# Introduction to multifractal spacetimes
## II – *Applications, complex measures, field theory*

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# Outline

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- Initial condition $\delta(x-x') \to f(x,x')$ [Modesto & Nicolini 2010].

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Stochastic model in the UV: Iterated Brownian motion
$X(\sigma) = B_1(|B_2(\sigma)|)$:

$$\left(\partial_\sigma^{1/2} - \nabla_x^2\right) P = 0\,, \quad \Leftrightarrow \quad \left(\partial_\sigma - \nabla_x^4\right) P = \frac{1}{\sqrt{\pi\sigma}} \nabla_x^2 P(x, x', 0)\,.$$

## 03/30– QEG: RG flow and multifractal geometry

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- RG scaling stems from comparison of any given scale $1/k = L \propto \ell^\alpha$ with a classical scale $1/k_0 = \ell$. To get finite results, the rod to use must be $k$-adapted ($q$-rod), yet $k_0$-dependent ($q = q(x)$).

# 04/30– Mapping QEG into multifractional geometry

| *QEG* | | *fractional theory* |
|---|---|---|
| $k = p_{\text{QEG}}$ (phys. momentum in a meas. at scale $L$) | $\longleftrightarrow$ | $(p_{\text{frac}})^{\alpha} \sim L^{-1} \leftrightarrow q^{-1}$ ($q$-scale) |
| IR limit $\downarrow$ | | |
| momenta/rods adapted $\updownarrow$ | | momenta fixed, rods adapted via measure $\updownarrow$ |
| $k_0 = p_{\text{cl}}$ (phys. momentum in a meas. at classical scale $\ell$) | $\longleftrightarrow$ | $p_{\text{frac}} \sim \ell^{-1} \leftrightarrow x^{-1}$ ($x$-scale) |

# 05/30– QEG vs. multifractional picture

| | Multifractional spacetimes | | QEG spacetimes |
|---|---|---|---|
| | | Mapping | |
| Coordinate | $x$ | $q(\ell) = \sum_n g_n \, \varrho_{\alpha_n}$ | $q$ |
| Physical momentum | $p_{\text{frac}}$ | $p(\ell)$ | $p_{\text{QEG}}$ |
| Scale dependence | implicit | explicit | implicit |
| Probed scale | $\ell = p_{\text{frac}}^{-1}$ | $L = p^{-1}$ | $L = p_{\text{QEG}}^{-1}$ |
| Rods adapted via | measure | momenta | momenta |
| Laplacian | $\mathcal{K}_\alpha \sim \partial_x^2$ | | $\Box^{1/\alpha} \sim \partial_q^{2/\alpha}$ |

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The very concept of dimensional flow is, in fact,

the notion of **adapted rod**

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HL multiscale geometry from hierarchy of differential Laplacian operators, from order $2z$ (UV) to $2$ (IR).
Different symmetries, physics inequivalent. $\mathcal{K}_\alpha \sim -p_t^2 + \partial_x^2$ vs. $\mathcal{K} \sim -\partial_t^2 + \partial_q^{2/\alpha}$.

## 08/30– $\alpha = 1 - \epsilon/D \sim 1$

| $D$ | $\Omega_{D,1-\epsilon/D}$ | $\Omega_{D-\epsilon,1}$ |
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| 2 | $\pi(1-0.42\epsilon)$ | $\pi(1-0.36\epsilon)$ |
| 3 | $\frac{4\pi}{3}(1-0.54\epsilon)$ | $\frac{4\pi}{3}(1-0.22\epsilon)$ |
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One can use bounds in dimensional regularization as a first approximation.

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CMB black-body spectrum [Caruso & Oguri 2009]:

$$|\epsilon| < 10^{-5}\,, \qquad \ell \sim 14.4\,\mathrm{Gpc}\,.$$

# 10/30– Bounds on dimensional flow – UV

Oscillations of neutral $B$ mesons and of the muon $g-2$: At mass scales $M > 300 \div 400\ \mathrm{GeV}$, any $2 < d_{\mathrm{H}} < 5$ is compatible with experiments [Shevchenko 2009].

