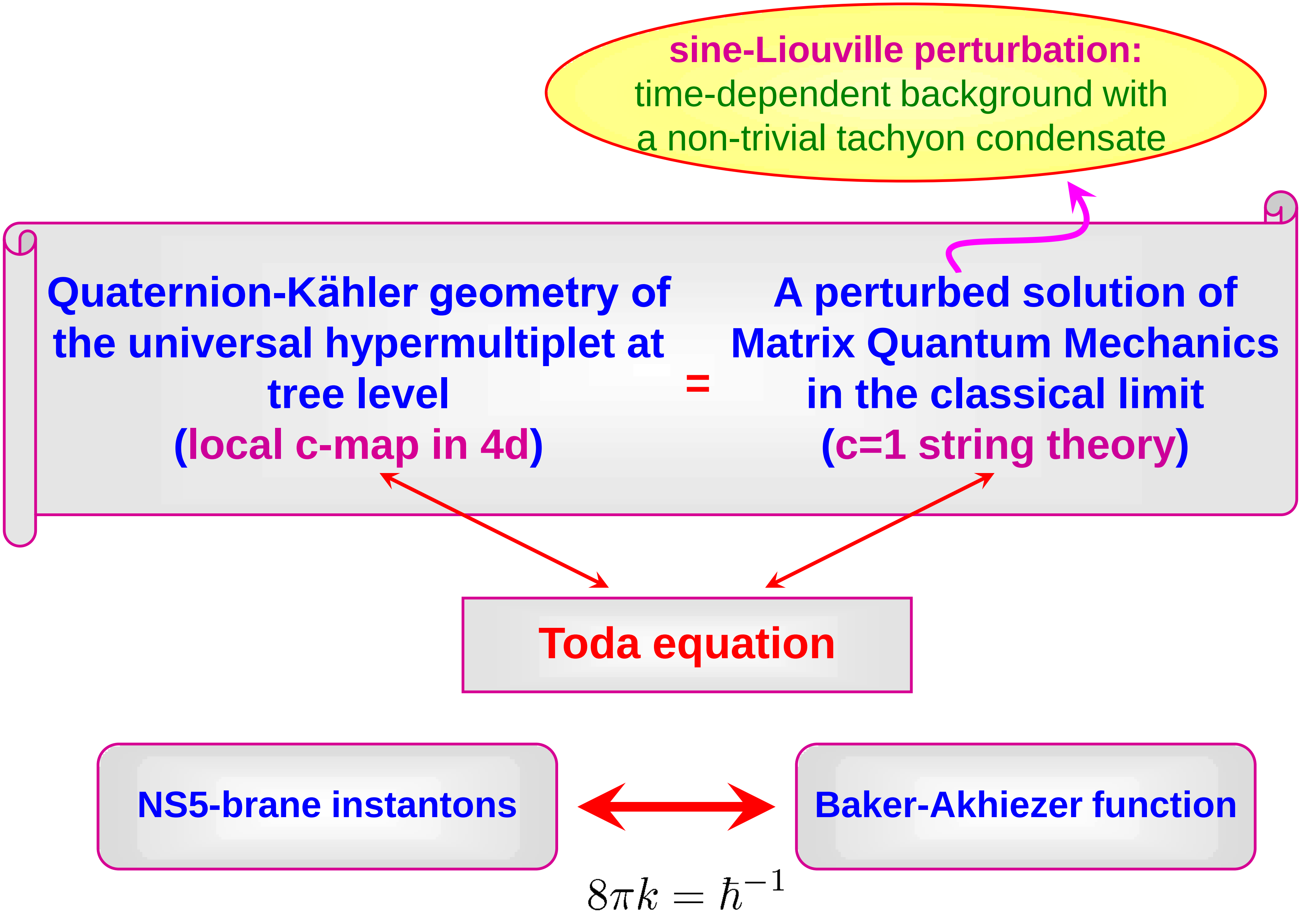
C-map as C=1 string

Sergei Alexandrov

Laboratoire Charles Coulomb
Montpellier

S.A. arXiv:1201.2761


sine-Liouville perturbation:
time-dependent background with
a non-trivial tachyon condensate
Quaternion-Kähler geometry of
the universal hypermultiplet at
tree level
(local c-map in 4d)
=
A perturbed solution of
Matrix Quantum Mechanics
in the classical limit
(c=1 string theory)
Toda equation
NS5-brane instantons
Baker-Akhiezer function
$8\pi k = \hbar^{-1}$

