Integrability properties of Wilson loops in AdS/CFT

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Summary

- **Introduction**

  String / gauge theory duality (**AdS/CFT**)

  Wilson loops in AdS/CFT =
  Minimal area surfaces in hyperbolic space

- **Minimal area surfaces in hyperbolic space**

  Simple examples: Surfaces ending on a circle, parallel lines, a cusp.

- **New examples**

Relation to Willmore surfaces in flat space.

Integrability, flat currents and the dressing method.

Solutions in terms of theta functions associated with hyperelliptic Riemann surfaces.

Computation of the area. Analogy to monodromy matrix.

Correlators of two Wilson loops.

- **Conclusions**

Gauge theory $\Longleftrightarrow$ **String theory**

λ small $\rightarrow$ gauge th.

λ large $\rightarrow$ string th.

Strings live in curved space, e.g.

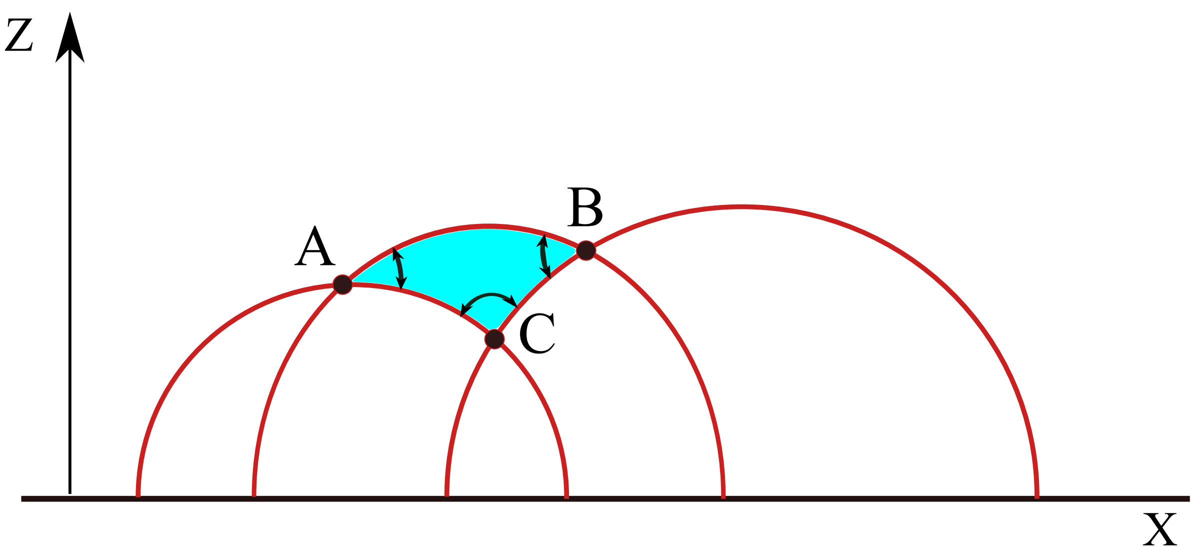
$AdS_5 x S^5$

S^5 : $X_1^2+X_2^2+\ldots+X_6^2 = 1$

AdS_5: $Y_1^2+Y_2^2+\ldots-Y_5^2-Y_6^2 =-1$ **(hyperbolic space)**

Hyperbolic space

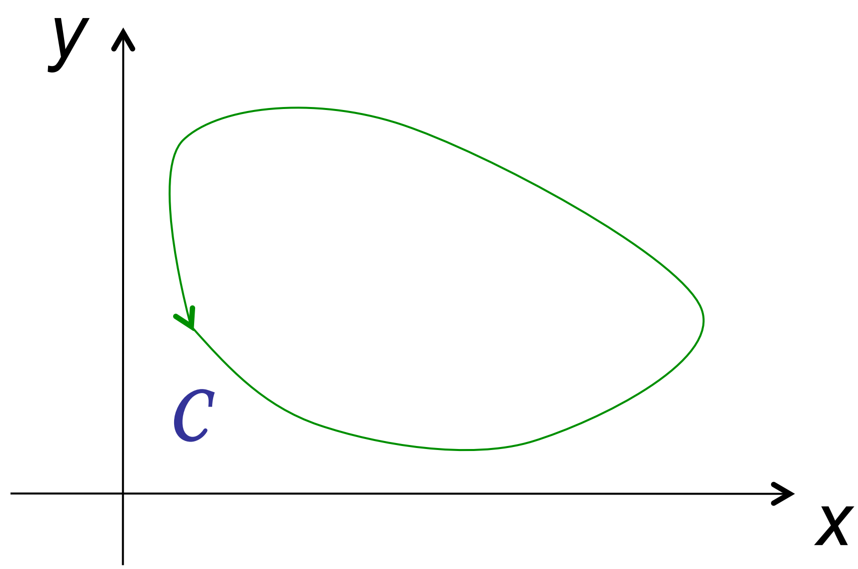
2d: Lobachevsky plane, Poincare plane/disk



$$ds^2 = \frac{dx^2 + dz^2}{z^2} \Rightarrow ds^2 = \frac{-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2}{z^2}$$

AdS metric in Poincare coordinates

Wilson loops: associated with a closed curve in space. Basic operators in gauge theories. E.g. $q\bar{q}$ potential.

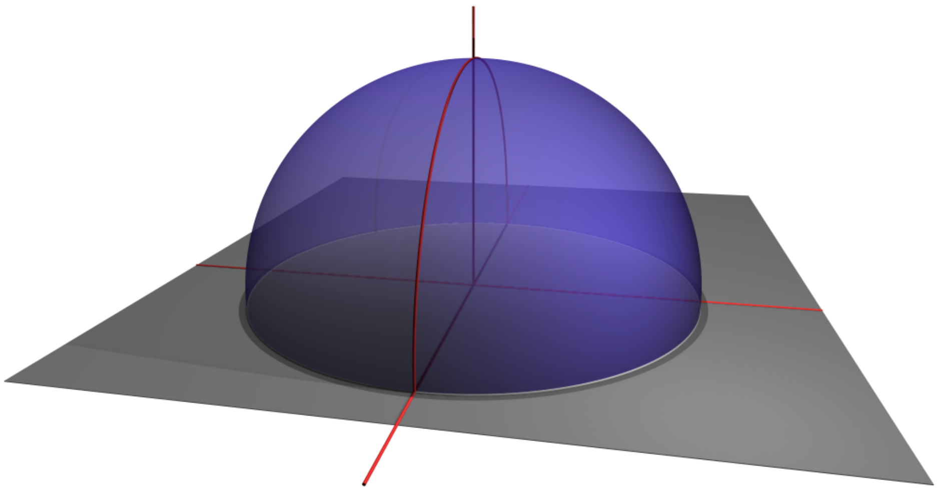


$$W = \frac{1}{N} \mathrm{Tr}\, \hat{P} \exp\left\{ i \oint_C \left(A_\mu \frac{dx^\mu}{ds} + \theta_0^I \Phi_I \left| \frac{dx^\mu}{ds} \right| \right) ds \right\}$$

Simplest example: single, flat, smooth, space-like curve (with constant scalar).