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Rough upper bound for $\ell_*$:

$$\ell_* < 10^{-18}\,\mathrm{m}\,.$$

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Return probability on deterministic fractals displays *ripples* [Lapidus & van Frankenhuysen 2006; Teplyaev 2007; Akkermans et al. 2009]:

$$\mathcal{P}(\sigma) = \frac{1}{(4\pi\sigma)^{\frac{d_S}{2}}} F(\sigma)\,, \qquad F \text{ periodic in } \ln\sigma.$$

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Complex fractional integrals approximate integrals on fractals. Their average over a log-period are real-order fractional integrals (which better approximate random fractals) [Nigmatullin & Le Méhauté 2005].

# 12/30– Complex fractional measures

$$\varrho_\alpha(x) \to \varrho_{\alpha,\omega} = c_+|x|^{\alpha+\mathrm{i}\omega} + c_-|x|^{\alpha-\mathrm{i}\omega}, \qquad \omega \geq 0\,.$$

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Summing over $\alpha$, $\omega$ and imposing $S$ to be real,

$$S = \int \mathrm{d}\varrho(x)\,\mathcal{L}\,, \qquad \mathrm{d}\varrho(x) = \prod_\mu \left[\sum_\alpha g_\alpha \sum_\omega \mathrm{d}\varrho_{\alpha,\omega}(x^\mu)\right]$$

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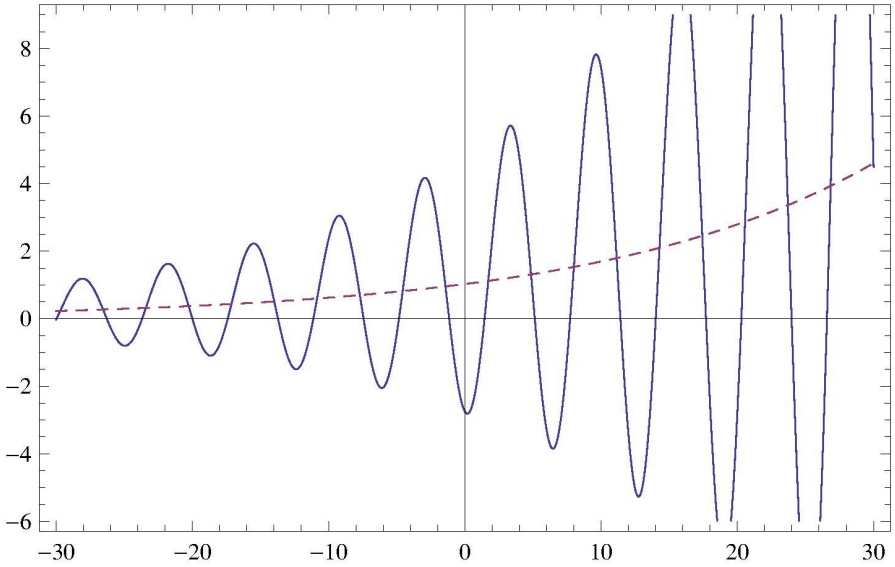
where

$$\boxed{\varrho_{\alpha,\omega}(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}\left[1 + A_{\alpha,\omega}\cos\left(\omega\ln\frac{|x|}{\ell_\infty}\right) + B_{\alpha,\omega}\sin\left(\omega\ln\frac{|x|}{\ell_\infty}\right)\right]}$$

$A_{\alpha,\omega}$ and $B_{\alpha,\omega} \in \mathbb{R}$. Form of measure also dictated by fractal geometry arguments.

# 13/30– Log-oscillating measure $\varrho_{\alpha,\omega}(x)$



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# 14/30– Discrete scale invariance

Oscillatory part of $\varrho$ log-periodic under the transformation

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DSIs appear in chaotic systems [Sornette 1998].