Plan of the talk

1. Toda hierarchy in the Lax formalism
2. Matrix formulation of c=1 string and its exact solution
3. Four-dimensional QK spaces: twistors vs. Toda
4. C-map and duality
5. NS5-barne instantons and the Baker-Akhiezer function

Toda hierarchy I

shift operator
$\hat{\omega} = e^{\hbar\partial/\partial s}$

Lax operators:

$$L = r(s)\hat{\omega} + \sum_{k=0}^{\infty} u_k(s)\hat{\omega}^{-k} \qquad \bar{L} = \hat{\omega}^{-1}r(s) + \sum_{k=0}^{\infty} \hat{\omega}^k \bar{u}_k(s)$$

Evolution laws:

$$\hbar\frac{\partial L}{\partial t_{\pm k}} = [H_{\pm k}, L] \qquad \hbar\frac{\partial \bar{L}}{\partial t_{\pm k}} = [H_{\pm k}, \bar{L}]$$

Hamiltonians:

$$H_k = (L^k)_> + \frac{1}{2}(L^k)_0 \qquad H_{-k} = (\bar{L}^k)_< + \frac{1}{2}(\bar{L}^k)_0$$

Discrete Toda equation:

$$\hbar^2\frac{\partial^2 \log r^2(s)}{\partial t_1 \partial t_{-1}} = 2r^2(s) - r^2(s+\hbar) - r^2(s-\hbar)$$

Toda hierarchy II

Orlov-Shulman operators:

$$M = \sum_{k\geq 1} k t_k L^k + s + \sum_{k\geq 1} v_k L^{-k} \qquad \bar{M} = -\sum_{k\geq 1} k t_{-k} \bar{L}^k + s - \sum_{k\geq 1} v_{-k} \bar{L}^{-k}$$

$$[L, M] = \hbar L \qquad [\bar{L}, \bar{M}] = -\hbar \bar{L}$$

$$[\hat{\omega}, s] = \hbar\, \hat{\omega}$$

Three ways to encode the solution:

- One can show

$$\frac{\partial v_k}{\partial t_l} = \frac{\partial v_l}{\partial t_k} \quad \Rightarrow \quad v_k(s,t) = \hbar^2 \frac{\partial \log \tau_s[t]}{\partial t_k}$$

τ-function of Toda hierarchy

- Baker-Akhiezer function

$$x\Psi = L\Psi \qquad \hbar x \frac{\partial \Psi}{\partial x} = M\Psi \qquad \hbar \frac{\partial \Psi}{\partial t_k} = H_k \Psi$$

- String equations

$$\bar{L} = f(L, M) \qquad \bar{M} = g(L, M)$$

Dispersionless Toda hierarchy

Classical limit: $\hbar \to 0$

$$\hat{\omega} \to \omega \; : \quad \boxed{\{\omega, s\} = \omega}$$

phase space

- τ-function: $\log \tau = \sum_{n \geq 0} \hbar^{-2+2n} F_n$

F_0 — dispersionless free energy

Continuous Toda equation:

$$\boxed{\frac{\partial^2 F_0}{\partial t_1 \partial t_{-1}} + \exp\left(\frac{\partial^2 F_0}{\partial s^2}\right) = 0}$$

- Baker-Akhiezer function:

$$\Psi \sim \exp\left\{\frac{1}{\hbar}\int^{\log x} M \mathrm{d} \log L\right\}$$