String theory: Wilson loops are computed by finding a minimal area surface **(Maldacena, Rey, Yee)**

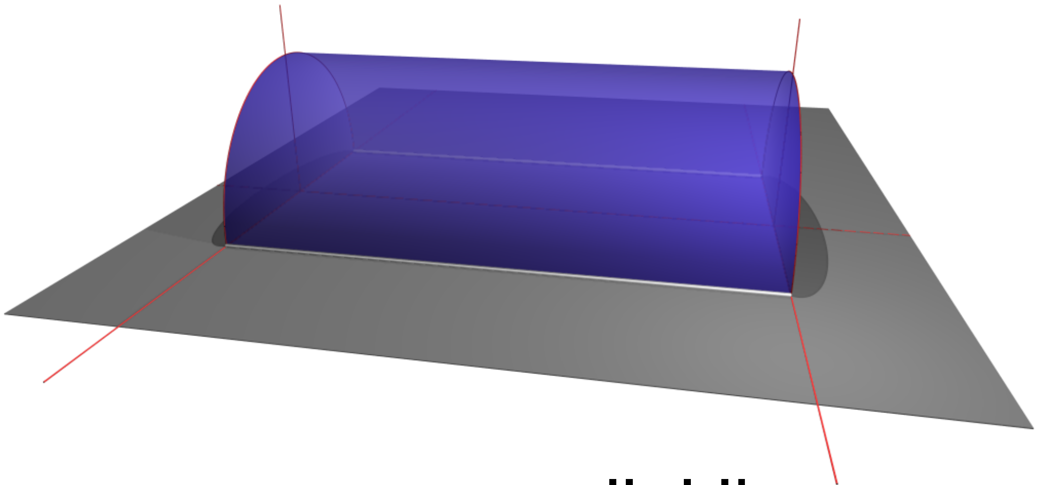
Circle:



circular (~ Lobachevsky plane)

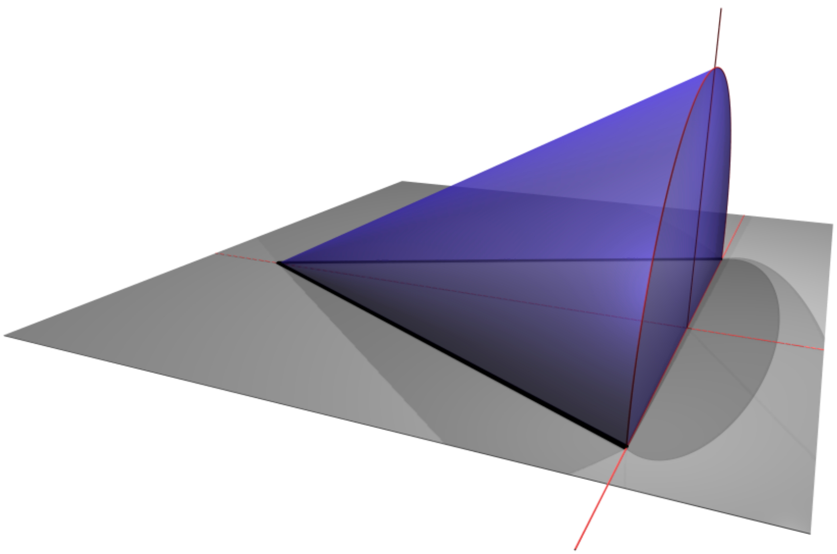
Berenstein Corrado Fischler Maldacena
Gross Ooguri, Erickson Semenoff Zarembo
Drukker Gross, Pestun

$$z = \sqrt{1 - r^2}$$



$$z = z(x)$$

Maldacena, Rey Yee parallel lines



$$z = r f(\theta)$$

Drukker Gross Ooguri cusp

Other cases

Many interesting and important results for Wilson loops with non-constant scalar and for Minkowski Wilson loops (lots of recent activity related to light-like cusps and their relation to scattering amplitudes).

New examples (R. Ishizeki, S. Ziama, M.K.)

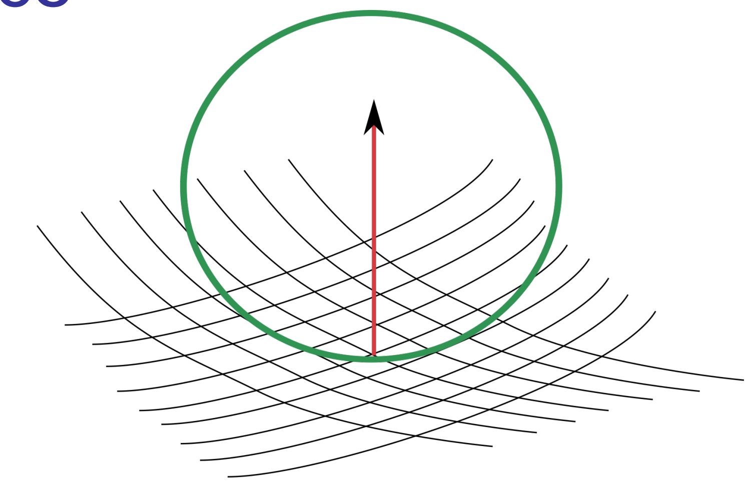
More generic examples for Euclidean Wilson loops can be found using Riemann theta functions.
Corresponds to single, flat, smooth, space-like curve (with constant scalar). In fact an infinite parameter family of solution is given. The renormalized area is given by a one-dimensional integral over the contour.

Babich, Bobenko. **(our case)**
Kazakov, Marshakov, Minahan, Zarembo (sphere)
Dorey, Vicedo. **(Minkowski space-time)**
Sakai, Satoh. **(Minkowski space-time)**

Relation to Willmore surfaces: Babich, Bobenko

Motivation: Willmore tori in flat space

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{\cancel{z^2}}$$



Surface: $\kappa_1 = \frac{1}{R_1}, \ \kappa_2 = \frac{1}{R_2},$ $R_{1,2}$ max. and min. R

Gauss curvature: $K = \kappa_1 \kappa_2$

Mean curvature: $H = \frac{1}{2}(\kappa_1 + \kappa_2)$

Willmore functional: $\mathcal{W} = \frac{1}{4} \int (\kappa_1 - \kappa_2)^2 \ d\mathcal{A} = \int H^2 - \int K$

Minimal Area surfaces in $EAdS_3$

Equations of motion

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1 \qquad X + iY = \frac{X_1 + iX_2}{X_0 - X_3}, \qquad Z = \frac{1}{X_0 - X_3}$$

$$z = \sigma + i\tau,\ \bar{z} = \sigma - i\tau$$

$$\begin{aligned} S &= \frac{1}{2}\int \left(\partial X_\mu \bar{\partial} X^\mu - \Lambda(X_\mu X^\mu - 1)\right)\, d\sigma\, d\tau \\ &= \frac{1}{2}\int \frac{1}{Z^2}\left(\partial_a X \partial^a X + \partial_a Y \partial^a Y + \partial_a Z \partial^a Z\right)\, d\sigma\, d\tau \end{aligned}$$

$$\boxed{\begin{aligned} &\partial\bar{\partial} X_\mu = \Lambda X_\mu \qquad \Lambda = -\partial X_\mu \bar{\partial} X^\mu \\ &\partial X_\mu \partial X^\mu = 0 = \bar{\partial} X_\mu \bar{\partial} X^\mu \end{aligned}}$$

We can also use:

$$\mathbb{X} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} = X_0 + X_i \sigma^i$$

$$\mathbb{X}^\dagger = \mathbb{X}, \quad \det \mathbb{X} = 1, \quad \partial\bar{\partial}\mathbb{X} = \Lambda \mathbb{X}, \quad \det(\partial \mathbb{X}) = 0 = \det(\bar{\partial}\mathbb{X})$$

The current: $J = \mathbb{X}^{-1} d\mathbb{X}$

satisfies
$$\begin{aligned} dJ + J \wedge J &= 0 \\ d * J &= 0 \end{aligned}$$

which allows us to construct a flat current (KMMZ, BPR):

$$a_z = \frac{1}{2}(1+\lambda) J_z, \quad a_{\bar{z}} = \frac{1}{2}\left(1 + \frac{1}{\lambda}\right) J_{\bar{z}}$$

Finding Solutions: Dressing method

$$\boxed{\begin{aligned} d\Psi(\lambda) &= \Psi(\lambda)a(\lambda) \\ \Psi(-1) &= \mathbb{I} \\ \Psi(\lambda)^\dagger &= \Psi(1)\Psi(-\frac{1}{\bar{\lambda}})^{-1} \end{aligned}}$$

Reality condition

$$\mathbb{X} = \Psi(1)$$

Now we look for a matrix χ such that $\tilde{\Psi}(\lambda) = \Psi(\lambda)\chi(\lambda)$