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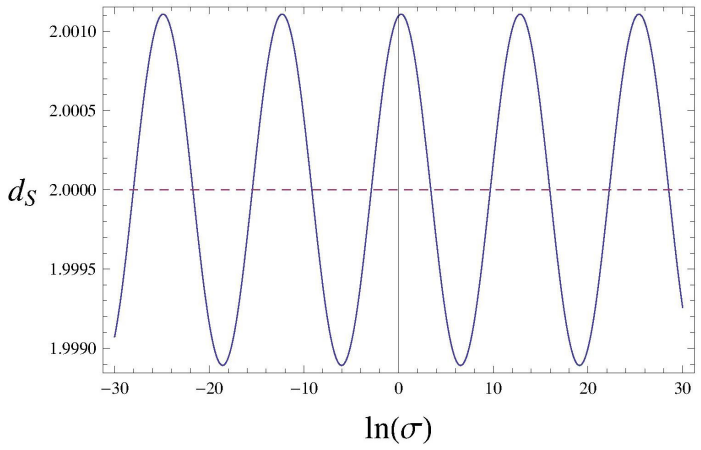


Figure : $\alpha = 1/2$ fixed, $\langle d_S \rangle = 2$, amplitudes $\sim 10^{-5}$.

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- Dimension of spacetime is $d_{\rm H} = d_{\rm S} = 4 - \epsilon$, Euclidean geometry in local inertial frames gets tiny corrections.

## 17/30– Classical regime ($\ell \gg \ell_*$)

- Ordinary Poincaré-invariant field theory on Minkowski spacetime recovered:, $\varrho(x) \sim \varrho_1(x) = x$.
- Dimension of spacetime is $d_{\rm H} = d_{\rm S} = 4 - \epsilon$, Euclidean geometry in local inertial frames gets tiny corrections.
- Bounds: $|\epsilon| < 10^{-8}$ at scales $\ell \sim 10^{-15}\,\mathrm{m}$. $\ell_* < 10^{-18}\,\mathrm{m}$ constrained by particle physics observations.

# 18/30– Fractional Minkwoski spacetime and generalizations

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$$v(x) = \prod_{\mu=0}^{D-1} v_\mu(x^\mu)\,, \qquad v_\mu(x^\mu) \geq 0\,.$$

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Guarantee unitary invertible momentum transform and consistent treatment of Noether currents.

# 19/30– Scalar field theory

arXiv:1210.2754 (with G. Nardelli)

$$S = \int_{-\infty}^{+\infty} \mathrm{d}\varrho(x) \left[ \frac{1}{2} \phi \mathcal{K}_\nu \phi - V(\phi) \right] = \int_{-\infty}^{+\infty} \mathrm{d}\varrho(x) \left[ -\frac{1}{2} \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi - V(\phi) \right]$$

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Field redefinition $\varphi(x) := \sqrt{v(x)}\, \phi(x)$, for $V(\phi) \propto \phi^n$ the aciton is

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*Formally*, one can treat multifractal FTs as non-autonomous FTs with spacetime-dependent couplings.

## 20/30– Fractional translations

Self-adjoint momentum operator and unitary $\infty$-dimensional repr. of "translations"

$$\hat{P}_\mu := -\mathrm{i}\mathcal{D}_\mu = \frac{1}{\sqrt{v}}\hat{p}_\mu\sqrt{v}\,, \qquad U_\epsilon := \mathrm{e}^{\mathrm{i}\epsilon^\mu\hat{P}_\mu} = \frac{1}{\sqrt{v}}\,\bar{U}_\epsilon\,\sqrt{v}\,.$$

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$\phi$ is a scalar density under $x \to x + \epsilon$:

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**Free** action invariant under fractional translations.

## 21/30– Poincaré symmetry of free theory

Free action (not measure or Lagrangian separately) Poincaré invariant.

