High energy asymptotics of scattering amplitudes in N=4 SUSY and the Dixon-Drummond-Henn ansatz

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1 Gluon reggeization in QCD

QCD Born amplitude at high energies $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,,\;\; \alpha_s \ln s \sim 1\,,\;\; \alpha_s = \frac{g^2}{4\pi} \ll 1$$

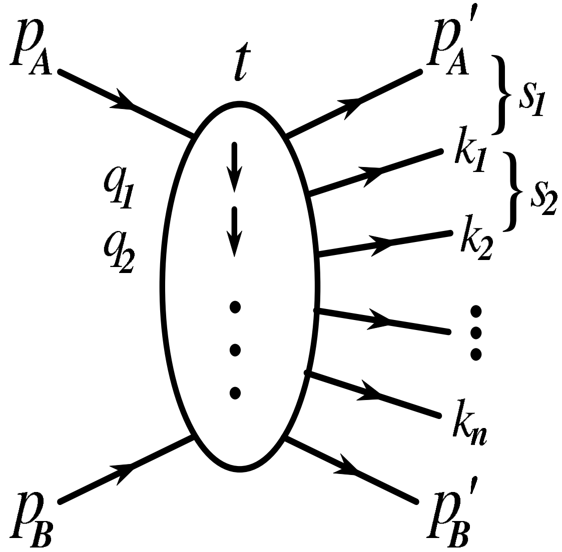
Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Mandelstum cut contribution

$$A(s,-|q|^2) \sim s \int d^2k\, \Phi^2(k,q-k)\, s^{\omega(-|k|^2)}\, s^{\omega(|q-k|^2)} \sim \frac{s^{j(t)}}{\ln s}$$

2 Amplitudes in multi-Regge kinematics



$$M_{2\to 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\, \ln\frac{|q_r^2|}{\lambda^2}\,,\; C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\; \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \sigma_t \sim s^{\Delta}\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,\min E$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$p_r = i\frac{\partial}{\partial\rho_r}\,,\ \rho_{12} = \rho_1 - \rho_2\,,\ \rho_r = x_r + iy_r\,,\ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

Möbius invariance and Pomeron intercept

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\ m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = 1/2 + i\nu\,,$$

$$E = \epsilon_m + \epsilon_{\widetilde{m}}\,,\ \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\ \Delta = \frac{g^2 N_c}{\pi^2}\ln 2 > 0$$

4 Integrability of the BFKL dynamics

Holomorphic separability of H_{BKP} at $N_c \to \infty$ (L.)

$$H = \frac{1}{2}(h+h^*)\,,\ \ h = \sum_{k=1}^{n} \left(\ln(p_k p_{k+1}) + \frac{1}{p_k}\ln \rho_{k,k+1}\, p_k + \frac{1}{p_{k+1}}\ln \rho_{k,k+1}\, p_{k+1} - 2\psi(1)\right)$$

Holomorphic factorization of wave functions (L.)

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

Monodromy matrix and Yang-Baxter equation (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}\,,\ \ L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix},$$

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)\,,\ \ \hat{l} = u\,\hat{1} + i\,\hat{P}$$

5 Effective action in gauge models

Locality in the rapidity space

$$y=\frac{1}{2}\ln\frac{\epsilon_k+|k|}{\epsilon_k-|k|}\,,\;\;|y-y_0|<\eta\,,\;\;\eta<<\ln\,s$$

Gluon and Reggeized gluon fields

$$v_\mu(x)=-iT^a v_\mu^a(x)\,,\;\;A_\pm(x)=-iT^a A_\pm^a(x)\,,\,\delta A_\pm(x)=0$$

Effective action for the reggeon interactions (L., 1995)

$$S=\int d^4x\left(L_{YM+matter}+Tr(V_+\partial_\mu^2 A_-+V_-\partial_\mu^2 A_+)\right)\,,$$

$$V_+=-\frac{1}{g}\partial_+\,P\exp\left(-g\int_{-\infty}^{x^+}v_+(x')d(x')^+\right)=v_+-gv_+\frac{1}{\partial_+}v_++...$$

