Deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring

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Outline

Introduction

The AdS/CFT correspondence relates strings on AdS spaces to (conformal) gauge theories [Maldacena'97]

Prototype

IIb string theory on $AdS_5 \times S^5 \Longleftrightarrow \mathcal{N} = 4$ Super Yang-Mills

Under the correspondence

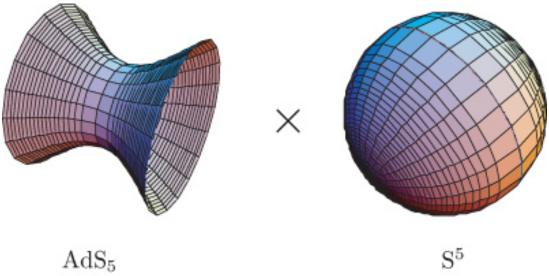
$$\text{energy } E \Leftrightarrow \Delta \text{ conformal dimensions}$$

Plan 1: compute spectra in planar limit, for arbitrary 't Hooft coupling λ

Plan 2: Extend discussion to less 'special' theories

The $AdS_5 \times S^5$ superstring

Closed string moving in $\mathrm{AdS}_5 \times \mathrm{S}^5$ background



$$\mathcal{L} = -\frac{g}{2}\left\{\gamma^{\alpha\beta} G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \ + \ \text{fermions}\right\}$$

- Under AdS/CFT $g \sim \sqrt{\lambda}$

Planar limit: zero string coupling $\Rightarrow$ world sheet is cylinder

Introduction

Problem: Spectrum of QFT in finite volume.

Way around this, integrability

Integrable theories are exactly solvable. Symmetry plays an important role.

Guiding principle:

Symmetries & Integrability $\Rightarrow$ Spectrum.

$\mathrm{AdS}_5 \times \mathrm{S}^5$ string hints at quantum integrability

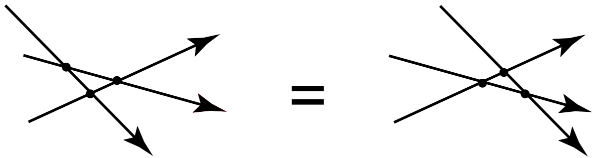
Top down approach

Assume quantum integrability and see where it leads

Integrability

Scattering for integrable theories

- no particle production/annihilation
- scattering permutes momenta
- any scattering process is a product of two-particle scattering processes
- two-particle S-matrix satisfies Yang-Baxter equation



Momenta are conserved and *all* scattering info in two-particle S-matrix

The $AdS_5 \times S^5$ superstring

Work in light-cone gauge [Arutyunov, Frolov '04], [Frolov, Plefka, Zamaklar '06]

- use AdS time t and S^5 equator-angle ϕ
- Conserved quantities $E = \int d\sigma p_t, J = \int d\sigma p_\phi$
- Hamiltonian $H = E - J$

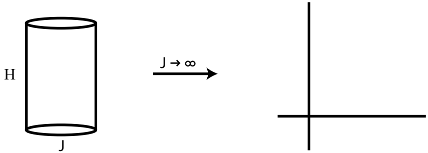
Remaining fields

- 4 bosons from AdS_5
- 4 bosons from S^5
- 8 fermions

Circumference of world-sheet is proportional to J

The $AdS_5 \times S^5$ superstring

Decompactification limit:



- Asymptotic states and S-matrices
- Symmetry $\mathfrak{su}(2|2)^2_{ce}$
- S-matrices give asymptotic spectrum via Bethe Ansatz

  [Beisert, Staudacher '05]

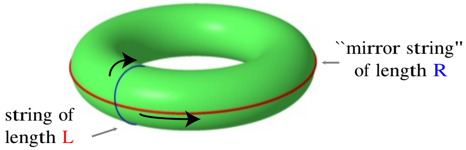
Problem: away from infinite volume?

TBA

An approach to finite size: Thermodynamic Bethe Ansatz (TBA)

TBA [Zamolodchikov '90]

- Define a mirror model [Arutyunov, Frolov '07]
- Finite size effects ⇔ finite temperature at infinite size



Bottom line: Asymptotics exact, *but* now thermodynamics.

