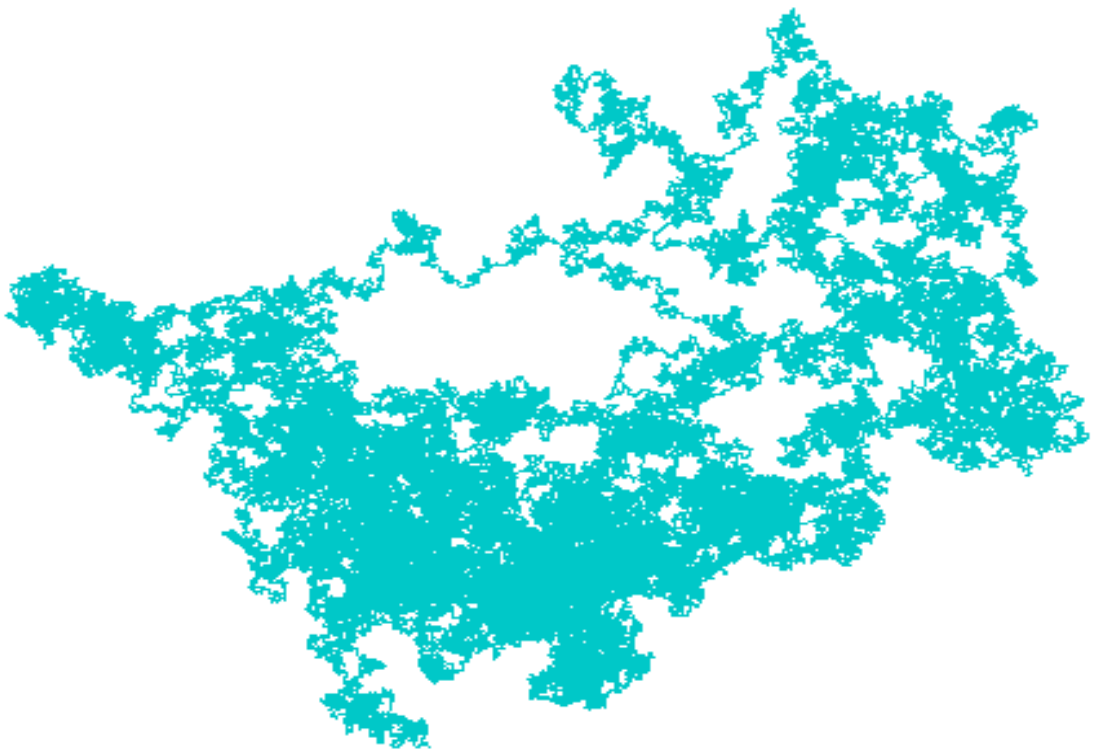

Figure : Brownian motion ($d_{\mathrm{H}} = 2$ in $D \geq 2$) [RANDOM FRACTAL]

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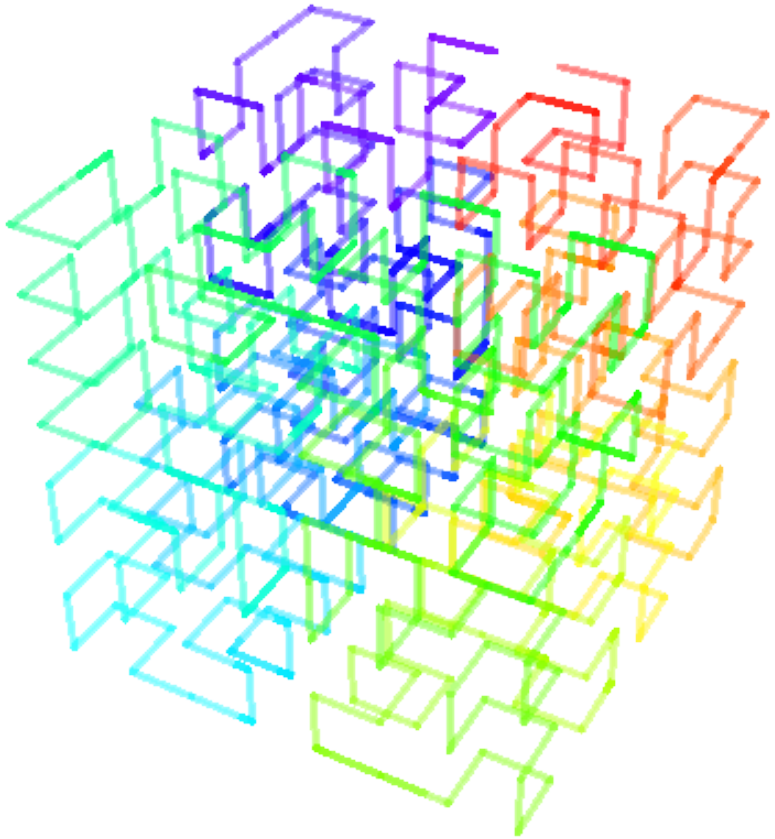

Figure : $3D$ Moore curve ($d_{\mathrm{H}} = 3$)