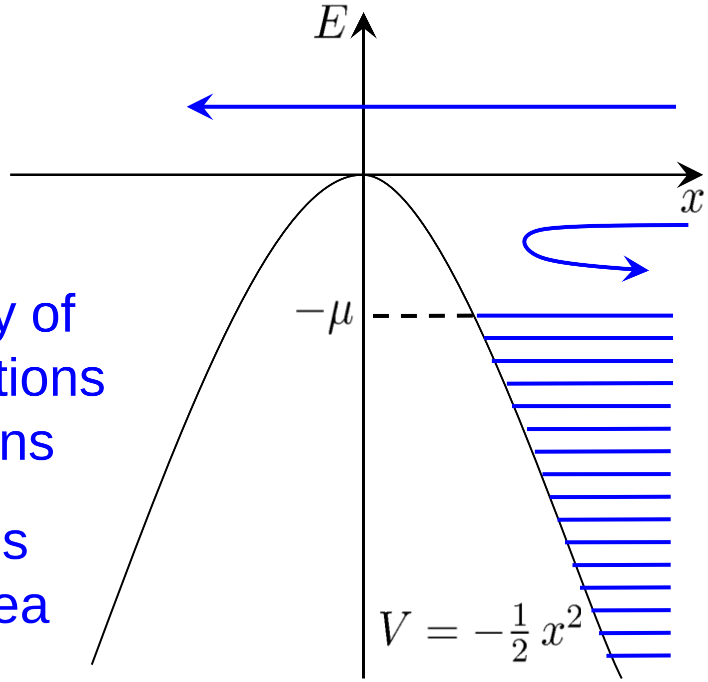
2D string theory in MQM formulation

$$Z_N = \int \mathcal{D}M(t) \exp\left[-N\mathrm{tr}\int dt\, \left(\frac{1}{2}\,\dot{M}^2 + V(M)\right)\right],$$

- diagonalize $M(t) = \Omega^\dagger(t)x(t)\Omega(t), \quad x = \mathrm{diag}(x_1, \ldots, x_N)$ and integrate out the angular degrees of freedom $\Omega(t) \in U(N)$
- take the double scaling limit: $N \to \infty, \quad \lambda \to \lambda_c$

Free fermions in the *inverse oscillator* potential

$$V(x) = -\frac{1}{2}\,x^2$$



Target space description of 2d string theory ⟷ Effective theory of collective excitations of free fermions

Different backgrounds ⟷ Different states of the Fermi sea

Light-cone representation of MQM

S.A.,Kazakov, Kostov '01

$$x_\pm = \frac{x \pm p}{\sqrt{2}}$$ — light-cone coordinates in the phase space $\{x_+, x_-\} = 1$

$$H_0 = -\frac{1}{2}(\hat{x}_+\hat{x}_- + \hat{x}_-\hat{x}_+)$$ — light-cone hamiltonian

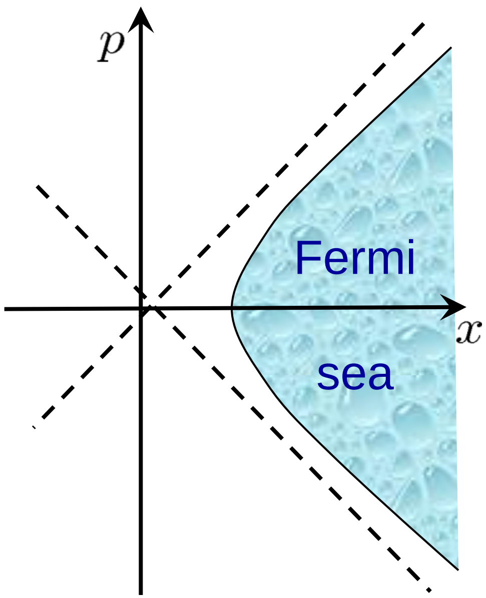
$$\psi^E_\pm(x_\pm) = \frac{1}{\sqrt{2\pi}} x_\pm^{\pm\frac{i}{\hbar}E - \frac{1}{2}}$$ — one-fermion eigenfunctions