Defines $\tilde{a} = \tilde{\Psi}^{-1} d\tilde{\Psi}$ with the same properties as a.

$$\begin{aligned} \tilde{a}_z &= \frac{1}{2}(1+\lambda)\ \chi_\infty^{-1}\, J_z\, \chi_\infty \\ \tilde{a}_{\bar{z}} &= \frac{1}{2}(1+\frac{1}{\lambda})\ \chi_0^{-1}\, J_{\bar{z}}\, \chi_0 \end{aligned}$$

In fact it turns out that

$$\chi = \mathbb{I} - \frac{(1+\lambda_1\bar{\lambda}_1)(1+\lambda)}{(1-\bar{\lambda}_1)(\lambda-\lambda_1)}\mathbb{P},$$

$$\mathbb{P} = \frac{v \otimes v^\dagger \Psi(1)}{v^\dagger \Psi(1) v}, \qquad \text{with} \quad v = \Psi(\lambda_1)^{-1} e_i$$

Satisfies all the properties, except it gives

$$\det \mathbb{X} = -1 \quad \text{or } X_0^2 - X_1^2 - X_2^2 - X_3^2 = -1$$

namely a solution in de Sitter space! It is not really a problem since we can apply the dressing method twice going back to EAdS.

Finding Solutions: Theta functions.

(w/ Riei Ishizeki, Sannah Ziama)

X hermitian can be solved by:

$$\mathbb{X} = \mathbb{A}\mathbb{A}^\dagger, \qquad \det \mathbb{A} = 1, \qquad \mathbb{A} \in SL(2,\mathbb{C})$$

Global and gauge symmetries:

$$\mathbb{X} \to U\mathbb{X}U^\dagger, \quad \mathbb{A} \to U\mathbb{A}, \quad U \in SL(2,\mathbb{C})$$

$$\mathbb{A} \to \mathbb{A}\mathcal{U}, \quad \mathcal{U}(z,\bar{z}) \in SU(2)$$

The currents:

$$J = \mathbb{A}^{-1}\partial\mathbb{A}, \qquad \bar{J} = \mathbb{A}^{-1}\bar{\partial}\mathbb{A}$$

$$\mathcal{A} = \frac{1}{2}(\bar{J} + J^{\dagger}), \quad \mathcal{B} = \frac{1}{2}(J - \bar{J}^{\dagger})$$

satisfy:

$$\begin{aligned} \mathrm{Tr}\mathcal{A} = \mathrm{Tr}\mathcal{B} &= 0\ , \\ \det\mathcal{A} &= 0\ , \\ \partial\mathcal{A} + [\mathcal{B}, \mathcal{A}] &= 0\ , \\ \bar{\partial}\mathcal{B} + \partial\mathcal{B}^{\dagger} &= [\mathcal{B}^{\dagger}, \mathcal{B}] + [\mathcal{A}^{\dagger}, \mathcal{A}]. \end{aligned}$$

$$\mathcal{A} \to \mathcal{U}^{\dagger}\mathcal{A}\mathcal{U}, \qquad \mathcal{B} \to \mathcal{U}^{\dagger}\mathcal{B}\mathcal{U} + \mathcal{U}^{\dagger}\partial\mathcal{U}, \quad \mathcal{U}(z, \bar{z}) \in SU(2)$$

Up to a gauge transformation (rotation) $\mathcal{A}$ is given by:

$$\mathcal{A} = \frac{1}{2} e^{\alpha(z,\bar{z})}(\sigma_1 + i\sigma_2) = e^{\alpha(z,\bar{z})}\sigma_+$$

$$\boxed{\begin{array}{l} \mathrm{Tr}\mathcal{A} = 0 \\ \det\mathcal{A} = 0 \\ \text{gauge} \end{array}}$$

Then: $\mathcal{B} = -\frac{1}{2}\partial\alpha\sigma_z + f(z)e^{-\alpha}\sigma_+$

$$\mathcal{A} = \bar{\lambda} e^{\alpha}\sigma_+ \ , \qquad |\lambda| = 1$$

$$\mathcal{B} = -\frac{1}{2}\partial\alpha\sigma_z + e^{-\alpha}\sigma_+ \ ,$$

$$\partial\bar{\partial}\alpha = 2\cosh(2\alpha) \ ,$$

Summary

Solve $\partial\bar{\partial}\alpha = 2\cosh 2\alpha$

plug it in $\mathcal{A}$, $\mathcal{B}$ giving:

$$J = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ \lambda e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad \bar{J} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & \bar{\lambda}e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}$$

Solve:

$$\begin{aligned} \partial\mathbb{A} &= \mathbb{A}J, \\ \bar{\partial}\mathbb{A} &= \mathbb{A}\bar{J}. \end{aligned} \quad \Longrightarrow \quad \mathbb{X} = \mathbb{A}\mathbb{A}^{\dagger}$$

Flat current

A one parameter family of flat currents can be found:

$$a = \frac{1}{\lambda} a_{-1} + a_0 + \lambda a_1$$

with the property: $*a_{-1} = -a_{-1}, \quad *a_1 = a_1$

This is equivalent to the equations of motion. The current is given by:

$$a_{1z} = 0, \quad a_{1\bar{z}} = \frac{1}{2}(J_{1\bar{z}} - J_{2\bar{z}})$$

$$a_{-1z} = \frac{1}{2}(J_{1z} - J_{2z}), \quad a_{-1\bar{z}} = 0$$

$$a_0 = \frac{1}{2}(J_1 + J_2)$$

$$\boxed{\begin{aligned} J_1 &= J \\ J_2 &= -J^\dagger \end{aligned}}$$

$$(a(\lambda))^\dagger = -a(-\frac{1}{\bar{\lambda}})$$

In Poincare coordinates the minimal area surfaces are given by functions:

$$Z = Z(z,\bar{z}), \quad X + iY = X(z,\bar{z}) + iY(z,\bar{z}), \qquad z = \sigma + i\tau$$

$$Z = \left|\frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})}\right| \frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)|\,|e^{\mu z+\nu\bar{z}}|^2}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2},$$

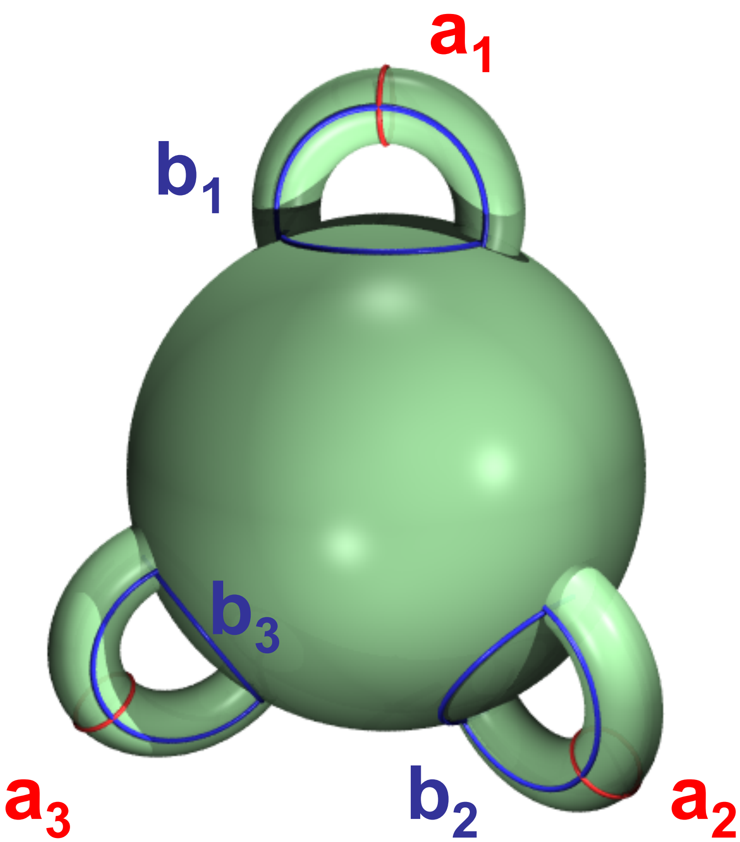
$$X + iY = e^{2\bar{\mu}\bar{z}+2\bar{\nu}z}\,\frac{\theta(\zeta-\int_{p_1}^{p_4})\overline{\theta(\zeta+\int_{p_1}^{p_4})}-\hat{\theta}(\zeta-\int_{p_1}^{p_4})\overline{\hat{\theta}(\zeta+\int_{p_1}^{p_4})}}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2}$$