09/35– Fractals with integer Hausdorff dimension

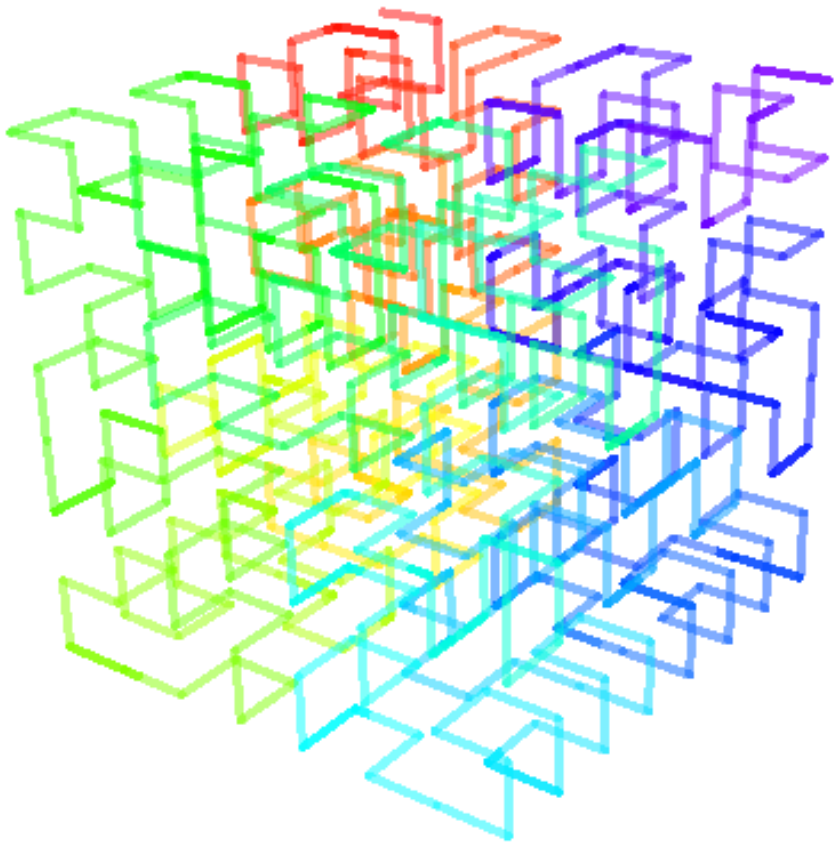

Figure : $3D$ Hilbert curve ($d_{\mathrm{H}} = 3$)

09/35– Fractals with integer Hausdorff dimension

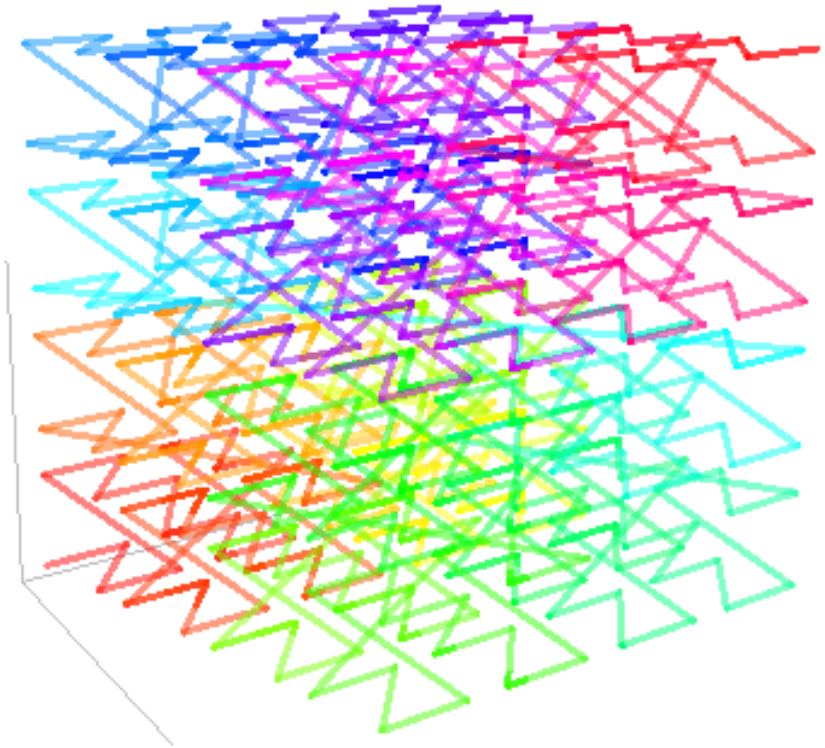

Figure : $3D$ Lebesgue curve ($d_{\mathrm{H}} = 3$)

09/35– Fractals with integer Hausdorff dimension

Figure : Lung surface ($d_{\mathrm{H}} \approx 2.97$) [NATURAL FRACTAL]

09/35– Fractals with integer Hausdorff dimension

Figure : Lung surface ($d_{\mathrm{H}} \approx 2.97$) [NATURAL FRACTAL]

10/35– Questions and caveats

Q3

Does $d_{\mathrm{H}} = 2$ in the UV guarantee renormalizability?

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Q3

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A3

No. Detailed RG analysis is required.

11/35– Questions and caveats

Q4

Is it really necessary to imagine this model on a “(multi)fractal”?

11/35– Questions and caveats

Q4

Is it really necessary to imagine this model on a “(multi)fractal”?

A4

No, the important thing is dimensional flow. But the fractal picture is natural and intuitive.

12/35– Questions and caveats

Q5

Are all spaces with anomalous dimension fractals?

12/35– Questions and caveats

Q5

Are all spaces with anomalous dimension fractals?

A5

NO! If $d_{\mathrm{S}} > d_{\mathrm{H}}$, they are associated with jump processes.