Ingredients: Scattering data of all particles in spectrum

Symmetry algebra

Light-cone Hamiltonian has symmetry: $\mathfrak{su}(2|2)^2_{c.e.}$ [Arutyunov, Frolov, Plefka, Zamaklar '06]

$\mathfrak{su}(2|2)_{c.e.}$ [Beisert '05]

$$[\mathbb{L}_a^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \tfrac{1}{2}\delta_a^b \mathbb{J}_c \qquad [\mathbb{R}_\alpha^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma$$

$$[\mathbb{L}_a^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \tfrac{1}{2}\delta_a^b \mathbb{J}^c \qquad [\mathbb{R}_\alpha^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma$$

$$\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C} \qquad \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger$$

$$\{\mathbb{Q}_\alpha^a, \mathbb{Q}_b^{\dagger\beta}\} = \delta_b^a \mathbb{R}_\alpha^\beta + \delta_\alpha^\beta \mathbb{L}_b^a + \tfrac{1}{2}\delta_b^a\delta_\alpha^\beta \mathbb{H}.$$

$\mathbb{L}, \mathbb{R}$ generate bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

$\mathbb{Q}, \mathbb{Q}^\dagger$ are susy generators

$\mathbb{H}, \mathbb{C}, \mathbb{C}^\dagger$ are central elements

Symmetry algebra

Features

- The central element $\mathbb{H}$ corresponds to the light-cone Hamiltonian $\mathbb{H}$
- The central elements $\mathbb{C}, \mathbb{C}^{\dagger}$ appear because of world-sheet momenta $\mathbb{C} \sim g(e^{iP} - 1)$
- Representations depend on p, g

The Hilbert space of states carries a representation of this algebra

The S-matrix should respect this symmetry

The S-Matrix [Beisert'05]

$$
\begin{aligned}
\mathbb{S}(p_1,p_2) = {} & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\left(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 - E_1^1\otimes E_2^2 - E_2^2\otimes E_1^1\right)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\left(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 + E_1^2\otimes E_2^1 + E_2^1\otimes E_1^2\right)\\
& - \left(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\right)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\left(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\right)\\
& + \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\,\frac{\eta_1}{\tilde\eta_1}\left(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\right)\\
& + \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{\eta_2}{\tilde\eta_2}\left(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\right)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\left(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\right)\\
& + \frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\left(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\right)\\
& + \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\,\frac{\eta_2}{\tilde\eta_1}\left(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\right)\\
& + \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\,\frac{\eta_1}{\tilde\eta_2}\left(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\right)
\end{aligned}
$$

$$
\frac{x^+}{x^-} = e^{ip}, \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}
$$

Bound state S-Matrices

Properties:

- Unitarity
- Crossing Symmetry [Janik '06]
- Yang-Baxter equation (YBE)

The scalar factor is also known [Arutyunov,Frolov,Staudacher'04] [Beisert,Hernandez,Lopez '06] [Beisert,Eden,Staudacher '06]

Similarly scattering data for all states can be found

[Arutyunov,MdL,Torrielli '09][Arutyunov,Frolov '09]

The Bethe Ansatz

NLSM

Example, the Nonlinear Schrödinger model

$$H = \int dx \, \partial\phi\partial\phi^* - c|\phi|^4$$

Number operator commutes with H; restrict to 2 particles
Ansatz: separated plane waves with momenta k_1, k_2

$$|k_1, k_2\rangle_1 = \int dx_1 dx_2 \, \theta(x_1 < x_2) e^{ik_n x_n} \phi^*(x_1)\phi^*(x_2)|0\rangle$$

and

$$|k_1, k_2\rangle_2 = \int dx_1 dx_2 \, \theta(x_2 < x_1) e^{ik_n x_n} \phi^*(x_1)\phi^*(x_2)|0\rangle$$