6 Effective action for gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta << \ln\, s$$

Reggeized graviton fields

$$\delta A^{++}(x) = \delta A^{--}(x) = 0\,, \;\; \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left(\sqrt{-g}\, R + \frac{1}{2} \left(\partial_+ j^-\, \partial^2_\mu A^{++} + \partial_- j_+\, \partial^2_\mu A^{--} \right) \right)$$

Hamilton-Jacobi equation for effective currents $j^{\pm} = 2x^{\pm} - \omega^{\pm}$

$$g^{\mu\nu} \partial_\mu \omega^{\pm} \partial_\nu \omega^{\pm} = 0\,, \;\; \partial_\pm j^{\mp} = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_\pm} h_{\pm\pm} \right)^2 + \ldots$$

7 Maximal helicity violation

BDS amplitudes in $N=4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,\ldots,a_n} = \sum_{\{i_1,\ldots,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} \ldots T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, \ldots, p_{i_n})\,,\;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right)\,,$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}\,,\;\; C^{(1)} = 0\,,\;\; C^{(2)} = -\zeta_2^2/2\,,\;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k f_k^{(l)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}\,,\;\; \sum_{l=1}^{\infty} a^l f_1^l = -a\zeta_3/2 + a^2(4\zeta_5 + 10\zeta_2\zeta_3/3) + \ldots$$

8 Elastic BDS amplitude

Its Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

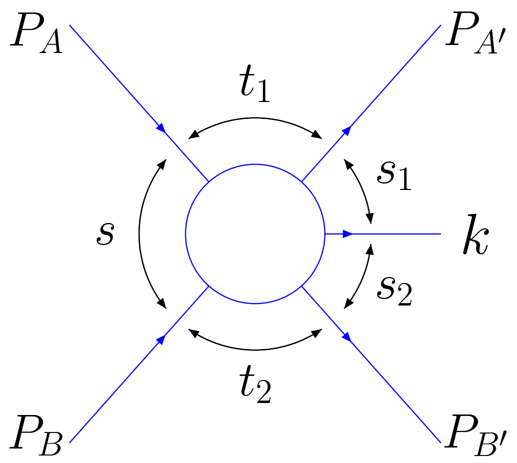
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

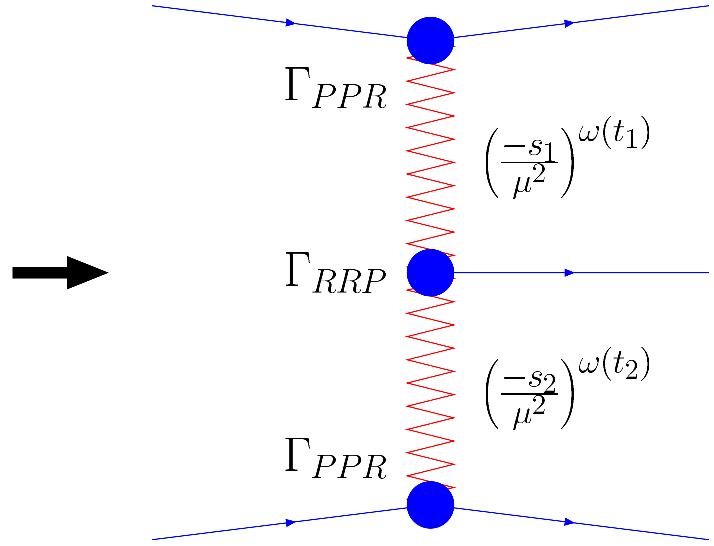
Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2} \zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

9 One particle production





$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2)-\int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2}-$$
$$\frac{\gamma_K(a)}{16}\left(\ln^2\frac{-\kappa}{\mu^2}-\ln^2\frac{-t_1}{-t_2}-\zeta_2\right)-\frac{1}{2}\int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

10 Steinmann relations

No simultaneous singularities in overlapping channels

$$(s_1, s_2)\ (2 \to 3)\,;\ \ (s_1, s_2),\ (s_2, s_3),\ (s_{012}, s_2),\ (s_{123}, s)\ (2 \to 4)$$