The classical (string tree level) limit is encoded in the profile of the Fermi sea in the phase space

Example: ground state $x_+x_- = \mu$ ⟷ linear dilaton background

the only static configuration

more general states $x_+x_- = \mu + \cdots$ ⟷ *time-dependent* backgrounds with a tachyon condensate



deformations corresponding to vertex operators of c=1 string theory

Perturbed backgrounds

Perturbed wave functions

$$\Psi^E_\pm(x_\pm) = e^{-\frac{i}{\hbar}\sum_{k\geq 1} t_{\pm k} x_\pm^{k/R} + \cdots} \psi^E_\pm(x_\pm)$$

vertex operators

$$V_k = \int d^2\sigma\, e^{ikX/R} e^{(|k|/R-2)\phi}$$

$$x_+ x_- = M_\pm(x_\pm) = \pm\frac{1}{R}\sum_{k\geq 1} k t_{\pm k}\, x_\pm^{k/R} + \mu \pm \frac{1}{R}\sum_{k\geq 1} v_{\pm k}\, x_\pm^{-k/R}$$

Correspondence with the Toda hierarchy

- lattice parameter *s* ↔ Fermi level μ
- Lax operators ↔ light-cone coordinates

  $L = x_+^{1/R} \qquad \bar{L} = x_-^{1/R}$
- Orlov-Shulman operators ↔ $M_\pm(x_\pm)$
- Baker-Akhiezer function ↔ $\Psi_\pm(x_\pm)$
- string equations ↔ equations for the profile of the Fermi sea

Solution for the sine-Liouville perturbation

$(t_{\pm 1} \neq 0 \qquad R = 1)$

Free energy

$$F_0 = \frac{1}{2}\mu^2\left(\log\mu - \frac{3}{2}\right) - \mu t_1 t_{-1}$$

Profile of the Fermi sea

$$x_\pm = \sqrt{\mu}\,\omega^{\pm 1} \mp t_\mp$$

4D QK spaces: Toda description

Quaternion-Kähler manifold **=** Holonomy group $Sp(n) \times SU(2)$

Four-dimensional quaternion-Kähler manifold **=** Self-dual Einstein space with $\Lambda \neq 0$

In the presence of one isometry

$$\mathrm{d}s^2_{\mathcal{M}} = -\frac{3}{\Lambda}\left[\frac{P}{\rho^2}\left(\mathrm{d}\rho^2 + 4\mathrm{e}^T \mathrm{d}z\mathrm{d}\bar{z}\right) + \frac{1}{P\rho^2}\left(\mathrm{d}\theta + \Theta\right)^2\right]$$

where Θ — a one-form, $P = 1 - \frac{1}{2}\rho\partial_\rho T$ and

$$\partial_z\partial_{\bar{z}}T + \partial_\rho^2 e^T = 0$$

Toda equation

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where Θ — a one-form, $P = 1 - \frac{1}{2}\,\rho\partial_\rho T$ and

$$\partial_z\partial_{\bar{z}} T + \partial_\rho^2\, e^T = 0 \quad \longleftrightarrow \quad \partial_{t_1}\partial_{t_{-1}} F_0 + e^{\partial_s^2 F_0} = 0$$

$$z, \bar{z} \leftrightarrow t_{\pm 1} \qquad \rho \leftrightarrow s \qquad T \leftrightarrow \partial_s^2 F_0$$

4D QK spaces: twistor space

Quaternionic structure:

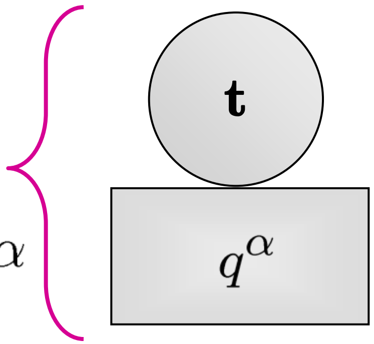
quaternion algebra of *almost* complex structures
$J^i J^j = \varepsilon^{ijk} J^k - \delta^{ij}$