Outline

13/35– Real-order fractional spaces

$$\boxed{\mathcal{E}_\alpha^D = (\mathbb{R}^D,\, \varrho_\alpha,\, \mathrm{Calc}^\alpha,\, \|\cdot\|,\, \mathcal{K})}$$

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- Action measure ϱ_α.
- Differential structure and calculus Calc^α.
- Natural norm $\|\cdot\|$.
- "Laplacian" $\mathcal{K}$.

14/35– Measure (bilateral and isotropic)

$$\mathrm{d}\varrho_\alpha(x) = \mathrm{d}^D x\, v_\alpha(x) = \mathrm{d}^D x \prod_\mu \frac{|x^\mu|^{\alpha-1}}{\Gamma(\alpha)}$$

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Other choices of boundary or measure do not lead to different physics. However, unilateral measures ($x^\mu \geq 0$) seem not to be fit for QM and QFT.

15/35– Measure (bilateral and isotropic)

"Geometric" coordinates:

$$q^\mu := \varrho_\alpha(x^\mu) = \frac{|x^\mu|^\alpha}{\Gamma(\alpha+1)} \qquad \Rightarrow \qquad \mathsf{d}\varrho_\alpha = \mathsf{d}^D q$$

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Anomalous scaling natural in fractional (Lebesgue–Stieltjes) integrals!

Outline

16/35– Fractional calculus

- Fractional calculus as old as ordinary calculus (Leibniz, Riemann, Liouville) but subtler.
- Care must be taken to represent fractional operators and define functional calculus.
- Applications: dissipative mechanics, chaos and percolation theory, statistics and long-memory processes such as weather and stochastic financial models, system modeling and control in engineering.

17/35– Fractional operators

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Left Caputo derivative:

$$(\partial^\gamma f)(x) := \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{\mathrm{d}x'}{(x-x')^\gamma} \partial_{x'} f(x') \,, \qquad 0 < \gamma \leq 1$$

Liouville derivative:

$$({}_\infty\bar{\partial}^\gamma f)(x) := -\frac{1}{\Gamma(1-\gamma)} \int_x^{+\infty} \frac{\mathrm{d}x'}{(x'-x)^\gamma} \partial_{x'} f(x') \,, \qquad 0 < \gamma \leq 1$$

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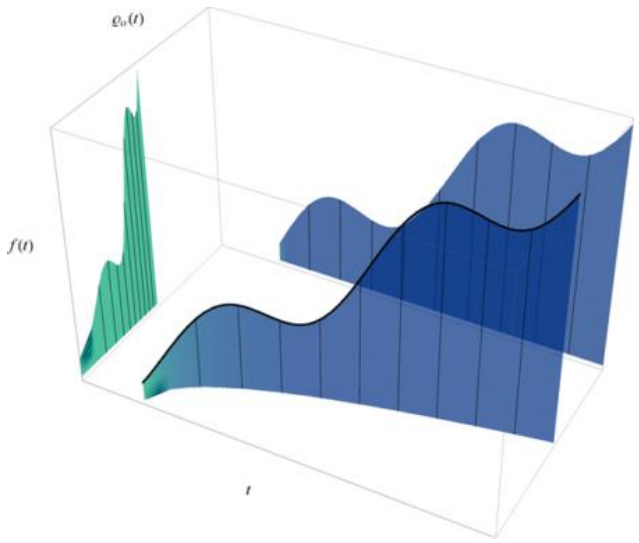
Weyl fractional integral:

$$(I^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \mathrm{d}x' \, (x'-x)^{\alpha-1} f(x') \,.$$

The measure ϱ_α defines a Weyl fractional integral.

18/35– Shadows on the wall

Bullock 1988, Podlubny 2002



- t-f plane.

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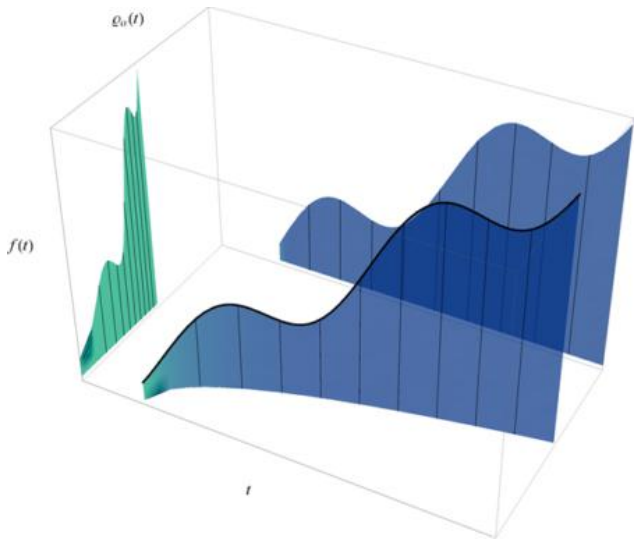

- t-f plane. Full-memory processes, $\alpha = 1$. Usual integral as "area under the curve $f(t)$".