Dispersion representation for $M_{2\to 3}$ in the Regge anzatz

$$M_{2\to 3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

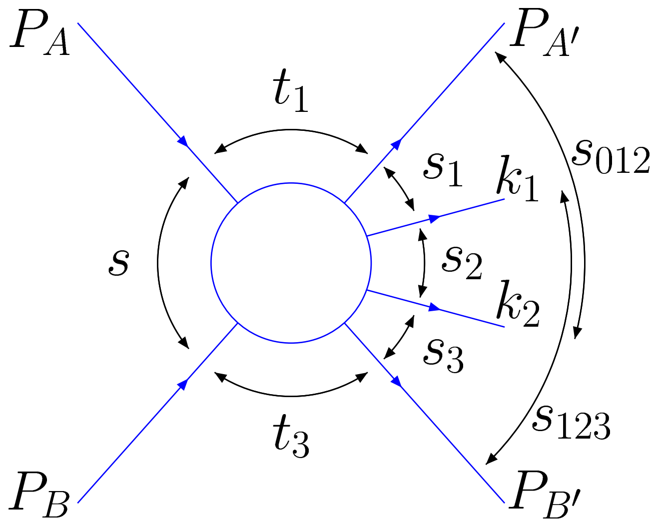
Violation of the dispersion representation for $M_{2\to 4}^{BDS}$

$$M_{2\to 4} \neq d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2}+d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$
$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$
$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\ \ j_r = j(t_r)$$

Mandelstam channels

$$\Phi \equiv \frac{(-s)(-s_2)}{(-s_{012})(-s_{123})}\,,\ \ a)s, s_2 > 0\,;\ \ s_{012}, s_{123} < 0\,,\ \ b)s, s_2 < 0\,;\ \ s_{012}, s_{123} > 0$$

11 Regge factorization violation



$$M_{2\to 4}|_{s,s_2>0;\, s_{012},s_{123}<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln \frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times \Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)} \Gamma(t_3)$$