Ansatz for eigenstate

$$|k_1, k_2\rangle = |k_1, k_2\rangle_1 + \mathcal{A}|k_1, k_2\rangle_2$$

NLSM

Explicit computation shows

$$H|k_1, k_2\rangle = (k_1^2 + k_2^2)|k_1, k_2\rangle$$

provided

$$\mathcal{A} = \frac{k_1 - k_2 - ic}{k_1 - k_2 + ic}$$

This is the S-matrix, $\mathcal{A} = \mathbb{S}$.

$$\theta(x_1 - x_2)e^{i(k_1 x_1 + k_2 x_2)}$$

$$\mathbb{S}\ \theta(x_2 - x_1)e^{i(k_1 x_1 + k_2 x_2)}$$

NLSM

Periodicity $\psi(x_1, x_2) = \psi(x_1 + L, x_2)$
Bethe equations (quantisation conditions)

$$e^{ik_i L} = \prod_{i \neq j} \mathbb{S}(k_i, k_j)$$

For systems with color

$$e^{ik_j L} = \mathbb{S}_{j+1,j}\mathbb{S}_{j+2,j} \dots \mathbb{S}_{N,j}\mathbb{S}_{1,j}\mathbb{S}_{2,j} \dots \mathbb{S}_{j-1,j}$$

Matrix equation $\Rightarrow$ diagonalize in steps

- apply Bethe like Ansatz on color (nesting)

Twisted Boundary Conditions

One can also do quasi-periodic boundary conditions (flux)

$$\psi(x_1, x_2) = e^{i\phi}\psi(x_1 + L, x_2)$$

Bethe equations get modified

$$e^{i(k_j L+\phi)} = \mathbb{S}_{j+1,j}\mathbb{S}_{j+2,j}\dots\mathbb{S}_{N,j}\mathbb{S}_{1,j}\mathbb{S}_{2,j}\dots\mathbb{S}_{j-1,j}$$

Question:

What kind of twist are compatible with integrable structure?

Best answered in Algebraic Bethe Ansatz framework.

Algebraic Bethe Ansatz

Lax operator $L_{a,i}(q, p_i)$ ($\sim S_{ai}$) acts on $V_a \otimes V_i$

$$V_a \otimes \underbrace{V_1 \otimes \ldots \otimes V_K}_{\text{physical space}}$$

It satisfies fundamental commutation relations ($\sim$YBE)

$$S_{ab} L_{a,i} L_{b,i} = L_{b,i} L_{a,i} S_{ab}$$

Monodromy and Transfer matrix

$$\mathbb{T}(q|\vec{p}) = L_{a,1} \ldots L_{a,K}, \qquad T(q|\vec{p}) = \mathrm{tr}_a \mathbb{T}$$

Generates family of conserved charges (FCR)

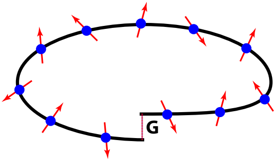
$$[T(q), T(q')] = 0, \qquad H \sim \frac{d}{dq} \ln T$$

Algebraic Bethe Ansatz

Suppose S has a symmetry $[S, G \otimes G] = 0$

Twisted Transfer Matrix: $T^G = \mathrm{tr}_a(G_a L_{a,1} \dots L_{a,K})$

G determines boundary conditions [Sklyanin '88]



Still defines commuting family of charges!

Integrable B.C. $\Leftrightarrow$ Symmetries of S-matrix

Algebraic Bethe Ansatz

The symmetry of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

$$G \in \underbrace{SU(2) \oplus SU(2)}_{AdS} \oplus \underbrace{SU(2) \oplus SU(2)}_{S}$$

W.l.o.g.

$$G = \mathrm{diag}(e^{i\alpha}, e^{-i\alpha} | e^{i\dot{\alpha}}, e^{-i\dot{\alpha}} | e^{i\beta}, e^{-i\beta} | e^{i\dot{\beta}}, e^{-i\dot{\beta}})$$