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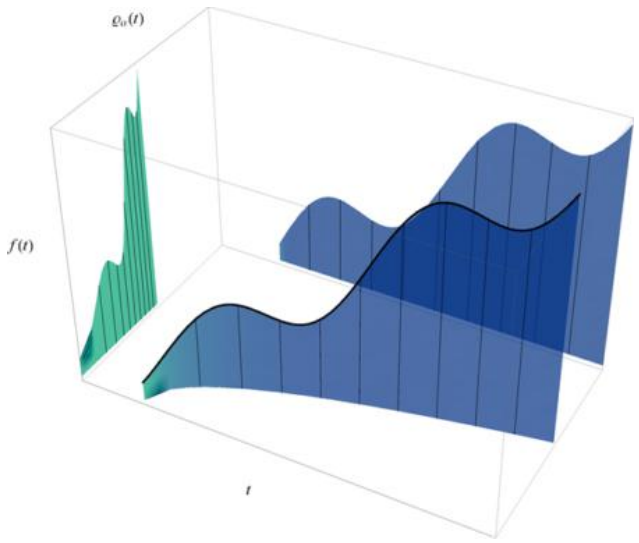

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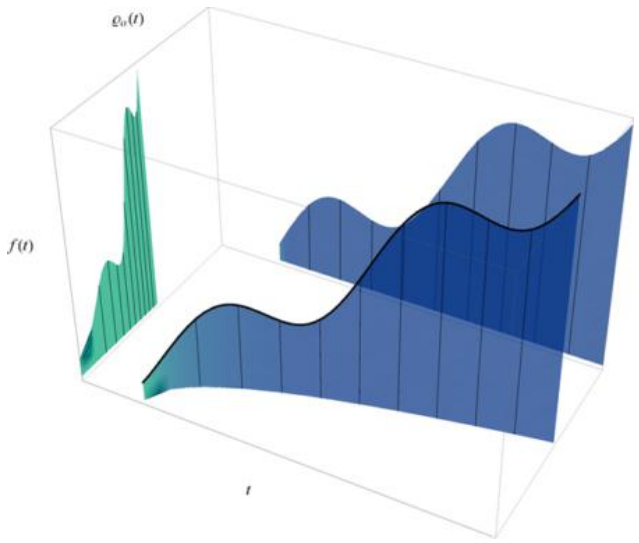

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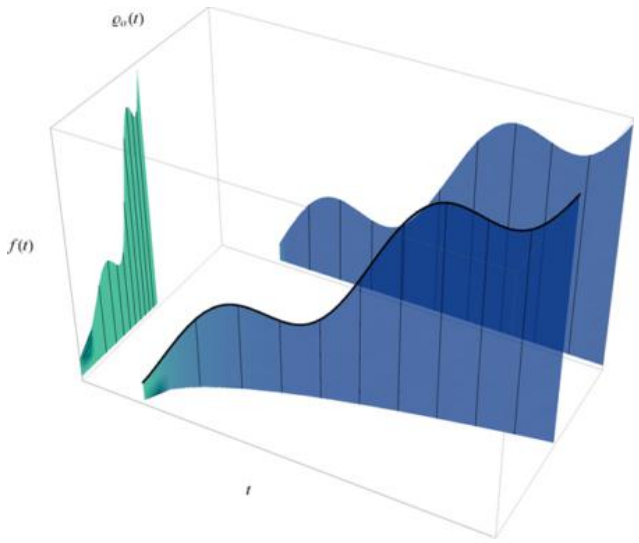

- t-f plane. Full-memory processes, $\alpha = 1$. Usual integral as "area under the curve $f(t)$".
- ϱ_α-f plane. Shadow is the fractional integral with $\alpha \neq 1$.
- Markov (no-memory) processes: $\alpha = 0$. $\alpha \sim$ fraction of states preserved at given time t.

19/35– Fractional approximates fractal

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- Integrals on net fractals (e.g., self-similar or cookie-cutter sets) can be approximated by fractional integrals with $\alpha \sim d_{\mathrm{H}}$ [Tatom 1995; Ren et al. 1996–2003; Nigmatullin & Le Méhauté 2003]:

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- Fractional integrals describe, e.g., anomalous transport systems [Zaslavsky 2002] and fractal media (analogy with dim. reg.) [Tarasov 2004–2007].

Outline

20/35– Norm

From the perspective of differential forms [Cottrill-Shepherd & Naber 2001; Tarasov 2008], the 2γ-norm is the natural distance:

$$\Delta_\gamma(x,y) := \sqrt{\sum_{\mu=0}^{D} \delta_{\mu\nu}(|x^\mu - y^\mu||x^\nu - y^\nu|)^\gamma}\,, \qquad [\Delta_\gamma] = -\gamma.$$