12 Regge cuts in j_2-plane

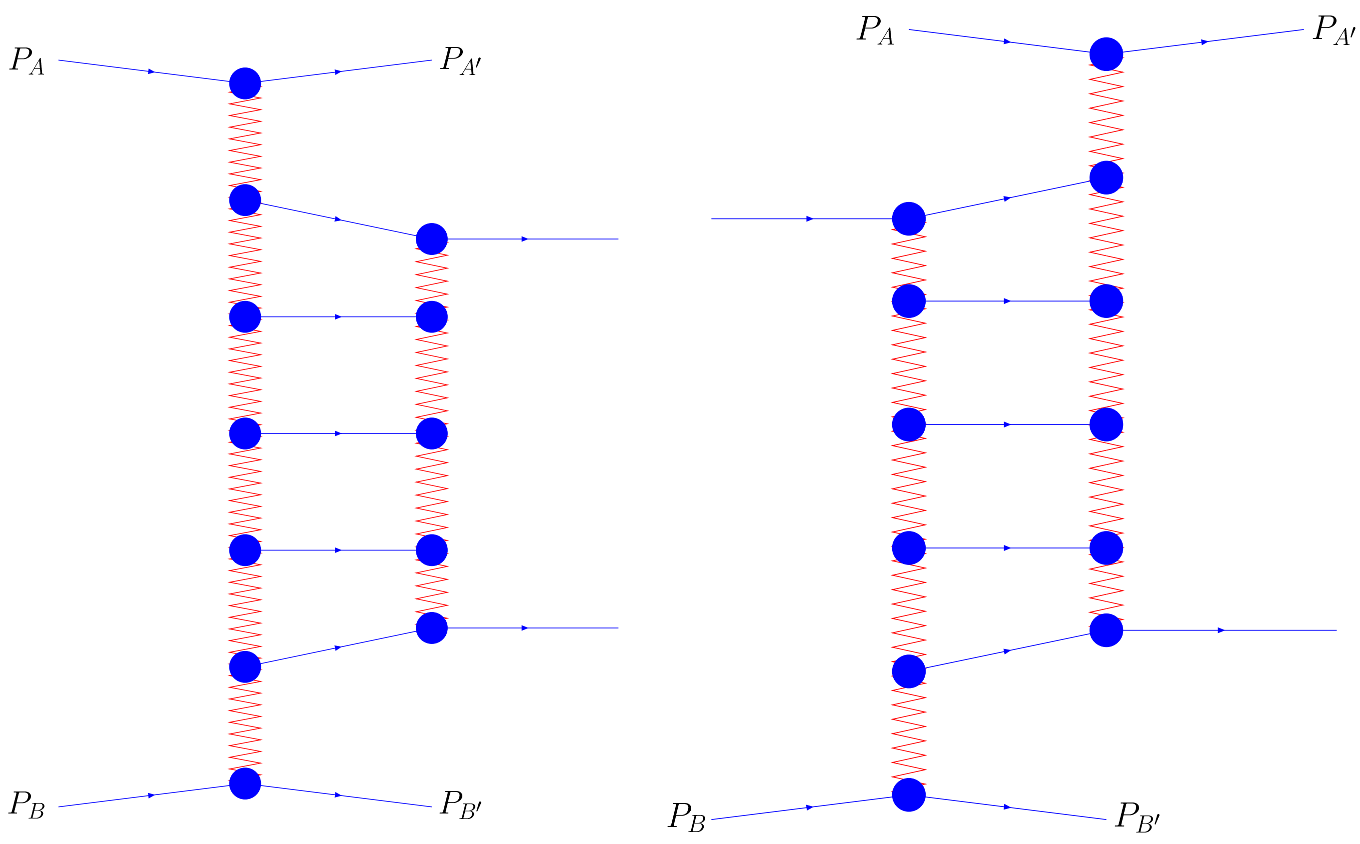


Figure 1: BFKL ladders in $M_{2\to4}$ and $M_{3\to3}$

13 BFKL equation in octet channels

Factorization of infrared divergencies in LLA

$$\lim_{\epsilon\to 0} M_{2\to 4}^{LLA} = f_{2\to 4}^{LLA} \lim_{\epsilon\to 0} M_{2\to 4}^{BDS}\,,$$

Renormalization of the intercept in the s_2-channel

$$\Delta_2 = -a\left(E + \ln\frac{t_2}{\mu^2} - \frac{1}{\epsilon}\right)$$

BFKL equation for the partial wave f_{j_2} (BLS, 2009)

$$E\Psi = H\Psi\,,$$

$$H = \ln\frac{|p_1 p_2|^2}{|p_1+p_2|^2} + \frac{1}{2}\frac{1}{p_1 p_2^*}\ln|\rho_{12}|^2 p_1 p_2^* + \frac{1}{2}\frac{1}{p_1^* p_2}\ln|\rho_{12}|^2 p_1^* p_2 + 2\gamma$$

Exact eigenfunctions and eigenvalues

$$\Psi \sim (\frac{p_1}{p_2})^{i\nu+\frac{n}{2}}(\frac{p_1^*}{p_2^*})^{i\nu-\frac{n}{2}}\,,\; E_{n,\nu} = Re\,\psi(1+i\nu+\frac{n}{2}) + Re\,\psi(1+i\nu-\frac{n}{2}) - 2\psi(1)$$

14 Möbius and conformal invariances

The remainder function $R = 1 + \Delta_{2\to 4}$ in the region $a \ln s_2 \sim 1$

$$\Delta_{2\to 4} = \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(V^*\right)^{i\nu - \frac{n}{2}} V^{i\nu + \frac{n}{2}} \left(s_2^{\omega(\nu,n)} - 1\right)$$

Duality transformation to the Möbius representation

$$V = \frac{q_3 k_1}{k_2 q_1} \to \frac{z_{03} z_{0'1}}{z_{0'3} z_{01}}$$

Perturbation theory expansion

$$i\Delta_{2\to 4} = -2i\pi\, a^2 \ln s_2 \ln \frac{|k_1 + k_2||q_2|}{|k_2||q_1|} \ln \frac{|k_1 + k_2||q_2|}{|k_1||q_3|} + ...$$