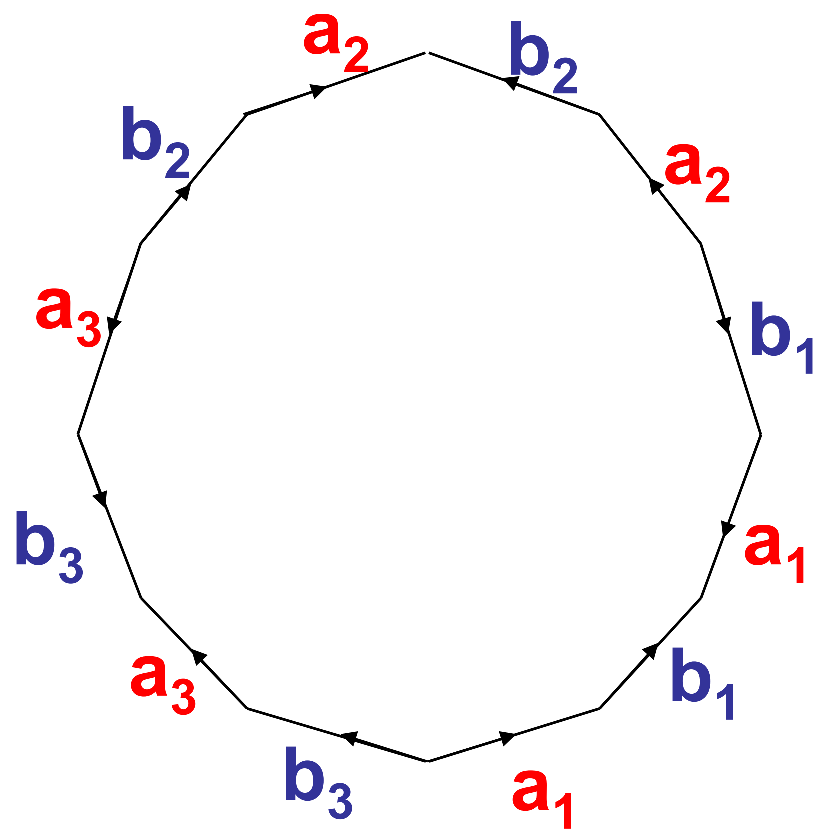
$$\zeta = 2\omega(p_1)\bar{z} + 2\omega(p_3)z$$

which we will now describe in detail.

Theta functions associated with (hyperelliptic) Riemann surfaces

Riemann surface:





hyperelliptic: $(\mu, \lambda), \quad \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i)$

Holomorphic differentials and period matrix:

$$\omega_{i=1\ldots g} \qquad \oint_{a_i} \omega_j = \delta_{ij}$$

$$\Pi_{ij} = \oint_{b_i} \omega_j$$

Theta functions:

$$\theta(\zeta) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}n^t\Pi n + n^t\zeta\right)}$$

Differential Equations

sin, cos, exp: harmonic oscillator (Klein-Gordon).

theta functions: sine-Gordon, sinh-Gordon, cosh-Gordon.

Trisecant identity:

$$\theta(\zeta)\,\theta\left(\zeta+\int_{p_2}^{p_1}\omega+\int_{p_3}^{p_4}\omega\right)=\gamma_{1234}\,\theta\Big(\zeta+\int_{p_2}^{p_1}\omega\Big)\,\theta\Big(\zeta+\int_{p_3}^{p_4}\omega\Big)+\gamma_{1324}\,\theta\Big(\zeta+\int_{p_3}^{p_1}\omega\Big)\,\theta\Big(\zeta+\int_{p_2}^{p_4}\omega\Big)$$

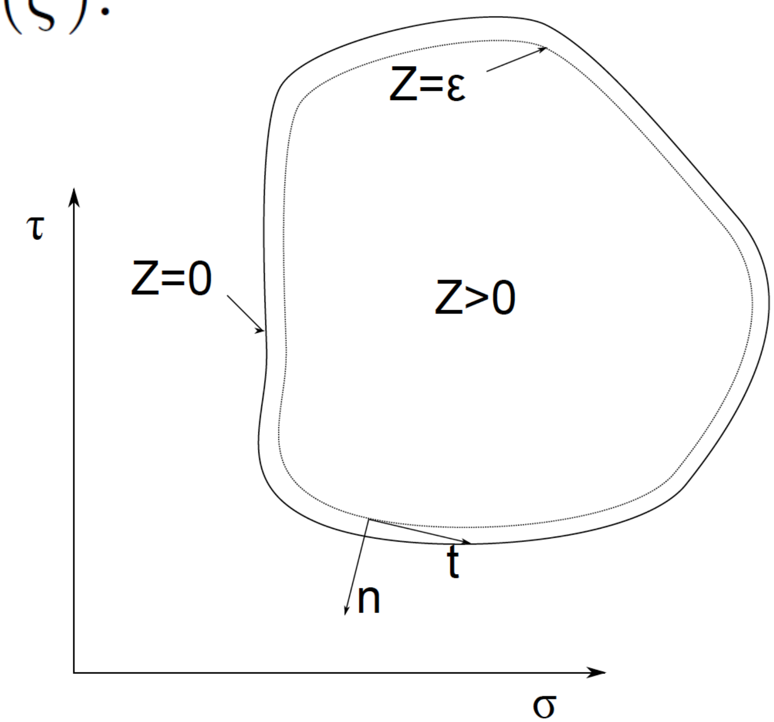
$$\gamma_{ijkl}=\frac{\theta(a+\int_{p_k}^{p_i}\omega)\,\theta(a+\int_{p_l}^{p_j}\omega)}{\theta(a+\int_{p_l}^{p_i}\omega)\,\theta(a+\int_{p_k}^{p_j}\omega)}$$

Renormalized area:

$$A = 2\int \partial X_\mu \bar{\partial} X^\mu d\sigma d\tau = 2\int \Lambda d\sigma d\tau = 4\int e^{2\alpha} d\sigma d\tau$$

$$\begin{aligned} e^{2\alpha} &= 4\left\{ D_{p_1p_3} \ln\theta(0) - D_{p_1p_3} \ln\hat{\theta}(\zeta) \right\} \\ &= 4D_{p_1p_3} \ln\theta(0) - \partial\bar{\partial}\ln\hat{\theta}(\zeta). \end{aligned}$$

$$A = \frac{L}{\epsilon} + A_f$$



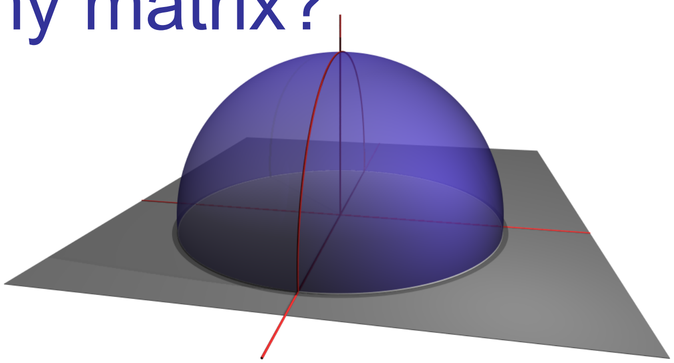
Subtracting the divergence gives:

$$A_f = -2\pi n + 4\Im\left\{ \oint D_1 \ln\theta(\zeta_\sigma) d\bar{z} - 2D_{13}\ln\theta(0) \oint z d\bar{z} \right\}$$