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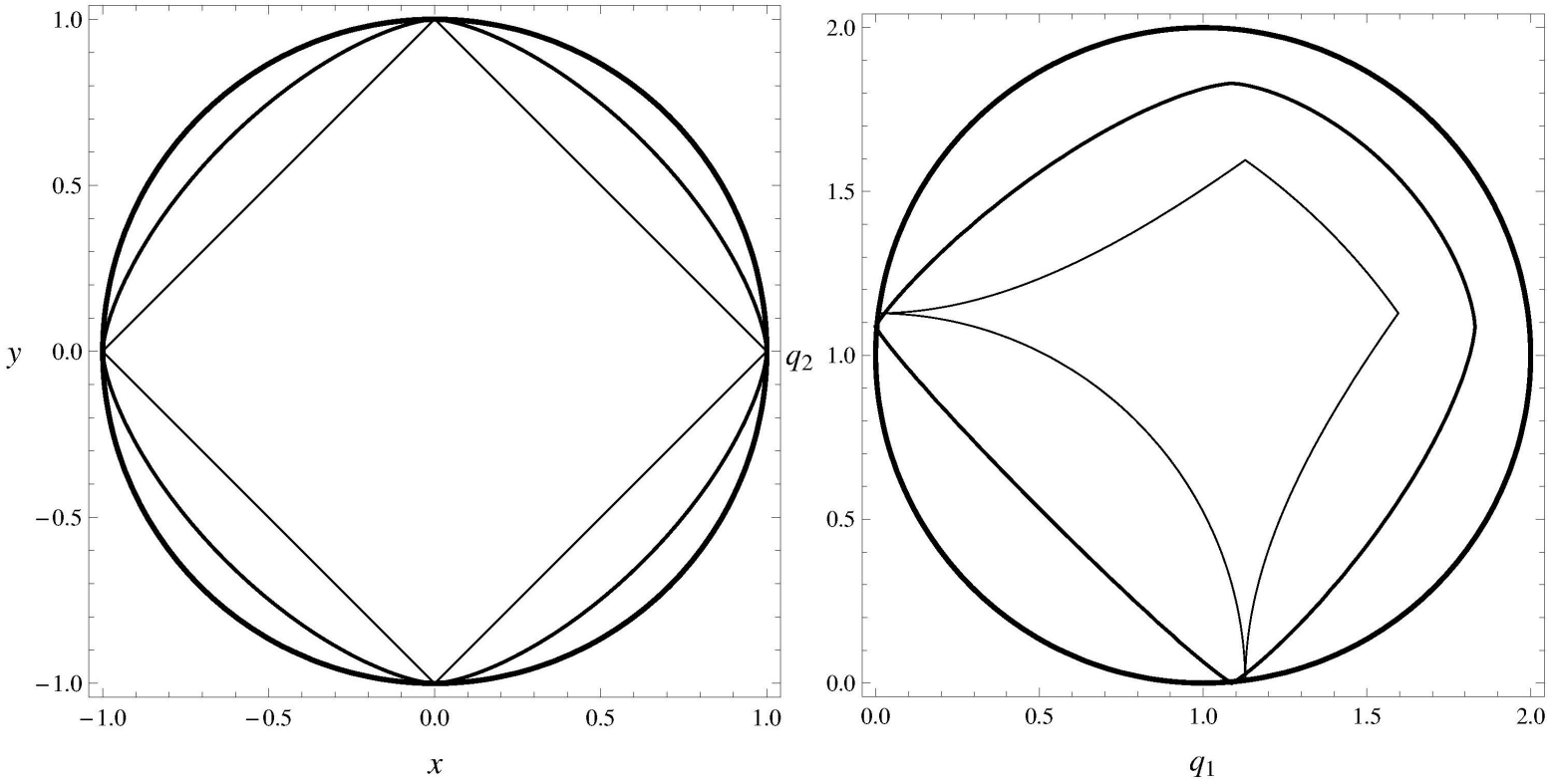
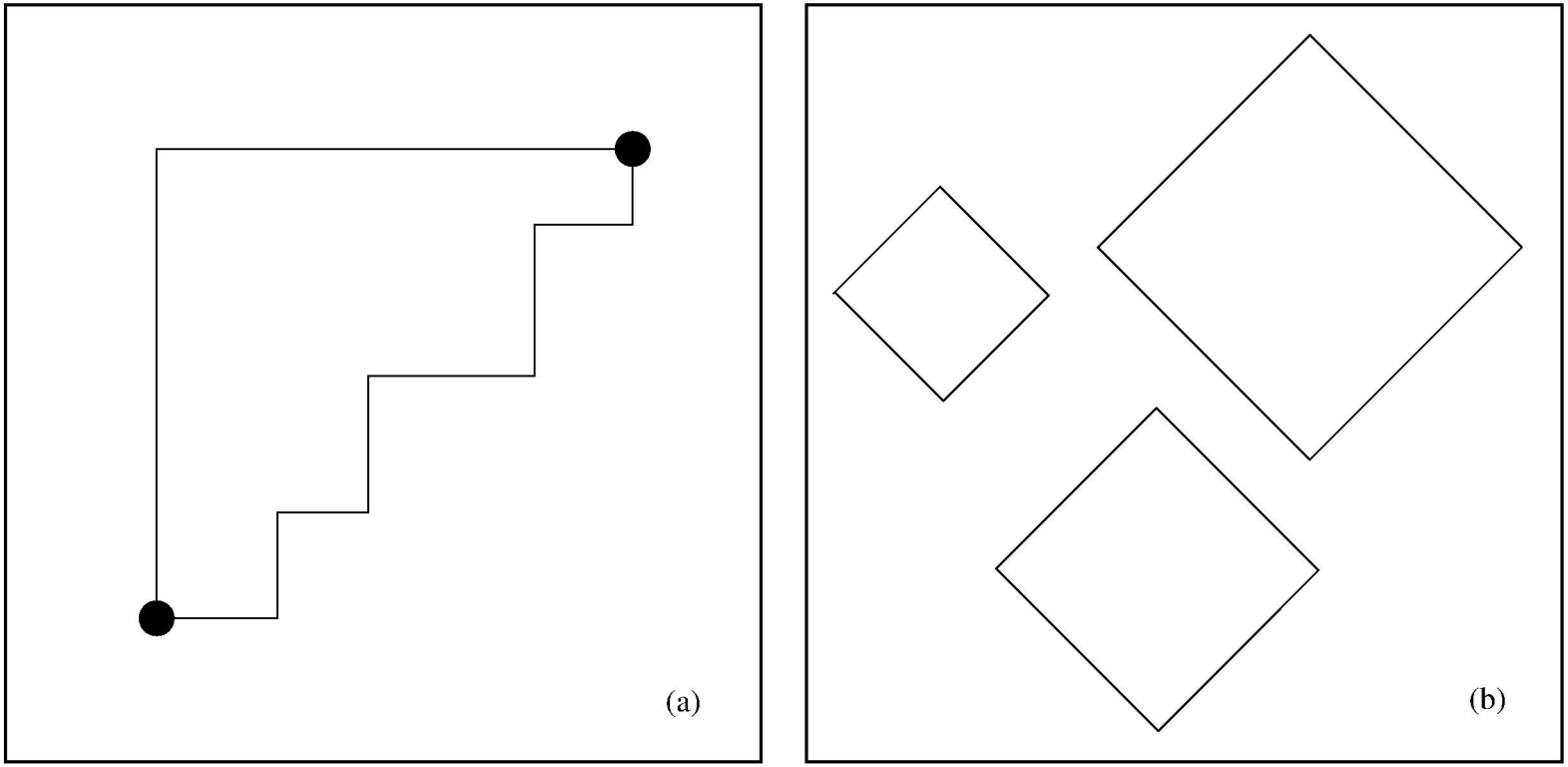
Skip examples