$\mathcal{Z}_\mathcal{M}$ is a *complex* space and carries *holomorphic contact structure*

Twistor space

$\mathcal{Z}_\mathcal{M}$

$\xi, \tilde\xi, \alpha$



$\mathbb{C}\mathrm{P}^1$ × $\mathcal{M}$

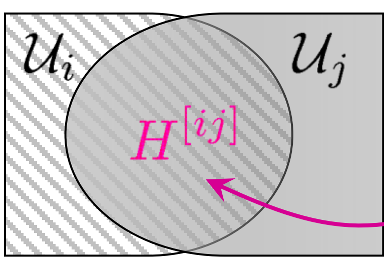
$$\mathcal{X} = \frac{2}{\mathrm{t}}\, e^{\phi} \left(\mathrm{dt} + p^+ - \mathrm{i} p^3\, \mathrm{t} + p^-\, \mathrm{t}^2 \right)$$

contact potential (dilaton)

Locally

$$\mathcal{X}^{[i]} = \mathrm{d}\alpha^{[i]} + \xi^{[i]} \mathrm{d}\tilde\xi^{[i]}$$

complex Darboux coordinates



holomorphic transition functions $H^{[ij]}(\xi, \tilde\xi, \alpha)$

gluing conditions

$$\xi_{[i]} = \xi_{[j]} + \partial_{\tilde\xi_{[j]}} H^{[ij]} - \xi_{[j]} \partial_{\alpha^{[j]}} H^{[ij]}$$
$$\tilde\xi^{[i]} = \tilde\xi^{[j]} - \partial_{\xi_{[i]}} H^{[ij]}$$
$$\alpha^{[i]} = \alpha^{[j]} - H^{[ij]} + \xi_{[i]} \partial_{\xi_{[i]}} H^{[ij]}$$

⇓

twistor lines

$$\xi_{[i]}(q^\alpha, \mathrm{t}), \;\; \tilde\xi^{[i]}(q^\alpha, \mathrm{t}), \;\; \alpha^{[i]}(q^\alpha, \mathrm{t})$$

⇓

metric on $\mathcal{M}$

Dictionary

Expansion in the patches around the north and south poles:

$$\xi^{[\pm]}(\mathrm{t}) = \mathcal{R}\mathrm{t}^{\mp 1} + \sum_{n=0}^{\infty} \xi_n^{[\pm]} \mathrm{t}^{\pm n}$$

$$\tilde{\xi}^{[\pm]}(\mathrm{t}) = \sum_{n=0}^{\infty} \tilde{\xi}_n^{[\pm]} \mathrm{t}^{\pm n}$$

$$\alpha^{[\pm]}(\mathrm{t}) = \sum_{n=0}^{\infty} \alpha_n^{[\pm]} \mathrm{t}^{\pm n}$$

?

Relation with the Toda description:

$$\rho = \mathrm{e}^{\phi} \qquad z = \frac{1}{2}\,\tilde{\xi}_0^{[+]} \qquad \theta = \frac{1}{2}\,\mathrm{Im}\alpha_0^{[+]}$$

$$T = 2\log\mathcal{R}$$

Correspondence with the Lax formalism

ω^{-1}	⟷	t
lattice parameter *s*	⟷	$\rho = e^{\phi}$
$t_{\pm 1}$	⟷	$z, \bar{z}$
$L, \bar{L}$	⟷	$\xi^{[\pm]}$
$M/L, \bar{M}/\bar{L}$	⟷	$\tilde{\xi}^{[\pm]}$
string equations	⟷	gluing conditions