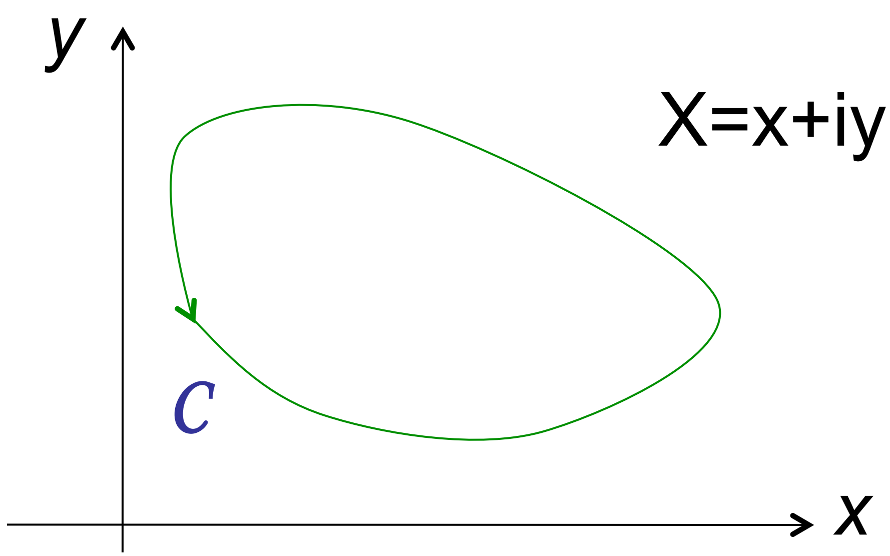
where *n* is an integer denoting the “winding number” of the loop. With the area, the expectation value of the Wilson loop is:

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda_t}}{2\pi} A_f}$$

Is there a formula with the monodromy matrix?
For one WL the monodromy is trivial.



In fact, we can construct something analogous to the monodromy matrix by defining a function $X_\sigma(\lambda)$



Namely funding a one (complex) parameter family of contours by solving the linear problem for Ψ. We get

$$\bar{X}_\sigma = (X - iY)_\sigma = -e^{2\bar{\mu}\bar{z}_\sigma + 2\bar{\nu}z_\sigma} \frac{\hat{\theta}(\zeta_\sigma + \int_1^4)}{\hat{\theta}(\zeta_\sigma - \int_1^4)}$$

This function has the property that, when λ crosses a cut:

$$X_\sigma(\lambda) \to \frac{1}{X_\sigma(\lambda)}$$

We can write conformally invariant quantities e.g. cross ratios. Interestingly, the area is given by:

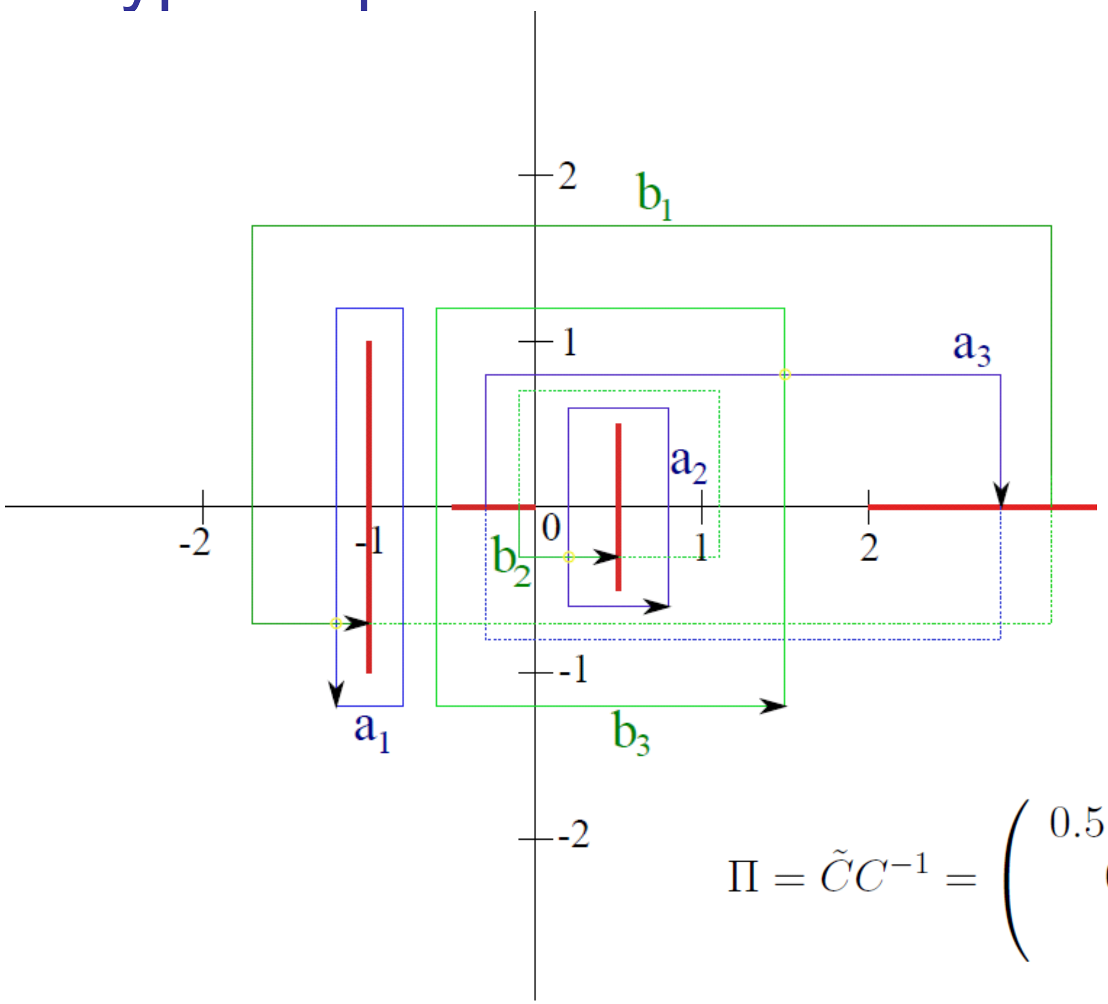
$$A_f = -2\pi + \frac{1}{\pi}\Re\left\{ \oint d\bar{z} \oint_0 \frac{d\lambda}{\lambda}\{X,\lambda\} \right\}$$

$$\{X,\sigma\} = \frac{2}{\lambda}(\partial_\sigma \bar{z})^2 + f(\sigma) - 2\lambda(\partial_\sigma z)^2$$

{ f , x } denotes Schwarzian derivative.