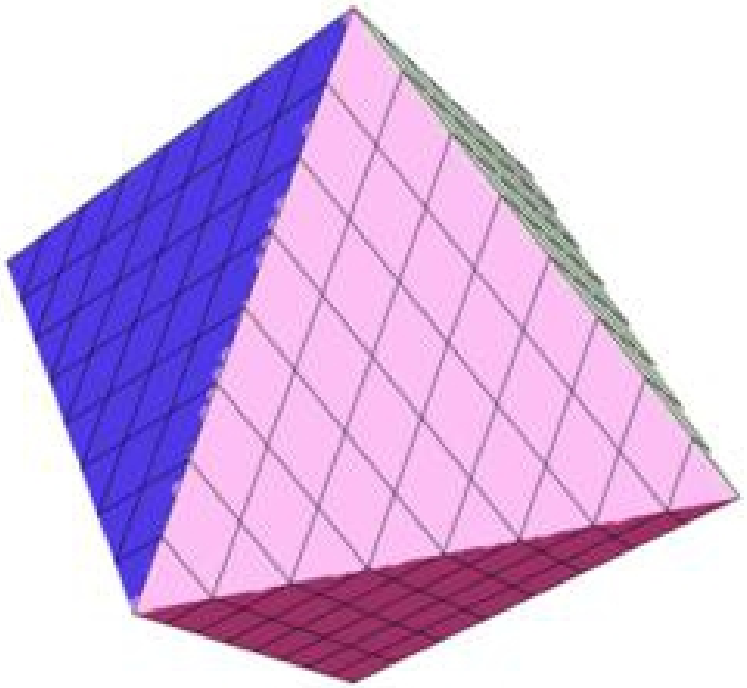
20/35– Circles ($D = 2$)



20/35– Taxicab geometry ($\alpha = 1/2$, $D = 2$)



20/35– Taxicab geometry ($\alpha = 1/2$, $D = 3$)



Outline

21/35– Laplacian

$$\mathcal{K}_\alpha = \frac{1}{\sqrt{v_\alpha(x)}} \delta^{\mu\nu} \partial_\mu \partial_\nu \left[\sqrt{v_\alpha(x)} \cdot \right], \qquad [\mathcal{K}_\alpha] = 2\,.$$

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Hermitian. Also fractional Laplacians are possible.

Outline

22/35– Are fractional spaces fractals?

Rigorous definition of fractal [Strichartz 2003]

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1. Fine structure (detail at every scale).

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23/35– Self-similarity and self-affinity

Similarities:

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but λ any $\rightarrow$ later this triviality will be fixed.
Actually, $\mathcal{E}_\alpha^D$ invariant under affinity:

$$q'^\mu = \Lambda(q^\mu) := \Lambda^\mu_\nu q^\nu + \mathrm{a}^\mu$$

24/35– Hausdorff dimension

Volume $\mathcal{V}^{(D)}$ of a D-ball of radius R defines d_{H} operationally:

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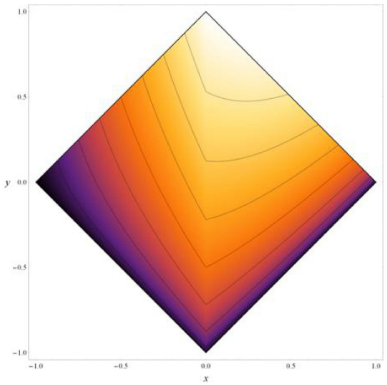

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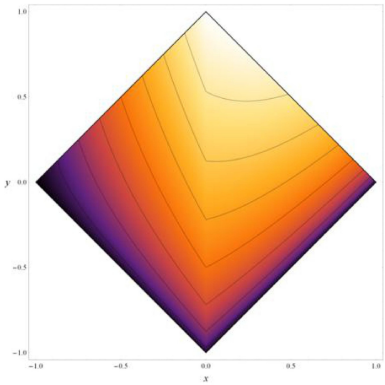

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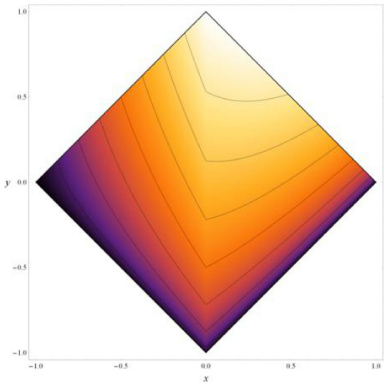

$$\boxed{d_{\rm H} = D\alpha}$$

Same result obtained via self-similarity theorem.