String compactifications and Hypermultiplets

Type II superstring theory compactified on a Calabi-Yau

$\mathcal{N}$=2 supergravity in 4d coupled to matter

Vector multiplets

$$\mathcal{L}_{\rm VM} = \mathcal{G}_{a\bar{b}}(z)\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \mathcal{N}_{\Lambda\Sigma}(z) F^\Lambda_{\mu\nu} F^{\Sigma,\mu\nu}$$

given by

holomorphic prepotential $F(z)$

no corrections in string coupling g_s

well-known

Hypermultiplets

$$\mathcal{L}_{\rm HM} = g_{\alpha\beta}(q)\partial_\mu q^\alpha \partial^\mu \bar{q}^\beta$$

non-linear **σ**-model

- $g_{\alpha\beta}$ describes a QK manifold
- receives all types of g_s-corrections

one-loop	D-brane instantons	NS5-brane instantons
Robles-Llana, Saueressig, Vandoren '06; S.A. '07	S.A., Pioline, Saueressig, Vandoren '09	not well understood

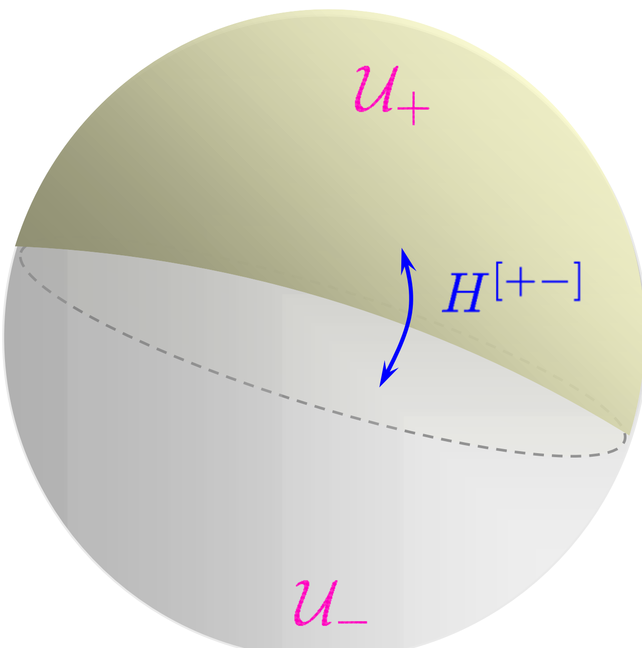
C-map

Special Kähler manifold → QK manifold

holomorphic prepotential | classical HM moduli space

Twistor description

$$H^{[+-]} = -4\,\mathrm{Im}\,F(\xi)$$



In four dimensions $(h_{2,1}(\mathfrak{Y}) = 0)$:

- prepotential $F(X) = -\frac{\mathrm{i}}{4} X^2$
- twistor lines

$$\xi^{[+]} = -(z+\bar z) + \sqrt{\rho}\left(\mathrm{t}^{-1} - \mathrm{t}\right)$$

$$\tilde\xi^{[+]} = 2\left(z + \sqrt{\rho}\,\mathrm{t}\right)$$

$$\alpha^{[+]} = \mathrm{i}\theta - \left[\rho - \frac{(z+\bar z)^2}{2} - 2\sqrt{\rho}(z+\bar z)\mathrm{t} - \rho \mathrm{t}^2\right]$$

Solution of Toda: $T = \log\rho$

Duality

4D c-map space and c=1 string theory with sine-Liouville perturbation at R=1 are described by the same solution of Toda hierarchy

$$\log\mu = \partial_\mu^2 F_0 \leftrightarrow T = \log\rho$$

Identifications:

- uniformization parameter $\omega^{-1} \leftrightarrow$ $\mathbb{C}\mathrm{P}^1$ variable t
- couplings $t_{\pm 1}$ $\leftrightarrow$ RR-fields $z, \bar{z}$
- MQM chiral coords. ↔ Lax & OV ops. ↔ Darboux coords.

$$x_+ = L = \xi^{[+]} + \frac{1}{2}\tilde{\xi}^{[+]} \qquad x_- = \bar{L} = -\xi^{[-]} + \frac{1}{2}\tilde{\xi}^{[-]}$$