Phase factors in Bethe equations and transfer matrix

Bethe equations

Twisted Bethe equations

$$1 = e^{i(\alpha+\dot{\alpha})} \left(\frac{x_k^+}{x_k^-}\right)^J \prod_{l\neq k}^{K^{\rm I}} S_0(p_k,p_l)^2 \prod_{l=1}^{K^{\rm II}} \frac{x_k^- - y_l}{x_k^+ - y_l}\sqrt{\frac{x_k^+}{x_k^-}} \prod_{l=1}^{\dot{K}^{\rm II}} \frac{x_k^- - \dot{y}_l}{x_k^+ - \dot{y}_l}\sqrt{\frac{x_k^+}{x_k^-}}$$

and they are suplemented by the auxiliary Bethe equations

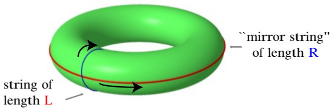
$$1 = e^{i(\beta-\alpha)} \prod_{l=1}^{K^{\rm I}} \frac{y_k - x_l^+}{y_k - x_l^-}\sqrt{\frac{x_k^-}{x_k^+}} \prod_{l=1}^{K^{\rm III}} \frac{y_k + \frac{1}{y_k} - w_l + \frac{i}{g}}{y_k + \frac{1}{y_k} - w_l - \frac{i}{g}}$$

$$1 = e^{-2i\beta} \prod_{l=1}^{K^{\rm II}} \frac{w_k - y_k - \frac{1}{y_k} + \frac{i}{g}}{w_k - y_k - \frac{1}{y_k} - \frac{i}{g}} \prod_{l\neq k}^{K^{\rm III}} \frac{w_k - w_l - \frac{2i}{g}}{w_k - w_l + \frac{2i}{g}}$$

where

$$x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{2i}{g}, \qquad \frac{x_k^+}{x_k^-} = e^{ip_k}$$

Beyond leading order



Partition functions agree $Z = Z_{mir}$

Basic idea of TBA

Spectrum of original theory $\Leftrightarrow$ thermodynamic quantities mirror model

Taking thermodynamic limit on Bethe equations $\Rightarrow$ infinite set of coupled integral equations

Twisted TBA

TBA for deformed string models [(Arutyunov),MdL, v. Tongeren '10,'11,'12]

twists ⇔ chemical potential

Chemical potentials can be projected out

- Simplified TBA same for all deformations
- Y-system same for all deformations

Twists manifest themselves in

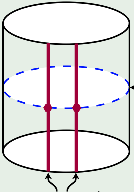
- Analytic behaviour
- Asymptotic solutions ⇔ twisted T^G.

TBA equations are the same, conditions on functions change

TBA and Lüscher

Lüscher's perturbative approach [Baknok,Janik'08]

- Corrections from virtual particles in compact direction



- Bound states go around

Naive formula is correct

$$E \sim \sum_Q \int dp e^{-J\mathcal{E}} \mathrm{tr} S_{Q1} S_{Q2}$$

Recognise $T \Rightarrow T^G$

Susy-Breaking Deformations

Orbifolds

Consider strings in (super) space $\mathcal{M}$, action of discrete group Γ
Want to consider $\mathcal{M}/\Gamma \Rightarrow$

$$X \equiv hX$$

Project Hilbert space on invariant subspaces

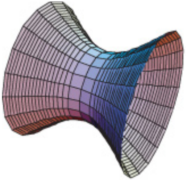
Closed strings can close up to group elements

$$X(0,\tau) = g\,X(2\pi,\tau)$$

For such B.C. twisted sector H_g
For non-Ablian project onto conjugacy classes.

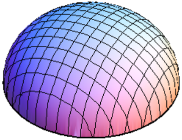
Orbifolds

We have $\mathcal{M} = \mathrm{AdS}_5 \times \mathrm{S}^5$ and take $\Gamma \subset SU(4)$ $\rightarrow$ orbifold S^5



AdS_5

$\times$



S^5/Γ

For simplicity we restrict $\Gamma = \mathbb{Z}_S$:

$$h = \mathrm{diag}(e^{-2\pi i \frac{t_1}{S}}, e^{2\pi i \frac{t_1 - t_2}{S}}, e^{2\pi i \frac{t_2 - t_3}{S}}, e^{2\pi i \frac{t_3}{S}})$$