25/35– Momentum transform

$$\tilde{f}(k) := \int \mathrm{d}^D x\, v_\alpha(x) f(x)\, \mathbb{e}^*_\nu(k,x)\,, \qquad \mathbb{e}_\nu(k,x) = \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{\sqrt{v_\alpha(x) v_{\alpha'}(k)}}\,,$$

$$f(x) = \int \mathrm{d}^D k\, v_{\alpha'}(k) \tilde{f}(k)\, \mathbb{e}_\nu(k,x)\,, \qquad \mathcal{K}_\alpha \mathbb{e}_\nu = -k^2 \mathbb{e}_\nu\,.$$

Unitary and invertible, not necessarily automorphism.

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Unitary and invertible, not necessarily automorphism. There exists a infinite discrete class of transforms.
Fractional delta distribution (not translation invariant):

$$\delta_\alpha(x,x') = \frac{\delta(x-x')}{\sqrt{v_\alpha(x) v_\alpha(x')}}\,, \qquad \delta_{\alpha'}(k,k') = \frac{\delta(k-k')}{\sqrt{v_{\alpha'}(k) v_{\alpha'}(k')}}$$

Outline

26/35– Diffusion equation

$$(\partial_\sigma^\beta - \mathcal{K}_\alpha) P(x, x', \sigma) = 0\,, \qquad P(x, x', 0) = \delta_\alpha(x, x')\,, \qquad 0 < \beta \leq 1$$

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27/35– Spectral dimension

Return probabilty

$$\mathcal{P}(\sigma) := \frac{1}{\int \mathsf{d}\varrho_\alpha(x)} \int \mathsf{d}\varrho_\alpha(x)\, P(x,x,\sigma)$$

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Walk dimension [Havlin & Ben-Avraham 1987,2000]

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Anomalous diffusion if $\beta \neq 1$. Super-diffusion or jump processes ($d_{\mathrm{W}} < 2$) do *not* correspond to fractals! [Barlow, Grigor'yan, Kumagai 2008,2009]

Outline

28/35– Self-similar measures

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Self-similar measures describe multifractals, which have scale-dependent dimension.

29/35– Multifractional action

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Option 1:

$$S = \sum_n g_n \int \prod_\mu \mathrm{d}\varrho_{\alpha_n}(x^\mu)\mathcal{L}_{\alpha_n} \,.$$

Example: scalar field

$$\mathcal{L}_\alpha = \tfrac{1}{2}\phi\mathcal{K}_\alpha\phi - V(\phi) \,.$$

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Option 2:

$$S = \int \mathrm{d}^D x\, v(x)\, \mathcal{L} = \int \prod_\mu \left[\sum_n g_n \mathrm{d}\varrho_{\alpha_n}(x^\mu)\right] \mathcal{L} \,.$$

$$\mathcal{K}_\alpha \to \mathcal{K}_v = \frac{1}{\sqrt{v}}\partial_\mu\partial^\mu\left(\sqrt{v}\,\cdot\right)$$

30/35– Scale-dependent Hausdorff dimension

Toy model, two terms (binomial measure):

$$I_D = I_D^{\alpha_1} + \ell_*^{D(\alpha_1-\alpha_2)} I_D^{\alpha_2}\,, \qquad [I_D] = -D\alpha_1\,, \qquad \tfrac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1\,.$$

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$$\mathcal{V}^{(D)}(R) = \ell_*^{D\alpha_1}\left[\Omega_{D,\alpha_1}\left(\frac{R}{\ell_*}\right)^{D\alpha_1} + \Omega_{D,\alpha_2}\left(\frac{R}{\ell_*}\right)^{D\alpha_2}\right]\,.$$

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$R \ll \ell_*$: $\quad \mathcal{V}^{(D)} \sim R^{D\alpha_1}$

$R \gg \ell_*$: $\quad \mathcal{V}^{(D)} \sim \tilde{R}^{D\alpha_2}, \qquad \tilde{R} = R\ell_*^{-1+\alpha_1/\alpha_2}$

31/35– Multiscale diffusion equation

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$$(\partial_\sigma - \mathcal{K}_v)\, P(x, x', \sigma) = 0\,, \qquad P(x, x', 0) = \ldots$$

$$v(x) = \prod_\mu \left[\sum_{n=1}^{N} g_n v_{\alpha_n}(x^\mu) \right] = \prod_\mu \left[\sum_{n=1}^{N} \ell^{\alpha_n - 1} g_n v_{\alpha_n}(\tilde{x}^\mu) \right]$$

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Claim: $N-1$ characteristic scales $\ell_1 < \ell_2 < \cdots < \ell_{N-1}$, $\ell_N = \ell$ and $\zeta_n := g_n^{1/(1-\alpha_n)} / \sigma^{\beta/2}$ dimensionless such that

$$\zeta_1(\ell) = \frac{\ell_1}{\ell}\,, \qquad \zeta_n(\ell) = \frac{\ell_n}{|\ell - \ell_{n-1}|}\,, \qquad \zeta_N \equiv 1\,.$$

Skip derivation

31/35– ζ_n

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- The nth term must dominate over the others at $l_{n-1} \lesssim \ell \ll \ell_n$, so $l_{A,n} = \ell_n$ and, tentatively, $l_{B,n} = \ell$.

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- However, at $\ell \ll \ell_{n-1}$ the $(n-1)$th term dominates, so $l_{B,n} = \ell - \ell_{n-1}$.