Functions of 4-dimensional anharmonic ratios

$$i\Delta_{2\to 4} = \frac{a^2}{4} Li_2(\chi) \ln \frac{\chi t_2 s_{13}}{s_3 t_1} \ln \frac{\chi t_2 s_{02}}{t_3 s_1} + ... , \;\; \chi = 1 - \frac{s s_2}{s_{012} s_{123}}$$

15 Open integrable spin chain

Equation for composite states with octet quantum numbers

$$H\Psi = E\Psi\,,\ \ H = h + h^*\,,\ \ h = \ln\frac{p_1 p_n}{q^2} + \sum_{r=1}^{n-1} h^t_{r,r+1}\,,\ \ p_r = Z_{r,r-1}$$

Integrals of motion: $[D, h] = 0$: open spin chain (L. (2009))

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k\,,\ \ q'_k = -\sum_{0<r_1<\ldots<r_k<n} Z_{r_1} \prod_{s=1}^{k-1} Z_{r_s,r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}$$

Sklyanin ansatz and Baxter equation (L. (2009))

$$\Omega = \prod_k Q(\hat{u}_k)\,\Omega_0\,,\ \ \Omega_0 = \prod_{l=1}^{n-1} \frac{1}{|Z_l|^4}\,,$$

$$D(u)Q(u) = (u+i)^{n-1} Q(u+i)$$

16 Analyticity and factorization

Analyticity constraint and factorization hypothesis

$$M_{2\to4} = M_{2\to4}^{pole} + M_{2\to4}^{cut} = c\, M_{2\to4}^{BDS}$$

BDS ansatz at $s, s_2 > 0, s_1, s_3 < 0$

$$M_{2\to4}^{BDS} = |M_{2\to4}^{BDS}|\, e^{-i\pi\omega_2} e^{i\delta}\,,\;\; \delta = \frac{\gamma_K}{4}\, \ln \frac{|q_1 q_3 k_a k_b|}{|k_a+k_b|^2 |q_2^2|}$$

Regge pole and cut contributions at $s, s_2 > 0, s_1, s_3 < 0$

$$c\, e^{i\delta} = \cos \pi\omega_{ab} + i \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i}\, (-s_2)^{\omega}\, f(\omega)\,,\;\; \omega_{ab} = \frac{\gamma_K}{4}\, \ln \frac{|k_a q_3|}{|k_b q_1|}$$

Prediction from the analyticity constraint (L. 2010)

$$c-1 = \frac{\ln s_2}{i\pi}\left(\frac{\delta^2}{2} - \frac{\pi^2\omega_{ab}^2}{2}\right) = -2\pi i \frac{a^2}{4}\, \ln s_2\, \ln \frac{|k_b|^2|q_1|^2}{|k_a+k_b|^2|q_2|^2}\, \ln \frac{|k_a|^2|q_3|^2}{|k_a+k_b|^2|q_2|^2}$$

Factor c is not a phase

17 Two loop expression for $M_{2\to4}$

GSVV remainder function based on the symbol theory

$$M^{(2)}_{2\to4} = a^2 R(u_1,u_2,u_3)\, M^{BDS}_{2\to4}\,,\; u_1 = \frac{ss_2}{s_{012}s_{123}}\,,\; u_2 = \frac{s_1 t_3}{t_3 s_{012}}\,,\; u_3 = \frac{s_3 t_1}{t_2 s_{123}}\,,$$

$$R = \sum_{i=1}^{3}\left(L_4 - \frac{1}{2}Li_4(1-1/u_i)\right) - \frac{1}{8}\left(Li_2(1-1/u_i)\right)^2 + \frac{J^2}{24} + \frac{\pi^2}{12}(J^2 - \zeta_2)\,,$$

$$L_4 = \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\ln^m(x^+x^-)\left(l_{4-m}(x^+) + l_{4-m}(x^-)\right) + \frac{1}{8!!}\ln^4(x^+x^-)\,,$$

$$J = \sum_{i=1}^{3}(l_1(x_i^+) - l_1(x_i^-))\,,\; l_n = \frac{1}{2}(Li_n(x) - (-1)^n Li_n(1/x))\,,$$

$$x_i^{\pm} = u_i x^{\pm}\,,\; x^{\pm} = \frac{u_1+u_2+u_3-1 \mp \sqrt{(u_1+u_2+u_3-1)^2 - 4u_1u_2u_3}}{2u_1u_2u_3}$$