Example of closed Wilson loop for g=3

Hyperelliptic Riemann surface



$$\nu_k = \frac{\lambda^{k-1}}{\mu} d\lambda, \qquad k = 1 \ldots 3.$$

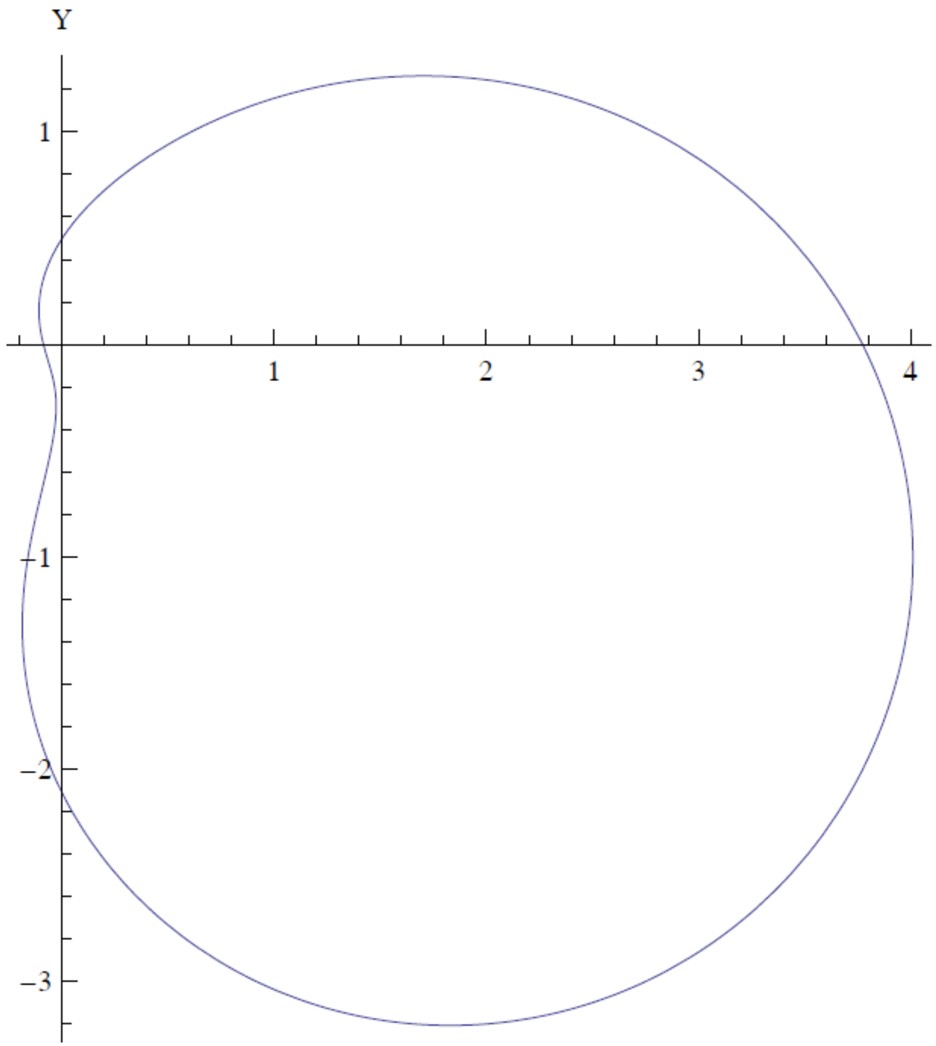
$$C_{ij} = \oint_{a_i} \nu_j, \qquad \tilde{C}_{ij} = \oint_{b_i} \nu_j \; ,$$

$$\omega_i = \nu_j \left(C^{-1}\right)_{ji} \; ,$$

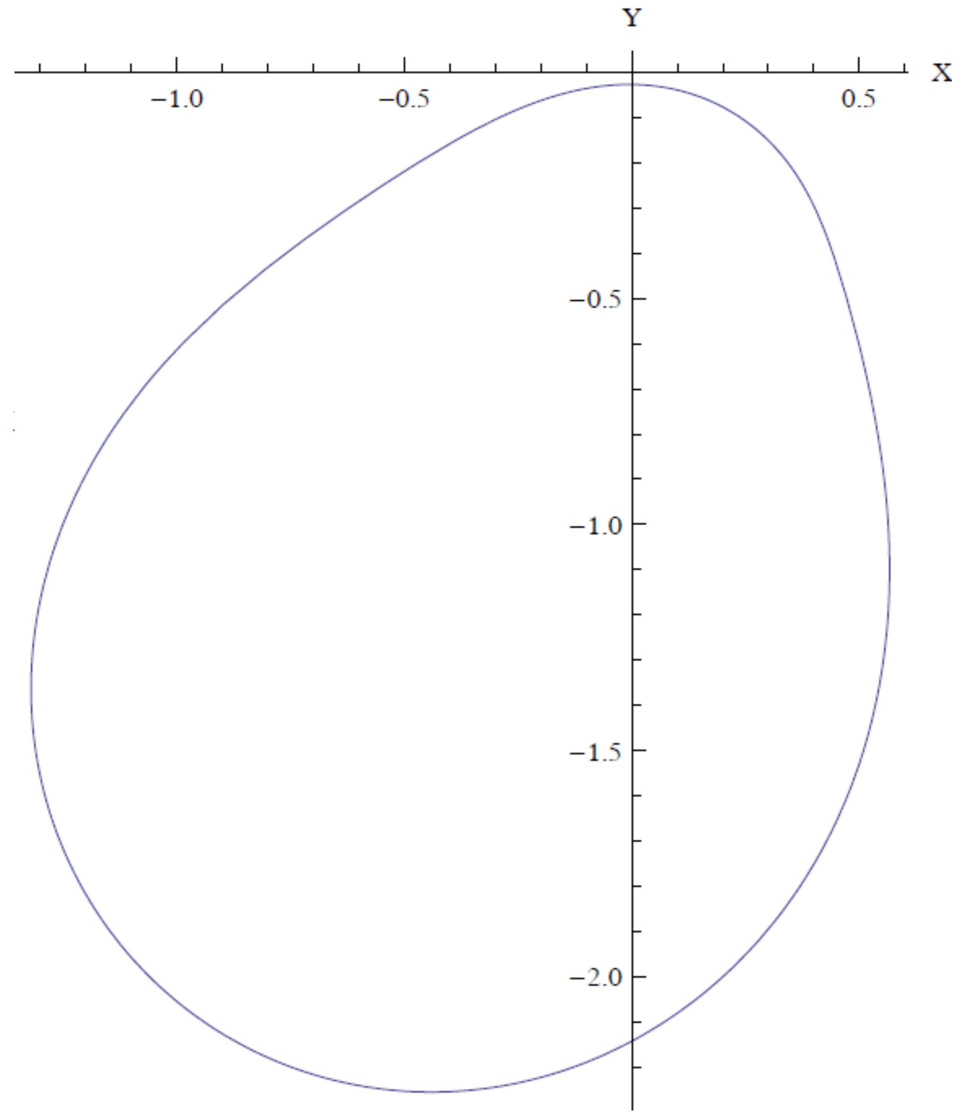
$$\Pi = \tilde{C} C^{-1} = \begin{pmatrix} 0.5 + 0.64972i & 0.14972i & -0.5 \\ 0.14972i & -0.5 + 0.64972i & 0.5 \\ -0.5 & 0.5 & 0.639631 \end{pmatrix}$$

$$\mu = i\sqrt{-i(\lambda+1-i)}\sqrt{-i(\lambda+1+i)}\sqrt{-i(\lambda-\frac{1+i}{2})}\sqrt{-i(\lambda-\frac{1-i}{2})}\sqrt{2-\lambda}\sqrt{\lambda}\sqrt{\lambda+\frac{1}{2}}$$

Shape of Wilson loop:

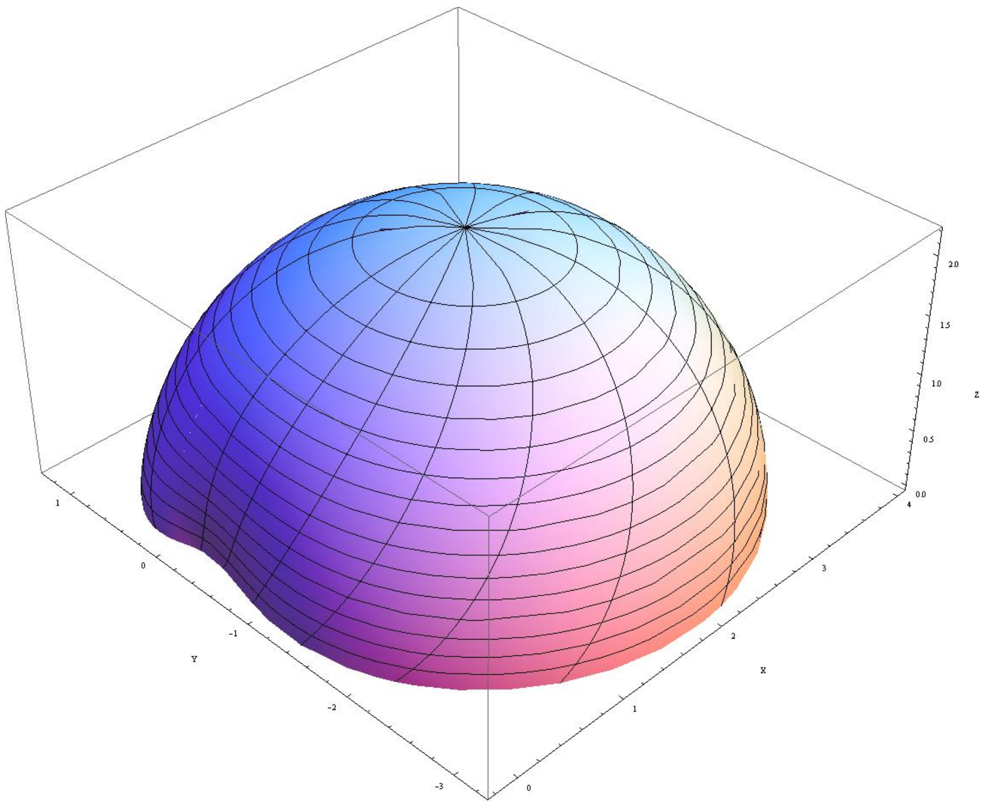


$$\lambda = i$$

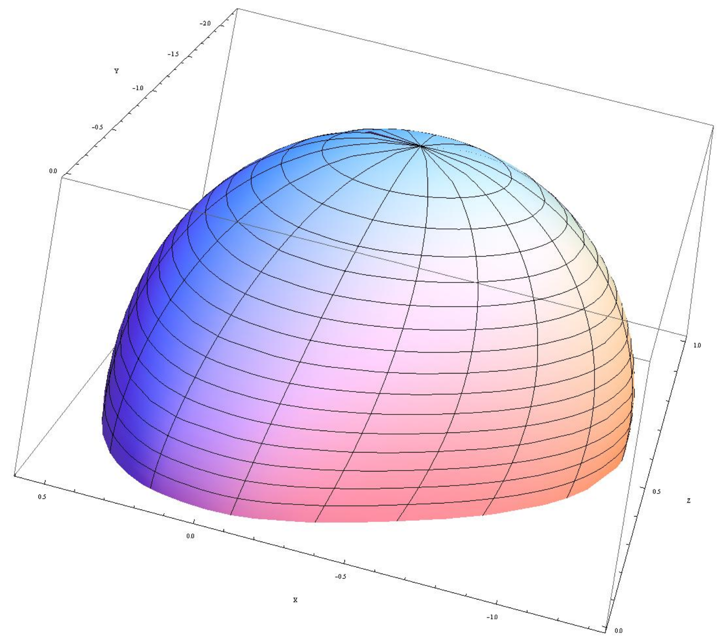


$$\lambda = -\frac{1+i}{\sqrt{2}}$$

Shape of dual surface:



$$\lambda = i$$



$$\lambda = -\frac{1+i}{\sqrt{2}}$$

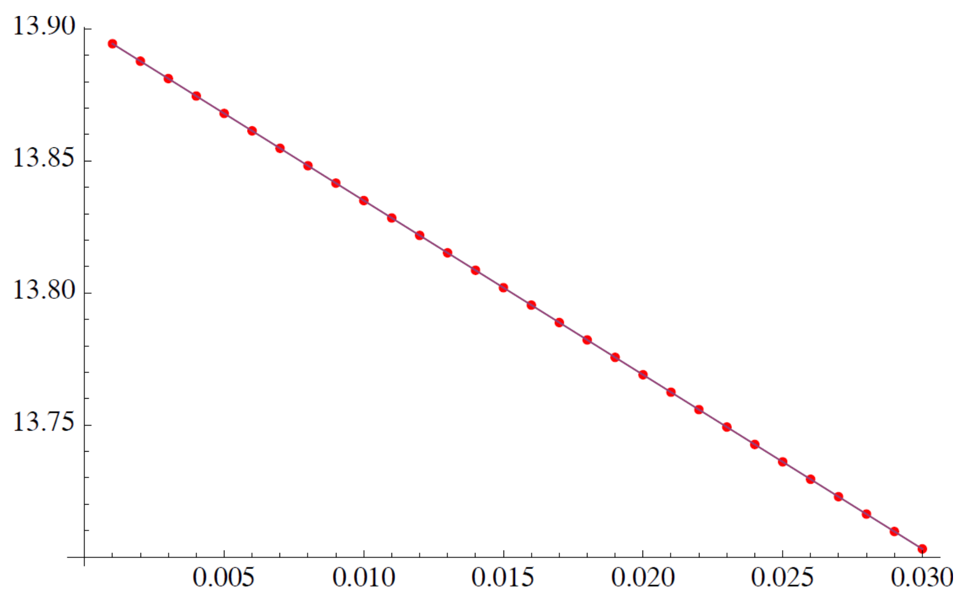
Computation of area:

Using previous formula

$$L_1 = 13.901, \quad L_2 = 6.449$$
$$A_f = -6.598 \quad \text{for both.}$$

Direct computation:

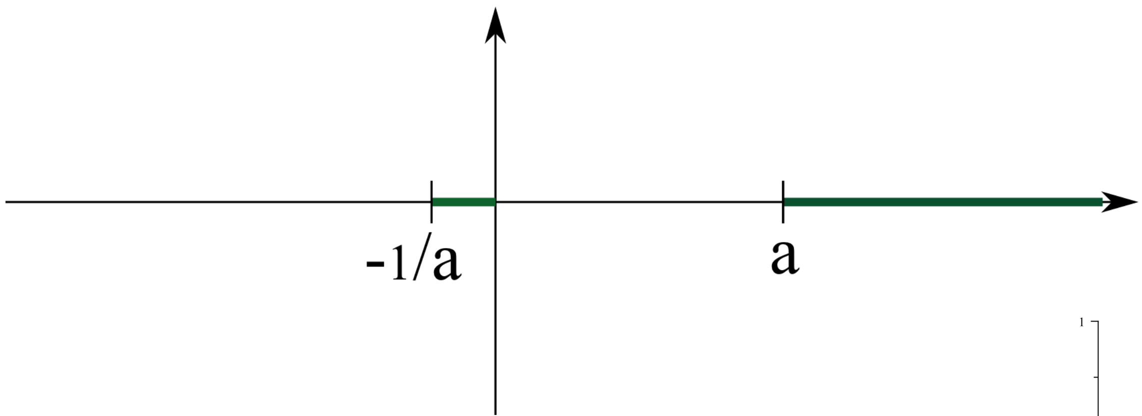
$$\varepsilon A = L + A_f \varepsilon$$

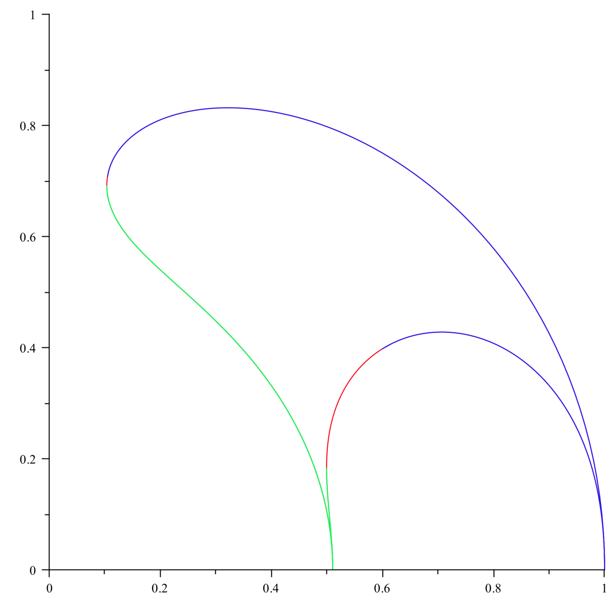


Circular Wilson loops , maximal area for fixed length.