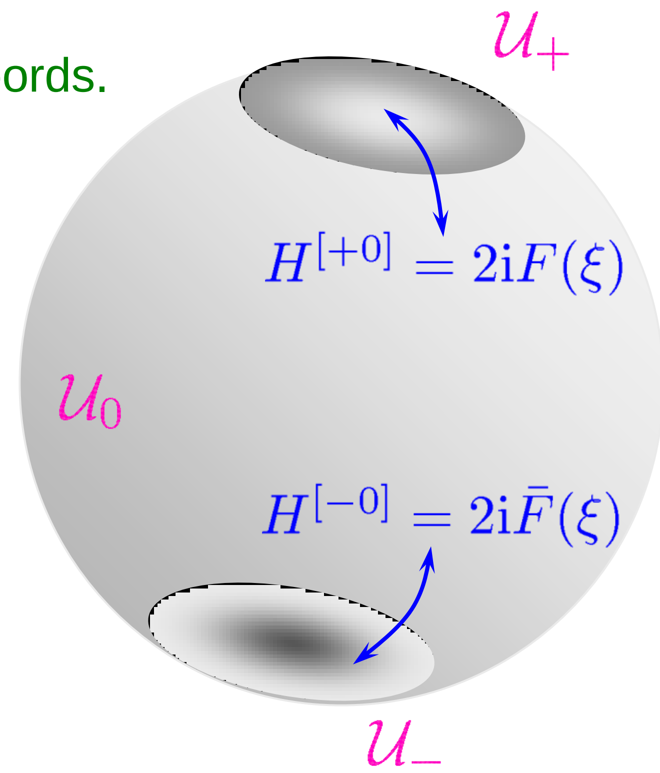
$$ML^{-1} = \frac{1}{2}\tilde{\xi}^{[+]} \qquad \bar{M}\bar{L}^{-1} = \frac{1}{2}\tilde{\xi}^{[-]}$$

- profile of the Fermi sea ↔ gluing conditions

$$\tilde{\xi}^{[+]} - \tilde{\xi}^{[-]} = -2\xi^{[\pm]}$$

In $\mathcal{U}_0$:

$$\xi = x_+ - x_- = \sqrt{2}p$$
$$\tilde{\xi} = x_+ + x_- = \sqrt{2}x$$



NS5-brane instantons and BA-function

What about α ?

Quasiclassical Baker-Akhiezer function: $\Psi \sim \exp\left\{ \frac{1}{\hbar} \int x_- \mathrm{d}x_+ \right\}$

NS-axion

$$-\frac{\mathrm{i}}{8}\,\sigma + \hbar \log \Psi(x_+) = \frac{1}{8}\left(-\mathrm{i}\tilde{\alpha} + \tilde{\xi}^2 - \xi^2\right) - \rho \log \mathrm{t}$$

the phase of the Baker-Akhiezer function ↔ Darboux coordinate α

holomorphic function incorporating NS5-brane instantons

S.A.,Pioline,Vandoren '09

$$H_k^{\mathrm{NS5}} \sim e^{-\pi \mathrm{i} k \tilde{\alpha} + \pi k \left(\tilde{\xi}^2 - \xi^2\right)}$$

NS5-brane charge

NS5-brane instanton as a fermion wave function

$$\hbar^{-1} = 8\pi k \qquad H_k^{\mathrm{NS5}} \sim \mathrm{t}^{8\pi k \rho}\, \Psi(x_+)\, e^{-\pi \mathrm{i} k \sigma}$$

Integration contour

The holomorphic function H_k^{NS5} is associated with a contour C_+ :