18 Continuation to multi-Regge region

Asymptotic behavior in the multi-Regge kinematics (L.,P. (2010))

$$R = \frac{i\pi}{2}\left(\ln(1-u_1)\,\ln|z|^2\,\ln|1-z|^2 + r(z)\right)\,,\;\; |z|^2 = \frac{u_2}{1-u_1}\,,\;\; |1-z|^2 = \frac{u_3}{1-u_1}$$

Next-to-leading correction

$$r(z) = \ln(|z|^2|1-z|^2)\;(\ln z\;\ln(1-z) - \zeta_2)$$

$$+\ln\frac{|1-z|^2}{|z|^2}\left(Li_2(z) - Li_2(1-z)\right) + 4\left(L_3(z) + Li_3(1-z)\right) + h.c.$$

Möbius representation

$$r = \sum_{n=-\infty}^{\infty}\int\frac{d\nu}{\nu^2+\frac{n^2}{4}}|w|^{2i\nu}\left(\frac{w}{w^*}\right)^{n/2}\left((E(\nu,n))^2 - \frac{n^2}{(\nu^2+\frac{n^2/4}{4})^2}\right)\,,\;\; w = \frac{z}{1-z}$$

19 Octet BFKL equation at two loops

Equation and kernel for $N = 4$ SUSY at $q \to \infty$ (F.,L. (2011))

$$\omega\, f(\vec{p}) = \int d^2p' K(\vec{p},\vec{p}')\, f(\vec{p}')\,, \quad \frac{4\pi^2}{\alpha N_c}\, K(\vec{p},\vec{p}') =$$

$$-\delta^2(\vec{p}-\vec{p}\,')\,|p|^2 \left(\left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right)\int d^2p\,' \left(\frac{2}{|p\,'|^2}+\frac{2(p\,',p-p\,')}{|p\,'|^2|p-p\,'|^2}\right) - 3\alpha\,\zeta(3)\right)$$

$$+\left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right)\left(\frac{|p|^2+|p\,'|^2}{|p-p\,'|^2}-1\right)+\frac{\alpha N_c}{8\,\pi}\,R(\vec{p},\vec{p}\,')\,,$$

$$R(\vec{p},\vec{p}\,') = \left(\frac{1}{2}-\frac{|p|^2+|p\,'|^2}{|p-p\,'|^2}\right)\ln^2\frac{|p|^2}{|p\,'|^2} - \frac{|p|^2-|p\,'|^2}{2|p-p\,'|^2}\,\ln\frac{|p|^2}{|p\,'|^2}\,\ln\frac{|p|^2|p\,'|^2}{|p-p\,'|^4}$$

$$+\left(-|p+p\,'|^2+\frac{(|p|^2-|p\,'|^2)^2}{|p-p\,'|^2}\right)\int_0^1 dx\,\frac{1}{|(1-x)p+xp\,'|^2}\,\ln\frac{|(1-x)p+xp\,'|^2}{x(1-x)|p-p\,'|^2}$$

20 Production amplitudes $A_{2\to 4}$

Remainder factor $R = A_{2\to 4}/A_{2\to 4}^{BDS}$ (L. (2009), F.,L. (2011))

$$R\,e^{i\pi\delta} = cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\phi n}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi(\nu,n)\left(\frac{-1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)}\,,$$

$$u_1 = \frac{ss_2}{s_{012}s_{123}}\,,\;u_2 = \frac{s_1t_3}{s_{012}t_2}\,,\;u_3 = \frac{s_3t_1}{s_{123}t_2}\,,\;|w|^2 = \frac{u_2}{u_3}\,,\;\cos\phi = \frac{1-u_1-u_2-u_3}{2\sqrt{u_2u_3}}\,,$$