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Dimensional flow is always measured starting from the lowest of two scales ℓ_{n-1} to the next ℓ_n, and relatively to the latter, which sets a gauge for the "rods".

32/35– Spectral dimension

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In each regime, $\sigma^{\beta/2} \sim \ell - \ell_{n-1}$,

$$\mathcal{P}(\sigma) \propto \prod_{\mu} \frac{1}{\sum_n g_n f_n^{(\mu)} \sigma^{\alpha_n \beta/2}} = \prod_{\mu} \frac{1}{\sum_n |\ell - \ell_{n-1}| \zeta_n^{1-\alpha_n}(\ell) f_n^{(\mu)}} \,.$$

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$f_n^{(\mu)} > 0$ dimensionless integrals $\int \mathrm{d}\varrho_{\alpha_n}(\tilde{x}_n^{\mu})$ regularized to a finite number.

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$f_n^{(\mu)} > 0$ dimensionless integrals $\int \mathrm{d}\varrho_{\alpha_n}(\tilde{x}_n^\mu)$ regularized to a finite number.

$$d_{\mathrm{S}}(\ell) = \beta \sum_\mu \frac{\sum_n \alpha_n \zeta_n^{1-\alpha_n}(\ell) f_n^{(\mu)}}{\sum_n \zeta_n^{1-\alpha_n}(\ell) f_n^{(\mu)}} \,.$$

Outline

33/35– One scale

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$$d_S \sim \begin{cases} D\,, & \ell \gg \ell_1 \quad \text{(IR)} \\ D\alpha_1\,, & \ell \ll \ell_1 \quad \text{(UV)} \end{cases}.$$

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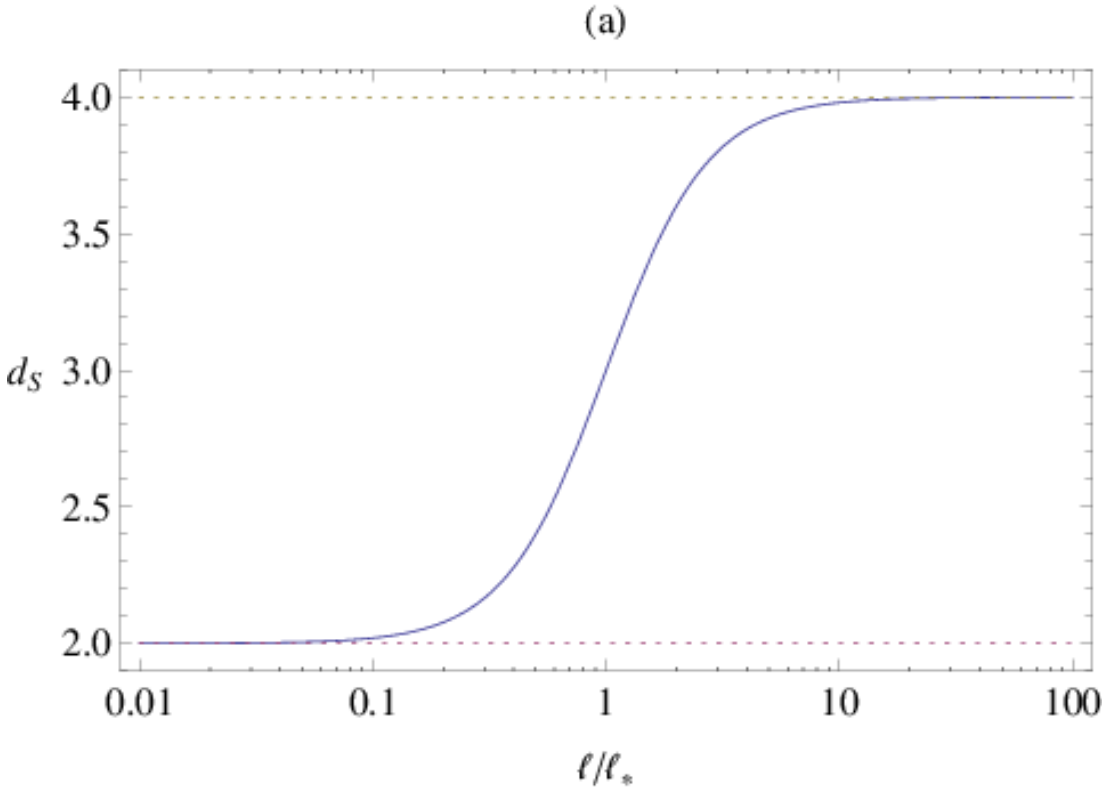
If $\alpha_1 = 1/2$ and $D = 4$, $d_S \sim 2$ in the UV.

33/35– One scale ($N = 2$)



34/35– Other single-scale system

Brownian-time telegraph process [Orsingher & Beghin 2009]:

$$\left(\partial_\sigma + \frac{\ell_1}{\ell}\,\partial_\sigma^{\frac{1}{2}} - \ell_1^2\nabla_x^2\right) P = 0\,,$$

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The spectral dimension d_S has the same profile of the previous figure.

35/35– Two scales ($N = 3$)

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$$d_{\rm S} \sim \begin{cases} D\,, & \ell \gg \ell_2 \gg \ell_1 \quad \text{(IR)} \\ D\alpha_2 = D/3\,, & \ell_1 \sim \ell \ll \ell_2 \quad \text{(intermediate)} \\ D\alpha_1 = D/2\,, & \ell \ll \ell_1 \ll \ell_2 \quad \text{(UV)} \end{cases} .$$

35/35– Two scales