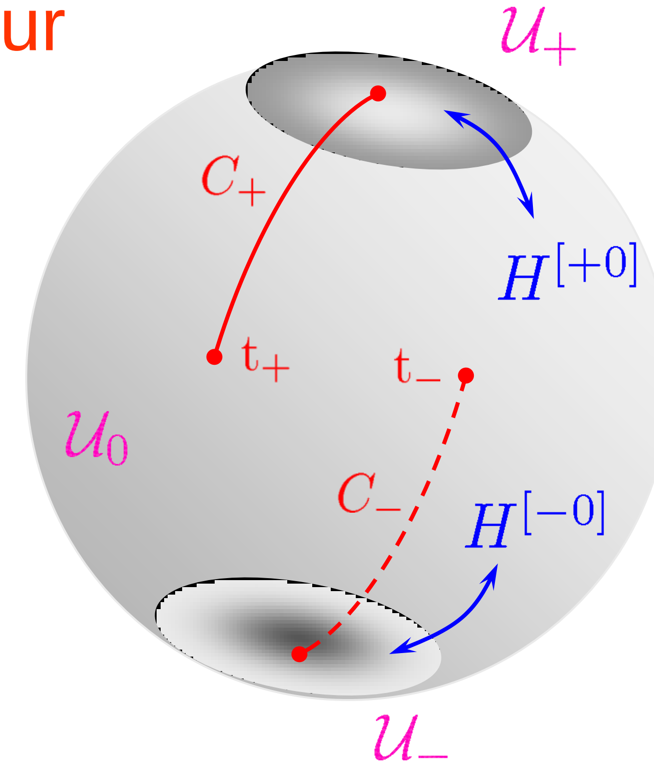
it joins the two points

$$\mathrm{t} = 0 \;\Leftrightarrow\; \xi = \infty$$

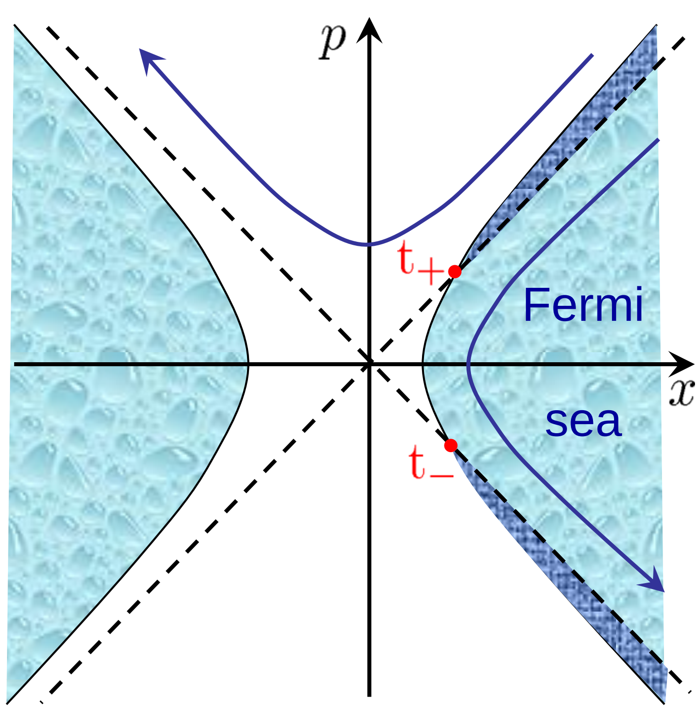
$$\mathrm{t} = \mathrm{t}_+ \equiv -\frac{z}{\sqrt{\rho}} \;\Leftrightarrow\; \xi = \tilde{\xi}$$

$$\Downarrow$$

$$x_+ = 0$$



What is its meaning?



t_+ is the intersection point of the Fermi level with the line of the perturbative stability

Conclusions

Results

- Duality between 4d local c-map and perturbed 2d string
- NS5-brane charge is associated with the inverse Planck constant, whereas the NS5-brane appears as a free fermion
- Interesting interpretation of the integration contours

Some questions

- Extension to higher dimensions?
- D-brane instantons?
- ZZ and FZZ-branes on the hypermultiplet side?
- Quantum Toda — inclusion of NS5-branes?