$$\delta = \frac{\gamma_K}{8}\,\ln\frac{|w|^2}{|1+w|^4}\,,\;\omega_{ab} = \frac{\gamma_K}{8}\,\ln|w|^2\,,\;\Phi = 1 - a\left(\frac{E_{\nu n}^2}{2}+\frac{3}{8}\,n^2/(\nu^2+\frac{n^2}{4})^2+\zeta(2)\right)$$

$$\omega(\nu,n) = -aE_{\nu,n} - a^2(\epsilon_{\nu n}^{FL} + 3\zeta(3))\,,\;E_{\nu n} = -\frac{|n|/2}{\nu^2+\frac{n^2}{4}} + 2\Re\psi(1+i\nu+\frac{|n|}{2}) - 2\psi(1)$$

Next-to-leading correction to ω (F.,L. (2011))

$$\epsilon_{\nu n}^{FL} = -\frac{\Re}{2}\left(\psi''(1+i\nu+\frac{|n|}{2}) - \frac{2i\nu\psi'(1+i\nu+\frac{|n|}{2})}{\nu^2+\frac{n^2}{4}}\right) - \zeta(2)\,E_{\nu n} - \frac{1}{4}\,\frac{|n|\,\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}$$

21 DDH anzatz and collinear limit

Perturbative expansion of R and collinear anomalous dimension

$$R = 1+i\,a^2(\widetilde{b}_1 \ln \frac{1}{\sqrt{u_2 u_3}} + \widetilde{b}_2) + a^3 \left(i\widetilde{c}_1 \ln^2 \frac{1}{\sqrt{u_2 u_3}} + (\widetilde{d}_1 + i\widetilde{c}_2) \ln \frac{1}{\sqrt{u_2 u_3}} + \widetilde{d}_2 + i\widetilde{c}_3 \right)$$

$$\widetilde{b}_1 = -\frac{\pi}{2} \ln|1+w|^2 \ln \frac{|1+w|^2}{|w|^2}\,, \quad \frac{\widetilde{c}_2}{\pi} = -\frac{\ln|w|^2}{4} \left(S_{1,2}(-w) + \ln(1+w)\, Li_2(-w) + h.c\right.$$

$$+\frac{\zeta(3)}{2} \ln|1+w|^2 - \ln \frac{|1+w|^2}{|w|} \left(Li_3(-w) - \frac{1}{2} \ln|w|^2\, Li_2(-w) + h.c. \right)$$

$$+\frac{1}{4} \ln|1+w|^2 (Li_3(-w) + h.c.) + \frac{1}{16} \ln^2|w|^2 \ln|1+w|^2 \ln \frac{|1+w|^2}{|w|^2}$$

$$+\frac{\ln^2|1+w|^2}{8} \ln^2 \frac{|1+w|^2}{|w|^2} + \frac{\ln^2|w|^2}{8} \ln(1+w)\, \ln(1+w^*) + \zeta(2) \ln|1+w|^2 \ln \frac{|1+w|^2}{|w|^2}\,,$$

$$\gamma_{col}(\omega) = \frac{a}{2} \left(\frac{1}{\omega} - 1 \right) - \frac{a^2}{4} \left(\frac{1}{\omega^2} + 2\,\frac{\zeta(2)}{\omega} \right) + \frac{a^3}{4\,\omega^2} (1 + 2\zeta(2) + \zeta(3)) + O(a^4)\,.$$

22 Discussion

1. Reggeized gluons as new degrees of freedom in QCD.
2. BFKL equation for the Pomeron wave function.
3. Effective actions for reggeized gluons and gravitons.
4. BDS ansatz and the Steinmann relations.
5. BFKL equation for the states in the adjoint representation.
6. Integrability of the multi-gluon interactions at large N_c.
7. Mandelstam cut contribution to $M_{2\to 4}$ in two loops.
8. Verification of the DDH ansatz based on the symbol theory.
7. Collinear anomalous dimension at all orders and MSV results.