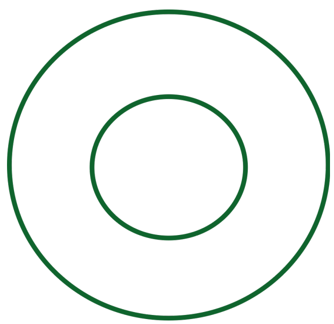
(Alexakis, Mazzeo)

Simpler case g=1

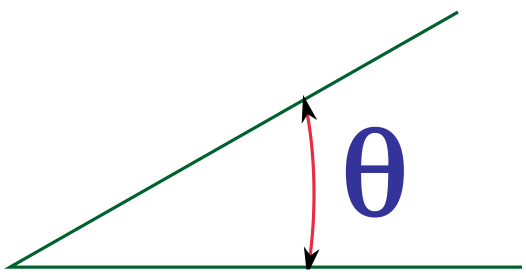




$a>1,\ \lambda=1$



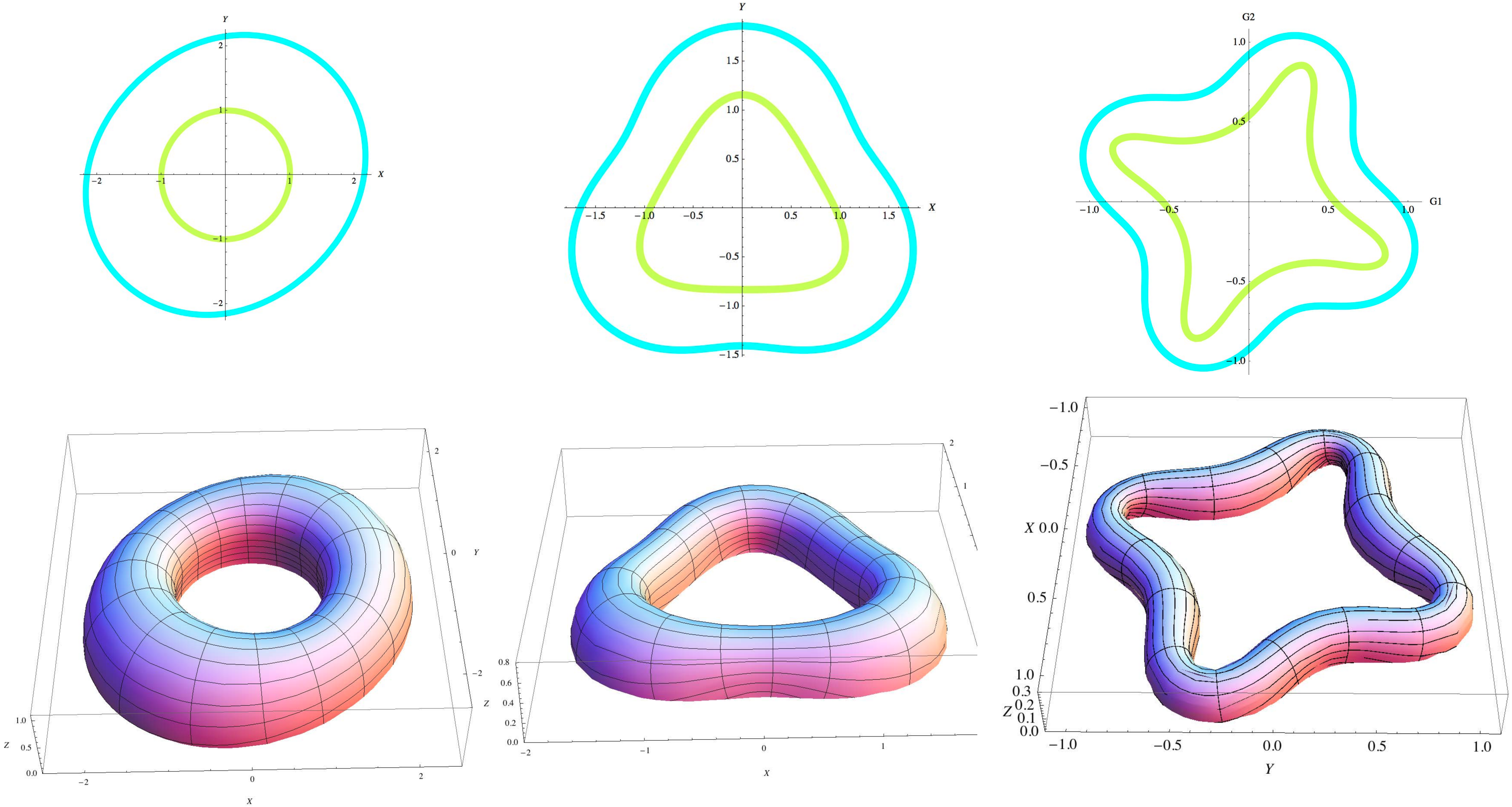
$a<1,\ \lambda=-1$



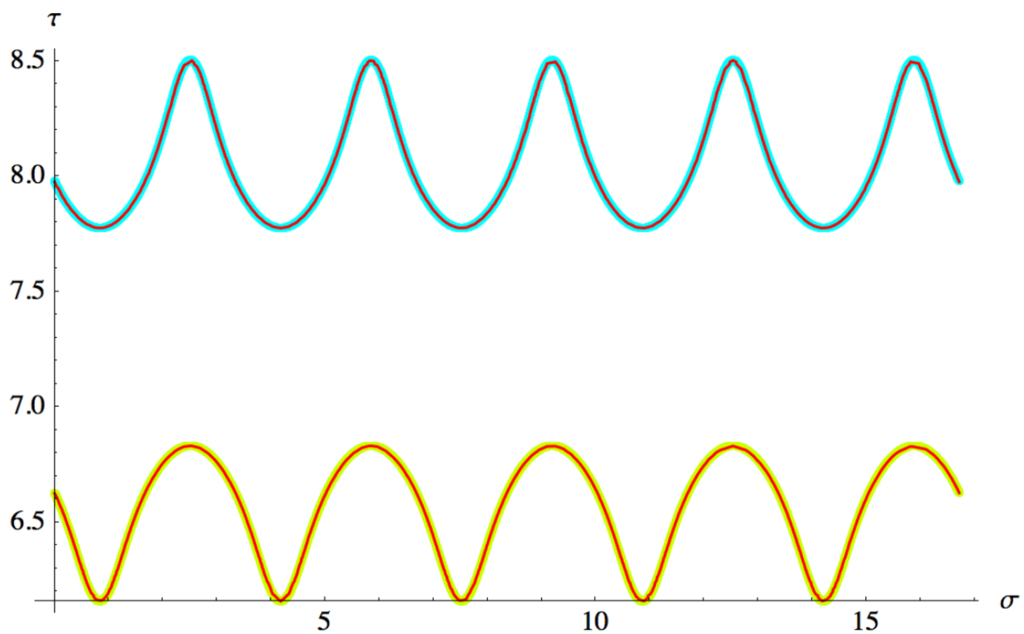
$a \to 1\ ,\ \ \theta \to 0$

$a \to 0\ ,\ \ \theta \to \pi$

Concentric curves by extending g=1 to g=3



In this case there is a non-trivial cycle. The world-sheet has the topology of a cylinder.



The formula for the area is still valid:

$$A_{finite} = -2\Im\left\{ D_{13}\log\theta(0)\oint z d\bar{z} + \int_{2-4} D_1 \log\theta(\zeta) d\bar{z} \right\}$$

Need to be related to the monodromy.

Conclusions

We review the duality between Wilson loops and minimal area surfaces in hyperbolic space.
We argue that there is an infinite parameter family of closed Wilson loops whose dual surfaces can be found analytically. The world-sheet has the topology of a disk and the renormalized area is found as a finite one dimensional contour integral over the world-sheet boundary. Also a world-sheet with the topology of a cylinder was described giving WL correlators.

Integrability properties of minimal surfaces in hyperbolic space and Euclidean Wilson loops constitute a beautiful subject that deserve further study.