Parameters t_i from different $SU(2)$s

Quasi-periodic B.C. $\quad X(0,\tau) = e^{i\phi} X(2\pi,\tau)$

Orbifolds

Dual QFT

$\mathcal{N} = 4$ SYM with orbifolded R-symmetry

So susy is broken

$\Gamma \subset$	SU(2)	SU(3)	SU(4)
$\mathcal{N} =$	2	1	0

The action is unchanged. Operators

$$\mathrm{tr}(\gamma D^n \Phi \ldots)$$

Explicit twist follows from B.C.

$$\beta = \pi \frac{2t_1 - t_2}{S} \qquad \dot{\beta} = \pi \frac{2t_3 - t_2}{S} \qquad P = 2\pi \frac{t_2}{S}$$

Orbifolds

$\mathcal{O} = \mathrm{tr}(D^2 Z^2)$

BAE: $e^{2ip} = S_0(p, -p)$

Same as for Konishi (same multiplet in $\mathcal{N} = 4$)→

$$E = 2 + 3g^2 - 3g^4 + \frac{21}{4}g^6 - g^8\left[\frac{9\zeta(3)}{8} + \frac{705}{64}\right]$$

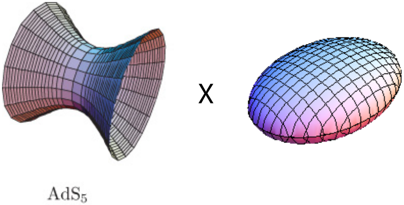
Only differs in finite size corrections

- $\mathcal{N} = 4$ gives $E_{f.s.} = \left[\frac{81}{64} + \frac{27\zeta(3)}{8} - \frac{45\zeta(5)}{8}\right] g^8$
- $\mathcal{N} = 1$ gives $E_{f.s.} = \frac{3}{8}\sin^2(\frac{\pi}{S})\, g^6$
- $\mathcal{N} = 0$ gives $E_{f.s.} = -\frac{1}{3}\sin^4(\frac{\pi}{S})\, g^4$

$\mathcal{N} = 2$ state is a bit special, but $E_{f.s.} \sim g^8$

TsT deformations

Deform background via TsT transformation [Lunin,Maldacena'05]



Can be mapped to $\mathrm{AdS}_5 \times S^5$ with twisted B.C. [Alday,Arutyunov,Frolov '05]:

$$\phi_i(2\pi) = \phi_i(0) - \epsilon_{ijk}\gamma_j J_k$$

ϕ angles of S^5 and J corresponding angular momentum

γ-deformed

Dual field theory

$\mathcal{N} = 4$ SYM with modified superpotential

Twists:

$$\beta = \pi(\gamma_1(J_2 - J_3) + J_1(\gamma_3 - \gamma_2))$$
$$\dot{\beta} = \pi(J_1(\gamma_2 + \gamma_3) - \gamma_1(J_2 + J_3))$$

For $\gamma_1 = \gamma_2 = \gamma_3$ reduces to β-deformed theory.

- Explicit FT computations [Fiamberti, et al. 08,09] match nicely with twist approach [Ahn et al.10,11] [Gromov et al. 10] [Arutyunov et al. 10]

Ground state

Interesting: ground state $\mathcal{O} = \mathrm{tr}(Z^J)$ not protected

$$E = \frac{(\cos\alpha - \cos\beta)(\cos\dot\alpha - \cos\dot\beta)}{\sin\alpha \sin\dot\alpha} \frac{\Gamma(J-\frac{1}{2})}{2\sqrt{\pi}\Gamma(J)} \times$$
$$\times \Big[\mathrm{Li}_{2J-1} e^{i(\alpha+\dot\alpha)} + \mathrm{Li}_{2J-1} e^{-i(\alpha+\dot\alpha)} - \mathrm{Li}_{2J-1} e^{i(\alpha-\dot\alpha)} - \mathrm{Li}_{2J-1} e^{-i(\alpha-\dot\alpha)} \Big]$$

Curious divergency for $J = 2$

Conclusions

Conclusions

- Spectrum of $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring from integrability
- Complete spectrum captured in TBA
- Techniques can be extended to deformations
- Twisted (T)BA describe spectrum of theories with less susy

Thank you