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\begin{aligned}
&[\hat{P}_\mu, \hat{P}_\nu] = 0\,, \\
&[\hat{P}_\mu, \hat{J}_{\nu\rho}] = \mathrm{i}(\eta_{\mu\rho}\hat{P}_\nu - \eta_{\mu\nu}\hat{P}_\rho)\,, \\
&[\hat{J}_{\mu\nu}, \hat{J}_{\sigma\rho}] = \mathrm{i}(\eta_{\mu\rho}\hat{J}_{\nu\sigma} - \eta_{\nu\rho}\hat{J}_{\mu\sigma} + \eta_{\nu\sigma}\hat{J}_{\mu\rho} - \eta_{\mu\sigma}\hat{J}_{\nu\rho})\,;
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- $\hat{P}$ and $\hat{J}$ do not generate ordinary Poincaré transformations.
- Interacting theory: Poincaré algebra deformed, invariance broken.

# 22/30– EOM and energy-momentum tensor

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$$\check{\mathcal{D}}_\mu T^\mu{}_\nu = \frac{\partial_\nu v}{v}\left(\frac{1}{2}\phi V_{,\phi} - V\right) =: s_\nu(x,\phi), \qquad \check{\mathcal{D}}_\mu := \frac{1}{v}\partial_\mu\left(v\,\cdot\right) .$$

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$$\boxed{\check{\mathcal{D}}_t P^\nu = \int \mathsf{d}\varrho(\mathbf{x})\, s^\nu(x,\phi)}$$

Momentum not conserved in time.

# 24/30– Deformed Poincaré algebra

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Fractional Poisson brackets:

$$\{A(\mathbf{x}), B(\mathbf{x}')\}_\nu := \int \mathrm{d}\varrho(\mathbf{y}) \left[ \frac{\delta_\nu A(\mathbf{x})}{\delta_\nu \phi(\mathbf{y})} \frac{\delta_\nu B(\mathbf{x}')}{\delta_\nu \pi_\phi(\mathbf{y})} - \frac{\delta_\nu A(\mathbf{x})}{\delta_\nu \pi_\phi(\mathbf{y})} \frac{\delta_\nu B(\mathbf{x}')}{\delta_\nu \phi(\mathbf{y})} \right] ,$$

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Does a quantum theory exist with well-defined concept of mass?

## 25/30– Adiabatic switching

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- I.e., mass and spin are understood for free-particle quantum field operators $\phi_{\rm in}$ and $\phi_{\rm out}$.
- Physical Hilbert spaces built through repeated action of asymptotic creation operators on the vacuum.

# 26/30– Quantum free theory and propagator

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$$\phi(x) = \frac{1}{\sqrt{v_0(t)}} \int \frac{\mathsf{d}\varrho(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} \left[ a^\dagger(\mathbf{p}) \mathbb{e}_\nu^*(\mathbf{p}, x) + a(\mathbf{p}) \mathbb{e}_\nu(\mathbf{p}, x) \right]_{p^0 = \omega(\mathbf{p})}$$

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$$\boxed{G(x, y) = \mathrm{i}\langle 0 \,|\mathcal{T}\,[\phi(x)\phi(y)]|\, 0\rangle_v = \int \mathrm{d}\varrho(k)\, \frac{\mathbb{e}_v(k,x)\mathbb{e}_v^*(k,y)}{k^2+m^2-\mathrm{i}\epsilon} = \frac{\bar{G}(x-y)}{\sqrt{v(x)v(y)}}}$$

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Fractional Green equation

$$(-\mathcal{D}_\mu \mathcal{D}^\mu + m^2) G(x, y) = \delta_\nu(x, y)\,.$$

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- Unitarity: no, loss of probability/states **but** expected and under control.
- Feynman diagrams under study.
- Gravity: Same mechanism applies with $\mathcal{L} \sim R$.

## 28/30– Conclusions

- "Quantum gravity" is a wide subject which can be explored in many different ways, not only in "big" frameworks such as string theory and LQG, but also in other theories sharing some universal characteristics.

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- They have just been formulated, many many things to check (field theory, RG analysis, gravity, cosmology, . . . ).
- Heavy use of fractal geometry and stochastic tools indispensable for their formulation.

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10. Quantum mechanics worked out (with Nardelli & Scalisi).

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8. Observational constraints on multifractional geometries (particle- and large-scale).